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AUGMENTED ELECTRIC AND MAGNETIC-FIELD INTEGRAL EQUATIONS

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Augmented electric and magnetic-field integral equations, which preserve the basic simplicity, solution capability, and pure electric and magnetic-field character of the original integral equations, are introduced to eliminate the spurious resonances from the exterior solution of the original integral equations. The exact dependence of the original and augmented integral equations on the geometry of the principal area (self patch) which excludes the singularity of their kernels is also determined; and alternate forms for the integral equations are provided that avoid integrals dependent upon the geometry of the

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principal area. Numerical results obtained for scattering from the perfectly conducting cube, sphere, and infinite circular cylinder confirm the theoretically predicted result that the augmented integral equations eliminate the spurious resonances for all perfectly conducting scatterers except the sphere.

Preface

This research was begun as a "Director's Reserve" project at the National Bureau of Standards, Boulder, Colorado and was completed as a "visiting scientist" at RADC/EEA Hanscom AFB, Massachusetts. My deep appreciation goes especially to R. J. Mailloux, C. J. Drane, W. Rotman, and A. C. Schell of RADC/EEA for providing the encouragement, facilities, and support to complete this work. Stimulating, detailed discussions with R. E. Kleinman at the University of Delaware led to the present integrated form of the 2-dimensional augmented integral equations. Helpful suggestions and references were also contributed by D. J. N. Wall, V. V. Varadan, and V. K. Varadan of Ohio State University.

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Contents

Contents

Illustrations

Augmented Electric and Magnetic-Field Integral Equations

1. INTRODUCTION

In 1949 Maue[1] derived the following two alternative integral equations for the surface current density excited by arbitrary harmonic fields incident upon a perfect electric conductor in free space:

$$\hat{n} \times \overline{E}_{inc} = \frac{\hat{n}}{4\pi i\omega\epsilon_o} \times \oint_S [k^2\overline{K}_s(\overline{r}')\phi - (\nabla_s' \cdot \overline{K}_s)\nabla'\phi]\, dS' \tag{1}$$

$$\hat{n} \times \overline{H}_{inc} = \pm \frac{1}{2} \overline{K}_s(\overline{r}) - \frac{\hat{n}}{4\pi} \times \oint_S \overline{K}_s(\overline{r}') \times \nabla'\phi\, dS' \quad . \tag{2}$$

(Strictly speaking, Eq. (1) is an integro-differential equation because it involves the divergence of $\overline{K}_s$ as well as $\overline{K}_s$. However, because the divergence of $\overline{K}_s$ lies under the integral sign and because it is possible to recast Eq. (1) in a form involving only $\overline{K}_s$ [but involving the highly singular operator $\nabla\nabla$] it will be referred

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1. Maue, A.W. (1949) On the formulation of a general scattering problem by means of an integral equation, Zeitschrift fur Physik 126(7):601-618.

to simply as an integr l equation. Also, Maue[1] and recently Kisliuk and Gozani[2] have pointed out that the MFIE Eq. (2) was presented without derivation by Fock[3] in 1946.) In these two equations, which are popularly referred to as the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE) respectively,[4] $\overline{K}_s$ is the unknown surface current density, $\phi(\overline{r}', \overline{r})$ is the free space scalar Green's function $\exp(ik|\overline{r}' - \overline{r}|)/|\overline{r}' - \overline{r}|$, $\overline{E}_{inc}$ and $\overline{H}_{inc}$ are the given electric and magnetic incident fields in the absence of the conductor, and $\overline{r}$ is the position vector to any nonedge point on the closed regular surface S[5] of the conducting body, with the unit normal $\hat{n}$ to S at $\overline{r}$ in a direction away from the interior region. As usual, ϵ_0 denotes the free space permittivity and k the free space propagation constant (equals ω/c where c is the speed of light); $\exp(-i\omega t)$ time dependence (with real ω) has been suppressed and the International System of Units (mksA) is used throughout. The surface integrations in Eqs. (1) and (2) are evaluated by excluding the singular point, $\overline{r}' = \overline{r}$, of the integrand by a circular "principal area" of diameter δ centered on $\overline{r}$ and letting δ approach zero. (In Sections 5 and 6 we generalize the integral equations to allow for arbitrarily shaped principal areas or self patches.)

The integral Eqs. (1) and (2) hold for both exterior and interior regions, with the only explicit change occurring in the sign of the $\overline{K}_s/2$ term of Eq. (2)—the positive sign applying for exterior (scattering or radiating) problems and the negative sign for interior (cavity) problems. An additional implicit change occurs, of course, in terms of the incident fields. The incident fields for the scatterer are generated by sources applied outside S; those for the cavity are generated by sources inside S.

Maue[1] noted two major difficulties with the integral equations: nonuniqueness of the exterior solution at interior or cavity resonant frequencies, and conditional convergence of the EFIE integral.

When applied to an exterior region, both Eqs. (1) and (2) fail to yield a unique solution for $\overline{K}_s$ at frequencies equal to the resonant frequencies of the corresponding interior cavity—even though the exterior solution to Maxwell's equations plus boundary and radiation conditions for a perfectly conducting scatterer exists

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2. Kisliuk, M. and Gozani, J. (1980) An alternate formulation of Maue's integral equation, Digest of URSI Symposium on EM Waves, Munich, 122B/1-2.

3. Fock, V. (1946) The distribution of currents induced by a plane wave on the surface of a conductor, J. of Phys. 10(2):130-136.

4. Poggio, A.J. and Miller, E.K. (1973) Integral equation solutions of three-dimensional scattering problems, Computer Techniques for Electromagnetics, R. Mittra, ed., Pergamon, New York, pp. 159-264.

5. Kellogg, O.D. (1929) Foundations of Potential Theory, Springer-Verlag, New York, p. 112.

uniquely.[6] (The solution to Eqs. (1) and (2) in an interior region enclosed by a perfect conductor also becomes nonunique at the cavity frequencies, but it can be shown (see Section 2 below) that the homogeneous solutions responsible for the nonuniqueness in the interior region are simply the proper cavity mode solutions exhibited by Maxwell's equations plus boundary conditions at the resonant frequencies).

Ever since the first numerical solutions were programmed using Eqs. (1) and (2) for both 2-dimensional (2-D)[7] and 3-dimensional (3-D) problems,[8, 9, 4] these spurious solutions in the exterior region at the interior resonant frequencies have interfered with the useful application of the integral equations. Although, in principle, the spurious solutions occur only at exactly the resonant frequencies, in numerical practice the solution to both the EFIE and MFIE for $\overline{K}_s$ deviates (coefficient matrices become ill-conditioned) within a significant bandwidth about (or near) the resonant frequencies and imitates a valid resonant phenomenon.[7, 8, 9, 10, 11, 12] Consequently, the solution for the fields and, in particular, the far fields computed from $\overline{K}_s$ become spurious in a bandwidth about the resonant frequencies for both the EFIE and MFIE, even though it can be shown analytically that the spurious currents of the EFIE do not, in theory, radiate (see Oshiro,[11] Mautz,[13] and Section 2). The problem accentuates at larger body dimension to wavelength ratios because the interval between successive resonant frequencies decreases as this ratio increases. For example, a spherical cavity of radius "a" has 4 resonant modes between ka equal to 0 and 5 but 15 resonant modes between ka equal to 5 and 10. Moreover, numerical filtering and

6. Saunders, W.K. (1952) On solutions of Maxwell's equations in an exterior region, Proc. Natl. Acad. Sci. 38(4):342-348.

7. Mei, K.K. and Van Bladel, J.G. (1963) Scattering by perfectly-conducting rectangular cylinders, IEEE Trans. Antennas Propagat. AP-11(2): 185-192.

8. Oshiro, F.K. (1965) Source distribution techniques for the solutions of general electromagnetic scattering problems, Proc. First GISAT Mitre Corp., 1:83-107.

9. Oshiro, F.K. and Su, C.S. (1965) A Source Distribution Technique for the Solution of General Electromagnetic Scattering Problems, Northrop Norair Rept. NOR 65-271.

10. Andreasen, M.G. (1964) Comments on scattering by conducting rectangular cylinders, IEEE Trans. Antennas Propagat. AP-12(2):235-236.

11. Oshiro, F.K., et al (1967) Calculation of Radar Cross Section, Pt I, Vol I, Air Force Tech. Rept. AFAL-TR-67-308.

12. Oshiro, F.K., Mitzner, K.M., and Locus, S.S. et al (1970) Calculation of Radar Cross Section Pt. II, Air Force Tech. Report AFAL-TR-70-21.

13. Mautz, J.R. and Harrington, R.F. (1978) H-field, E-field, and combined-field, solutions for conducting bodies of revolution, AEU Electronics and Communication 32(4):159-164.

regularization techniques have been shown to be inadequate to eliminate the spurious resonant solutions.[14, 15]

Three basic methods have been applied successfully for eliminating the spurious resonances from the electromagnetic exterior surface integral equation solution. They will be referred to as the combined field, combined source, and hybrid methods. In acoustics a fourth method, the modified Green's function method, has also been applied to eliminate the spurious resonances by introducing a more complicated Green's function into the kernels of the integral operators.[16, 17, 18, 19]

The combined field method constructs a single integral equation by adding $\hat{n} \times$ Eq. (1) (multiplied by an arbitrary constant) to Eq. (2). This linear combination of the EFIE and MFIE yields an integral equation which has a unique solution in the exterior region at all frequencies.[12, 4, 13] Burton and Miller[20] formulate another combined field method which has been used[21, 22] to eliminate the spurious resonances from the exterior Neumann acoustics problem. Specifically, after noting that the normal derivative of potential is not necessarily restricted to zero by the integral equation, they add to the double-layer integral equation a second integral equation formed by taking the normal derivative of the extended double layer equation. Kleinman, Roach and Angell combine, through augmentation

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14. Klein, C.A. and Mittra, R. (1975) An application of the "condition number" concept to the solution of scattering problems in the presence of the interior resonant frequencies, IEEE Trans. Antennas Propagat. AP-23(3):431-435; also 448-450.

15. Seidel, D.B. (1974) A new method for the detection and correction of errors due to interior resonance for the problem of scattering from cylinders of arbitrary cross section, M.S. Thesis, The University of Arizona.

16. Roach, G.F. (1967) On the approximate solution of elliptic self-adjoint boundary value problems, Arch. Ration. Mech. Anal. 27(3):243-254; also (1970) 36(1):79-88.

17. Ursell, F. (1973) On the exterior problems of acoustics, Proc. Camb. Phil. Soc. 74(1):117-125.

18. Jones, D.S. (1974) Integral equations for the exterior acoustic problem, Q. Jl. Mech. Appl. Math. 27(1):129-142.

19. Colton, D. and Windland, W. (1976) Constructive methods for solving the exterior Neumann problem for the reduced wave equation in a spherically symmetric medium, Proc. Roy. Soc. Edin. 75A(8):97-107; also (1978) SIAM J.Math-Anal. 9(5):935-942.

20. Burton, A.J. and Miller, G.F. (1971) The application of integral equation methods to the numerical solution of some exterior boundary-value problems, Proc. Roy. Soc. Lond., Series A, 323(1553):201-210.

21. Meyer, W.L., et al (1978) Boundary integral solutions of three dimensional acoustic radiation problems, J. of Sound and Vibration 59(2):245-262.

22. Meyer, W.L., et al (1979) Prediction of the sound field radiation from axisymmetric surfaces, J. Acoust. Soc. Am. 65(3):631-638.

rather than addition, a similar supplementary integral equation to obtain uniqueness for the exterior Dirichlet[23] and Robin[24] problems as well as the Neumann problem. When applied to 2-D problems (infinite cylinders), the augmentation introduced in the present paper becomes identical to using the 2-D EFIE and MFIE simultaneously, or identical mathematically to the Kleinman and Roach[23] augmentation for 2-D acoustic problems.

The combined source method derives an alternative integral equation to Eq. (1) which expresses the electric field in terms of ficticious electric and magnetic surface currents. The ficticious magnetic current is specified as equal to a constant times $\hat{n} \times$ the ficticious electric current; and the resulting integral equation is shown to have a unique solution in the exterior region at all frequencies.[25–33] When certain multiplicative constants are chosen, the combined source and combined field integral operators become the adjoint of each other.[25]

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23. Kleinman, R.E. and Roach, G.F. (1974) Boundary integral equations for the three-dimensional Helmholtz equation, SIAM Review 16(2):214-236.

24. Angell, T.S. and Kleinman, R.E. (1980) Boundary integral equations for the Helmholtz equation: the third boundary value problem, Applied Math. Inst. Tech. Report 73A, Univ. of Delaware, Newark, Delaware.

25. Mautz, J.R. and Harrington, R.F. (1979) A combined-source solution for radiation and scattering from a perfectly conducting body, IEEE Trans. Antenna Propagat. AP-27(4):445-454.

26. Brakhage, H. and Werner, P. (1965) Uber das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung, Arch. Math. 16(415):325-329.

27. Greenspan, D., and Werner, P. (1966) A numerical method for the exterior Dirichlet problem for the reduced wave equation, Arch. Ration. Mech. Anal. 23(4):288-316.

28. Kussmaul, R., and Werner, P. (1968) Fehlerabschatzungen für ein numerisches Verfahren zur Auflosung Linearer Integralgleichungen mit Schwachsingulaten Kernen, Computing (Arch. Elektron. Rechnen) 3(1): 22-46.

29. Kussmaul, R. (1969) Ein Numerisches Verfahren zur Losung des Neumannschen Aussenraumaufgabe für die Helmholtzsche Schwingungsgleichung, Ibid. 4(3):246-273.

30. Panic, O.I. (1965) On the solubility of exterior boundary value problems for the wave equation and for a system of Maxwell's equations, Uspehi Mat. Nauk 20(1):221-226.

31. Bolomey, J.C. and Tabbara, W. (1973) Numerical aspects on coupling between complementary boundary value problems, IEEE Trans. Antenna Propagat. AP-21(3):356-363.

32. Knauff, W. and Kress, R. (1979) On the exterior boundary-value problem for the time-harmonic Maxwell equations, J. Math. Anal. Appl. 72(1): 215-235.

33. Schenck, H.A. (1967) Improved integral formulation for acoustic radiation problems, J. Acoust. Soc. Am. 44(1):41-58.

The hybrid method, sometimes referred to as the method of Schenck, who used the scheme for the analogous acoustic integral equations,[33] supplements either Eq. (1) or (2) when applied in an exterior region with its corresponding "extended integral equation" which holds throughout the interior region.[34,35,36] If the extended integral equation is satisfied at a limited number of judiciously chosen interior points, the spurious resonant solutions will be suppressed in the EFIE and MFIE solution.[14,15,37] It should be mentioned that the extended integral equations alone satisfied within the interior region produce a unique solution and have been applied separately to solve both electromagnetic[35] and acoustic[38,39] problems. The idea of the hybrid approach, however, is to use the extended integral equations in a limited fashion to overdetermine the surface integral equations sufficiently to reject the spurious resonances from their exterior solutions.

Certain disadvantages accompany each of the basic methods which have been used to eliminate the spurious resonances. Some of these have been discussed in the papers by Jones[18,36] and Mautz and Harrington.[13,25]

In applying the hybrid method one has no reliable criterion for selecting the number and position of interior points at which the extended integral equations must be satisfied to assure convergence to the correct unique solution for the exterior EFIE and MFIE at all frequencies.* Moreover, because the extended integral equations are 3-component equations, unlike the 2-component surface EFIE and MFIE, there remains an uncertainty as to what components of the extended integral equations should be utilized.

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*For the acoustics problem Jones[18] has devised an alternative hybrid approach which suggests a systematic way of choosing the interior points. Recently, for 2-dimensional, scalar EM problems, Morita has also applied "some lower order equations of the extended boundary condition method" to help remove the arbitrariness in choosing the interior points.[40]

34. Stratton, J.A. (1941) Electromagnetic Theory, Sec. 8-14, McGraw-Hill, New York.

35. Waterman, P.C. (1965) Matrix formulation of electromagnetic scattering, Proc. IEEE 53(8):805-812.

36. Jones, D.S. (1974) Numerical methods for antenna problems, Proc. IEE 121(7):573-582.

37. Morita, N. (1978) Surface integral representations for electromagnetic scattering from dielectric cylinders, IEEE Trans. Antennas Propagat. AP-26(2):261-266.

38. Waterman, P.C. (1968) New formulation of acoustic scattering, J. Acoust. Soc. Am. 45(6):1417-1429.

39. Copley, L.G. (1967) Integral equation method for radiation from vibrating bodies, J. Acoust. Soc. Am. 41(4):807-816; also (1968) 44(1):28-32.

40. Morita, N. (1979) Resonant solutions involved in the integral equation approach to scattering from conducting and dielectric cylinders, IEEE Trans. Antennas Propagat. AP-27(6):869-871.

The combined field and combined source equations each present a unique solution to the exterior problem at all frequencies but at a substantial increase in complexity and programming over the original EFIE and MFIE. (Matrix fill time also increases but this is inconsequential for most scattering problems where matrix inversion time eventually dominates matrix fill time.) The increased complexity is especially pronounced in contrast to the MFIE which involves only $\overline{K}_s$, $\overline{H}_{inc}$, $\nabla\phi$, and conforms well to efficient, straightforward point matching or "subsectional collocation"[4] moment methods of solution. The combined field and combined source integrals involve all the operators present in both the EFIE and MFIE as well as the derivatives of $\overline{K}_s$ present in the EFIE which is not as receptive as the MFIE to an accurate solution by the simpler numerical techniques. The combined field equation demands input of the incident (or impressed) electric and magnetic fields, and thus becomes inconvenient for antenna or aperture problems which specify only the impressed electric field. The combined source equation requires only the incident electric field but yields a solution in terms of ficticious surface currents. To obtain the actual surface currents and fields, an indirect computation of $\hat{n} \times H$ must be performed at a sacrifice of simplicity and computer time.[25]

The primary objective of this paper is to augment the EFIE and MFIE separately to eliminate the spurious resonant solutions from the exterior region without sacrificing the basic simplicity, solution capability, and pure electric and magnetic field character of the original two equations. To accomplish this we begin by revealing exactly why the EFIE and MFIE are deficient in the exterior region, that is, why they do not yield the unique Maxwellian solution at frequencies equal to the interior resonant frequencies. In particular, it is proven that the electric field tangent to the scatterer is not restricted to zero by the MFIE, and the magnetic field tangent to the scatterer is not restricted to equal $\overline{K}_s \times \hat{n}$ in the EFIE solution at (and only at) the interior resonant frequencies. This, in turn, implies that $\hat{n} \cdot \overline{H}$ and $(\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_0)$ are not necessarily restricted to zero by the exterior MFIE and EFIE, respectively, at the interior resonant frequencies.

Except for helicoids, which for our purposes mean bodies of revolution and infinite cylinders, augmenting the MFIE with the equation $\hat{n} \cdot \overline{H} = 0$ and the EFIE with the equation $\hat{n} \cdot \overline{E} = \nabla_s \cdot \overline{K}_s / i\omega\epsilon_0$ is then prover to immediately remedy these deficiencies, eliminate the spurious resonances, and compel the solution to the augmented integral equations to equal the unique Maxwellian solution in the exterior region for all frequencies. Moreover, the augmentation transforms the electric field integral equation from an integral equation of the first kind to an integral equation of the second kind which, like the MFIE, is more amenable to a stable numerical solution than the original EFIE. (Integral equations of the first kind depend strongly upon the singular nature of their kernels for their solvability;[41] see also Jones [Sec. 5i].[36])

For finite bodies of revolution except the sphere, and for infinite cylinders, the augmented electric and magnetic field integral equations (AEFIE, AMFIE) also eliminate the spurious resonance solutions but only after the incident field is divided into E_z and H_z axial components and the AEFIE and AMFIE are applied separately. (Of course, this division of incident fields and separate applications of both integral equations is commonly done by choice when applying the original EFIE and MFIE to the infinite cylinders.)

Thus, the sphere (which has a simple eigenfunction solution) remains the one scatterer for which the augmented integral equations cannot be applied to eliminate all the spurious resonances. This, plus the necessity to use both the augmented electric and magnetic integral equations for bodies of revolution when the solution for both arbitrary E_z and H_z incidence is required, constitutes the main disadvantage of the augmented integral equations.

Most of the preceding conclusions on the uniqueness of solution for the augmented integral equations follow from a basic theorem and central result of the paper proven in Appendix A for the electromagnetic field interior to a cavity; namely, that Maxwell's equations inside a perfectly conducting cavity can be satisfied by electric modes transverse with respect to the normal to the surface of the cavity only if the surface is a helicoid. And the only helicoids which represent scatterers or antennas solvable directly by surface integral equations are bodies of revolution and infinite cylinders.

A second objective of this paper is to correct the second difficulty noted by Maue,[1] that of the conditional convergence of the EFIE, a subtlety which to the author's knowledge, has been ignored in subsequent treatments. The EFIE, unlike the MFIE, is conditionally convergent in that the form of Eq. (1) depends critically upon choosing a circle centered upon the singularity of the integrand as the principal area (self patch) used when evaluating the principal value integral. A side effect to augmenting the integral equations is to make the integral in the AMFIE as well as the EFIE (and AEFIE) conditionally convergent with respect to the geometry of the principal area. Thus, our second objective is actually to determine the exact dependence of the EFIE, AEFIE, and AMFIE on the geometry of the principal area and to provide alternative forms of these three integral equations that remain independent of the chosen principal area.

Finally, some numerical results are obtained to test the theory. First, the MFIE and AMFIE are applied to the problem of planewave scattering from a

41. Courant, R. and Hilbert, D. (1953) Methods of Mathematical Physics, Ch. 3, Sec. 10.11, Interscience, New York.

perfectly conducting cube, demonstrating that the augmented equation did indeed remove the spurious resonances which severely denigrated the original solution beyond a cube size parameter of about 2.5.

The original and augmented integral equations are applied to the problem of planewave incidence upon infinite circular cylinders for which the exact solution is known—again demonstrating the predicted elimination of the spurious resonances by the augmented integral equations.

Scattering from the perfectly conducting sphere is also determined numerically using the EFIE, MFIE, and their augmented counterparts, the AEFIE and AMFIE, with comparison being made to the exact eigenfunction (Mie) solution. The surface values of $\hat{n} \cdot \overline{H}$ for the MFIE and $(\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_0)$ for the EFIE were also monitored as the frequency was changed. As the theory of Section 3 predicts, the surface values of $\hat{n} \cdot \overline{H}$ and $(\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_0)$ for the MFIE and EFIE respectively became nonnegligible at the TM spherical cavity frequencies where the augmented integral equations eliminated the spurious resonances, but remained negligible at the TE spherical cavity frequencies where the augmented integral equations retained the spurious resonances.

In brief, the numerical results confirmed the theoretical predictions that, except for the sphere, the augmented integral equations eliminate the spurious resonances; and, for 3-D scatterers, require, as Section 3 explains, an insignificant increase in computer programming, run time, and central memory requirements over that of Maue's original integral equations. For 2-D problems, the augmented integral equation approach becomes identical mathematically to the method of Kleinman and Roach,[23] and equivalent to the combined field method.

Many of the results contained in this report were first presented at the 1980 International URSI Symposium on Electromagnetic waves.[42]

2. PRECISE DETERMINATION OF WHY THE MFIE AND EFIE ALLOW SPURIOUS SOLUTIONS

Maue provided in his original paper[1] that the EFIE and MFIE (and their acoustic analogues) have a unique solution except at the frequencies equal to the resonant frequencies of the interior cavity bounded by the perfectly conducting surface S. Nonunique solutions at the resonant frequencies are, of course, present when applying the integral equations to the interior region because Maxwell's equations (from which the integral equations are derived) allow homogeneous solutions (the

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42. Yaghjian, A.D. (1980) Augmented electric and magnetic-field integral equations which eliminate the spurious resonances, Proceedings of the 1980 Intl. URSI. Symp., 121B/1-121B/4.

cavity modes) at the resonant frequencies. However, for the exterior problem Maxwell's equations with the boundary conditions and outgoing radiation condition demand a unique solution for fields and currents scattered or radiated from a perfect conductor.[6] Thus, the homogeneous solutions to the exterior MFIE and EFIE are spurious solutions which do not satisfy entirely the Maxwellian boundary value problem. (A direct way to show that these spurious homogeneous solutions exist in the exterior region is to note that the homogeneous solutions to Eq. (1) are the same for both the interior and exterior problems, and that the homogeneous interior and exterior operators of Eq. (2) are effectively adjoints of each other[43, 44] — adjoint linear operators for an even determined set of equations having identical real eigenvalues; that is, resonant frequencies.)

Although Maxwell's equations and boundary conditions are used to derive Eqs. (1) and (2), it must be concluded that Eqs. (1) and (2), when applied in the exterior region, are not equivalent to Maxwell's equations plus boundary conditions. The initial step in eliminating the spurious resonances from the exterior problem will therefore be to determine the basic reasons for this nonequivalence. Concomitantly, it will be proven that Eqs. (1) and (2) applied to the interior problem are equivalent to the Maxwellian system even at the resonant frequencies; that is, the only homogeneous solutions to Eqs. (1) and (2) in the interior region are indeed the surface currents of the cavity modes.

As a preliminary to the analysis, we supply the expressions to compute the scattered fields ($\overline{E}_{sc}, \overline{H}_{sc}$) from $\overline{K}_s$ determined by Eqs. (1) and (2). For either the interior region with all applied sources inside S, or the exterior region with all applied sources outside S, the formal expressions are the same:

$$\overline{E}_{sc}(\overline{r}) = \frac{-1}{4\pi i\omega\epsilon_0} \int_S [k^2 \overline{K}_s \phi - (\nabla_s' \cdot \overline{K}_s)\nabla'\phi] \, dS' \tag{3a}$$

$$\overline{H}_{sc}(\overline{r}) = \frac{1}{4\pi} \int_S \overline{K}_s \times \nabla'\phi \, dS' \quad . \tag{3b}$$

Taking the curl of Eqs. (3a) and (3b) shows that the scattered fields obey Maxwell's homogeneous equations within their region of application. Since the incident fields obey Maxwell's equations in free space, the total fields

43. Muller, Claus (1969) Foundations of the Mathematical Theory of Electromagnetic Waves, Sec. 25, Springer-Verlag, New York.

44. Marin, L. (1973) Natural-mode representation of transient scattered field, IEEE Trans. Antennas Propagat. AP-21(6):809-818.

$$\overline{E} = \overline{E}_{sc} + \overline{E}_{inc} \quad , \quad \overline{H} = \overline{H}_{sc} + \overline{H}_{inc} \quad , \tag{4a, b}$$

obey Maxwell's equations within their region of application; specifically,

$$\nabla \times \overline{E} = i\omega\mu_o \overline{H} \tag{5a}$$

$$\nabla \times \overline{H} = -i\omega\epsilon_o \overline{E} + \overline{J}_{inc} \quad , \tag{5b}$$

where $\overline{J}_{inc}$ is the incident applied current density.

Let the observation point $\overline{r}$ in Eq. (3a) approach the surface S of the conductor from its region of validity, and convert the surface integration to a circular principal value integration using the following formula (derived by a straightforward integration near the singularity of ϕ):

$$\int_{S(\overline{r}\to S)} \nabla'\phi \, dS' = \oint_S \nabla'\phi \, dS' \pm 2\pi\hat{n} \quad , \tag{6}$$

where the + and - sign apply to the exterior and interior problem respectively. Combining this result with Eqs. (1) and (4a) shows immediately that the EFIE solution yields zero tangential E-field on S; that is,

$$\hat{n} \times \overline{E} = 0 \quad \text{on S} \quad \text{(for the EFIE)} \quad . \tag{7a}$$

Thus, in principle, the fields (but not necessarily the surface current) determined from the solutions to the EFIE are the correct Maxwellian fields.

Similarly letting the observation point $\overline{r}$ approach the surface S in Eq. (3b) reveals from Eqs. (2), (6), and (4b) that the solutions $\overline{K}_s$ to the MFIE will satisfy

$$\overline{K}_s = \pm \hat{n} \times \overline{H} \quad \text{on S} \quad \text{(for the MFIE)} \quad . \tag{7b}$$

(Again the + and - signs in Eq. (7b) are associated with the exterior and interior regions respectively.) It is emphasized that we have not proved, however, that the EFIE satisfies this latter boundary condition Eq. (7b), nor that the MFIE satisfies the former boundary condition Eq. (7a) of zero tangential E-field on S.

2.1 Proof That the Surface Tangential E-Field Determined by the Exterior MFIE is Unequal to Zero at the Interior Resonant Frequencies

Because the fields determined through Eqs. (3a) and (3b) from the $\overline{K}_s$ of Eq. (2) satisfy Maxwell's Eqs. (5a) and (5b) in the region of application (interior or exterior), the Stratton-Chu formulas[34] apply; and, in particular, Eq. (20) of Stratton-Chu[34] combines with Eqs. (3b) and (7b) to yield the following equation for the electric field tangential to S:

$$\int_S [k^2(\hat{n}' \times E)\phi - \nabla_s' \cdot (\hat{n}' \times \overline{E})\nabla'\phi] \, dS' = 0 \tag{8}$$

for all $\overline{r}$ within the region of application.

2.1.1 APPLICATION TO THE INTERIOR MFIE

Apply Eq. (8) to an interior region ($\overline{r}$ inside S), let $\overline{r}$ approach S, use Eq. (6), and take $\hat{n} \times$ the resulting equation to get

$$\hat{n} \times \oint_S [k^2(\hat{n}' \times \overline{E})\phi - \nabla_s' \cdot (\hat{n}' \times \overline{E})\nabla'\phi] \, dS' = 0 \quad , \quad \overline{r} \text{ on S} \quad . \tag{9}$$

Denoting the left side of Eq. (8) by $\overline{F}(\overline{r})$, we see that by taking the curl of Eq. (8) twice we get

$$\nabla \times \nabla \times \overline{F} - k^2\overline{F} = 0 \tag{10a}$$

in the exterior as well as interior region, and from Eq. (9)

$$\hat{n} \times \overline{F} = 0 \quad , \quad \overline{r} \text{ on S} \quad . \tag{10b}$$

In addition, $\overline{F}$ satisfies the outgoing radiation condition as $\overline{r} \to \infty$ and thus $\overline{F} = 0$ is is the only solution[6] for all $\overline{r}$ outside S as well. Because $\overline{F}$ is zero both inside and outside S, the curl of $\overline{F}$ is also zero inside and outside S. Thus, letting $\overline{r} \to$ S in the equation $\nabla \times \overline{F} = 0$ from inside and outside, converting the resulting integrals to principal value integrals via Eq. (6), and subtracting the two equations yields $\hat{n} \times \overline{E} = 0$ on S. Since we proved in Eq. (7b) that the boundary condition on current is satisfied by the MFIE, we have completed the proof that the MFIE applied to an interior region is equivalent to Maxwell's equations and all boundary conditions, even at the resonant frequencies of this interior region (cavity). In particular, the homogeneous solutions to Eq. (2) (with the minus sign) are simply the proper surface current densities of the cavity modes.

2.1.2 APPLICATION TO THE EXTERIOR MFIE

The same method of proof does not apply in total for the MFIE applied to an exterior region, because in that case when Eqs. (10a) and (10b) are considered for the corresponding interior region it is found to possess a nontrivial solution at (but only at) the cavity resonant frequencies. Thus, the above method of proof also shows the equivalence of the exterior MFIE and the Maxwellian system except at the cavity resonant frequencies.

We can complete the analysis for the exterior MFIE by returning to Eq. (9), which is equivalent to Eq. (8) in the exterior region because Eq. (9) also derives from Eq. (8) in an exterior region, and Eqs. (10a) and (10b) along with the radiation condition can be applied to derive Eq. (8) from Eq. (9) in an exterior region.* Moreover, Eq. (9) is merely the homogeneous EFIE with $\hat{n} \times \overline{E}$ replacing $\overline{K}_s$. In Section 2.2 it is proven that the only nontrivial solutions to the homogeneous EFIE are the cavity mode surface current densities. Moreover, Appendix B proves that Eq. (9) is also a sufficient as well as necessary condition for $\hat{n} \times \overline{E}$, obtained from the system of Eqs. (2) and (3a), to satisfy in the exterior region. Consequently, we conclude that <u>at and only at the resonant frequencies of the associated cavity, the exterior MFIE is not equivalent to the Maxwellian system of equations and boundary conditions simply because of these frequencies the exterior MFIE allows nonzero tangential electric fields ($\hat{n} \times \overline{E}$ on S) equal to the surface current densities of the interior</u> cavity modes.

It is emphasized that it is not the spurious exterior MFIE currents, but the spurious tangential electric fields ($\hat{n} \times \overline{E}$) which equal the cavity mode currents. Moreover the spurious exterior MFIE electric currents are not equal to the effective modal electric currents ($\hat{n} \times$ magnetization) for the cavity with perfect magnetic walls. There is a simple relationship between the modes of a perfect magnetic cavity and the spurious MFIE solutions but it is between tangential electric fields, that is, magnetic "currents," and not electric currents. Consider Eq. (9). Applying the same method of analysis used in Sections 2.1, 2.1.1, and 2.2, proves that the solutions to Eq. (9) for $\hat{n} \times \overline{E}$ are the homogeneous solutions to the tangential electric field ($\hat{n} \times \overline{E}_m$) on the surface of a cavity with perfectly magnetically conducting walls S; that is,

$$\nabla \times \nabla \times \overline{H}_m - k^2 \overline{H}_m = 0 \tag{11a}$$

$$\overline{E}_m = -\nabla \times \overline{H}_m / i\omega\epsilon_o \tag{11b}$$

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*To avoid confusion with the results obtained in Section 2.1.1 for the interior region, note that Eq. (9) is not equivalent to Eq. (8) for the MFIE applied to the interior region.

$$\hat{n} \times \overline{H}_m = 0 \quad \text{on } S \quad . \tag{11c}$$

There is no simple relationship between the spurious MFIE electric currents and the fields of either the perfectly electric or magnetic cavity resonator. Mathematically, the spurious MFIE current densities are simply the homogeneous solutions to the exterior MFIE given by Eq. (2) with the plus sign, or equivalently, essentially the adjoint solutions of the interior homogeneous Eq. (2) with the minus.[43, 44]

Physically, there is a fairly straightforward interpretation for the spurious MFIE surface currents. In the wall of the magnetic cavity with the solutions obeying Eqs. (11a), (11b), and (11c) there will be a surface magnetization $\overline{M}_s$ at the resonant frequencies equal to $-\hat{n} \times \overline{E}_m / i\omega\mu_o$ on S. Now suppose the inside of the cavity is filled with perfect electric conductor and $\overline{M}_s$ is maintained at its original value everywhere in the magnetic wall of the cavity. A surface current density will be excited on the surface of the electrical conductor which will reduce the tangential electric field inside S to zero and a tangential electric field equal to $\hat{n} \times \overline{E}_m$ will be produced on the outside of the cavity wall S. Thus, one can interpret physically the spurious MFIE currents as the electrical surface currents that the surface magnetization of the perfectly magnetic cavity modes would produce if the interior cavity were filled with perfect electric conductor and the surface magnetization maintained the same.

The spurious MFIE magnetic field just outside S will equal (as Eq. (7b) indicates) $-\hat{n} \times$, this spurious surface current density. However, the fundamental reason that the MFIE allows spurious currents and fields on S and therefore spurious fields throughout the exterior region is that at the cavity resonant frequencies the MFIE does not demand that the tangential E-field (calculated from Eqs. (3a) and (4a)) be zero as one approaches the surface S from the exterior region. In Section 3 the spurious MFIE solutions are eliminated by forcing the tangential E-field to equal zero even at the cavity resonant frequencies.

2.2 Proof that the Surface Current Density Determined by the Exterior EFIE is Unequal to the Surface $\hat{n} \times \overline{H}$ at the Interior Resonant Frequencies

Proceeding as we did in Section 2.1 with the Stratton-Chu formulas, Eq. (19) of Stratton[34] combines with Eqs. (3a) and (7a) to derive Eqs. (8), (9), (10a), and (10b) but with $\overline{K}_s \mp (\hat{n} \times \overline{H})$ replacing $\hat{n} \times \overline{E}$. (The upper and lower sign refer to the exterior and interior problem, respectively.) Thus the analysis of Section 2.1.1, can be invoked immediately to prove the following results for the interior EFIE.

The EFIE applied to an interior region is equivalent to Maxwell's equations and boundary conditions, even at the resonant frequencies of this interior region (cavity). In particular, the homogeneous solutions to Eq. (1) are simply the proper surface current densities of the cavity modes.

For the exterior EFIE (as for the exterior MFIE), the problem of determining uniqueness reduces to investigating the solutions to Eq. (9), specifically the homogeneous EFIE with $(\overline{K}_s - \hat{n} \times \overline{H})$ as the unknown surface vector. But we just proved that the solutions to the homogeneous EFIE were identical to the surface currents of the resonant cavity modes. Thus, we can conclude immediately that at and only at the resonant frequencies of the associated cavity, the exterior EFIE is not equivalent to the Maxwellian system of equations and boundary conditions simply because at those frequencies the exterior EFIE allows nonzero $(\overline{K}_s - \hat{n} \times \overline{H})$ on S equal to the surface current densities of the interior cavity modes. In other words, at the cavity resonant frequencies the tangential H-field calculated from Eqs. (3b) and (4b) for the exterior EFIE problem will not equal $\overline{K}_s \times \hat{n}$ (calculated from Eq. (1)) as the perfectly conducting surface S is approached from the exterior region and as the Maxwellian solution demands. In principle, since the EFIE spurious currents are identical to the cavity currents, they, unlike the MFIE spurious currents, produce zero $\overline{E}$ and $\overline{H}$ fields outside S – a fact reassured by Eq. (7a). In practice, numerical inaccuracies associated with the ill-conditioned matrices at the resonant frequencies contaminate the exterior fields with spurious solutions as well. In Section 3 we show how the spurious EFIE currents can be eliminated in principle and practice by forcing them equal to their associated zero $\hat{n} \times \overline{H}$ fields just outside S.

3. AUGMENTING THE MFIE AND EFIE TO YIELD THE UNIQUE EXTERIOR SOLUTION AT ALL FREQUENCIES

Answering the question in Section 2 of exactly why the exterior integral equations allow spurious solutions at the eigenfrequencies of the cavity leads one quite naturally to a method of eliminating these unwanted solutions. First consider the exterior MFIE. At the internal resonant frequencies the E-field tangential to S calculated from the MFIE surface currents are not zero; it follows that the H-field normal to S will not necessarily be zero. Thus, we immediately consider augmenting the exterior MFIE with the equation

$$\hat{n} \cdot \overline{H} = 0 \quad \text{on S} \tag{12a}$$

to hopefully eliminate its spurious resonances. By a similar argument we conclude that the exterior EFIE should be augmented by demanding that the normal component of $\epsilon_o \overline{E}$ on S equal the surface charge density; that is,

$$\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_o = 0 \qquad \text{on } S \tag{12b}$$

in hopes of forcing $\overline{K}_s$ equal to $\hat{n} \times \overline{H}$ on S for all frequencies and thus eliminating its spurious resonances. Note that the auxiliary boundary conditions Eqs. (12a) and (12b) are equivalent to $\nabla_s \cdot (\hat{n} \times \overline{E}) = 0$ and $\nabla_s \cdot (\hat{n} \times \overline{H} - \overline{K}_s) = 0$ on S, respectively.

The boundary conditions Eqs. (12a) and (12b) are written in terms of incident and scattered fields by substituting from Eqs. (3a), (3b), (4a), and (4b), to obtain

$$\hat{n} \cdot \overline{H}_{inc}(\overline{r}) = \frac{-\hat{n}}{4\pi} \cdot \oint_S \overline{K}_s \times \nabla'\phi \, dS' \tag{13a}$$

$$\hat{n} \cdot \overline{E}_{inc} = \frac{\hat{n}}{4\pi i\omega\epsilon_o} \cdot \oint_S [k^2 \overline{K}_s \phi - (\nabla_s' \cdot \overline{K}_s)\nabla'\phi] \, dS' + \frac{\nabla_s \cdot \overline{K}_s}{2i\omega\epsilon_o} \quad . \tag{13b}$$

If the original exterior integral Eqs. (1) and (2) are now augmented with Eqs. (12a) and (12b) by adding Eq. (13b) to Eq. (1) and Eq. (13a) to Eq. (2), we obtain the "augmented" exterior electric and magnetic field integral equations (AEFIE and AMFIE):

$$\overline{E}_{inc}(\overline{r}) = \frac{1}{4\pi i\omega\epsilon_o} \oint_S [k^2 \overline{K}_s \phi - (\nabla_s' \cdot \overline{K}_s)\nabla'\phi] \, dS' + \frac{(\nabla_s \cdot \overline{K}_s)\hat{n}}{2i\omega\epsilon_o} \tag{14}$$

$$-\overline{H}_{inc} = \frac{1}{2}\hat{n} \times \overline{K}_s + \frac{1}{4\pi} \oint_S \overline{K}_s \times \nabla'\phi \, dS' \quad . \tag{15}$$

Note that Eq. (1) and Eq. (2) in the exterior region are reclaimed by taking $\hat{n} \times$ Eq. (14) and Eq. (15), respectively. Also, the augmentation has transformed the EFIE from an integral equation of the first kind to an integral equation of the second kind, which, like the MFIE is more amenable to a stable numerical solution. (Integral equations of the first kind, unlike those of the second kind, depend strongly on the singular nature of their kernels for their solvability [Courant[41] and Jones,[36] Section 5i].)

Before leaving this section, we point out that Eqs. (12a) and (12b) can be obtained by taking the surface divergence ($\nabla_s \cdot$) of Eqs. (1) and (2), respectively. Thus, the AMFIE Eq. (15) is simply the original MFIE Eq. (2) augmented with $\nabla_s \cdot$, the original EFIE Eq. (1); the AEFIE Eq. (14) is simply the original EFIE Eq. (1) augmented with $\nabla_s \cdot$, the original MFIE Eq. (2). Alternatively, the augmented integral Eqs. (14) and (15) derive directly from the corresponding extended integral equations by letting the observation point $\bar{r}$ approach the surface S from the interior region (see Section 3.2).

3.1 Uniqueness of Solution for the Augmented Integral Equations

The augmented integral Eqs. (14) and (15) are very similar in form to the original Eqs. (1) and (2) and require, as Section 3.3 explains, very little increase in programming complexity, or computer run time and central memory. However, they provide no advantage unless we prove they eliminate the spurious resonance solutions.

As a first step in this proof, repeat the analysis of Section 2 using Eqs. (14) and (15) instead of Eqs. (1) and (2) in the exterior region, noted from Eqs. (12a) and (12b), which are derivable from Eqs. (14), (15), (3a), and (3b), that the $\nabla_s \cdot (\hat{n} \times \bar{E})$ and $\nabla_s \cdot (\hat{n} \times \bar{H} - \bar{K}_s)$ terms are zero. This procedure shows that both Eqs. (14) and (15) have spurious solutions if and only if the following system of equations have a nontrivial solution:

$$\nabla \times \nabla \times \bar{F} - k^2\bar{F} = 0 \quad \text{inside S} \tag{16a}$$

$$\bar{F} = 0 \quad \text{on S} \quad . \tag{16b}$$

In other words, Eqs. (16a) and (16b) indicate that the AEFIE Eq. (14) and the AMFIE Eq. (15) eliminate the spurious resonances from the exterior scattering or radiating problem unless a cavity resonant mode has zero normal field as well as zero tangential field on S.

At first thought one might expect that Eqs. (16a) and (16b) possess no nontrivial solutions because the cavity modes at the eigenfrequencies are the only solutions to Eqs. (16a) and (16b) even before the normal component of $\bar{F}$ is specified zero on the boundary S. However, the counterexample of a sphere shows that there exists at least one shape of cavity S for which some of the cavity modes satisfy Eqs. (16a) and (16b). Specifically, the TE modes of a perfectly electrically conducting spherical cavity resonator have zero normal as well as tangential E-field on the surface S. Thus, we know immediately that for a sphere Eqs. (14) and (15) will eliminate the spurious solutions only at the frequencies of the TM (but not the

TE) resonant modes. Fortunately, it may not be critical that the spherical scatterer be solved by the integral equation approach because it is the one 3-D geometry that has a simple eigenfunction solution, the familiar Mie solution. Thus, we remain undaunted by this one counterexample and ask if there are any shapes S besides the sphere for which solutions to Eqs. (16a) and (16b) exist.

Appendix A answers this question and thereby performs the proof of the central result of this paper: Except for the TE modes of the sphere, the augmented electric and magnetic field integral Eqs. (14) and (15), eliminate the spurious resonances and yield the unique solution to Maxwell's equations and boundary conditions in the exterior region at all frequencies. For bodies of revolution (other than the sphere) and for infinite cylinders, the theorem holds but requires a special procedure explained in Section 4.

This crucial proof detailed in Appendix A begins by expressing Eqs. (16a) and (16b) in orthogonal curvilinear coordinates formed by the family of surfaces parallel to S and the two families of surfaces generated by the normals to S along the lines of curvature. It is shown that Eqs. (16a) and (16b) can be satisfied if and only if the associated PEC solution can have TE modes with respect to the normal direction, and that this is possible if and only if either: 1) the principal radii of curvature of S are equal for all points on S, or 2) the principal radii of curvature and the fields depend only on one tangential curvilinear coordinate. The equations of Mainardi-Codazzi and Gauss[45] for the fundamental magnitudes of the surface S are then invoked to prove that case 1 holds only for the sphere and case 2 only for helicoids which, for our purposes, means bodies of revolution or infinite cylinders. Finally, Section 4 shows that, except for the sphere, these latter two geometries can also be solved by the augmented integral equations without introducing spurious resonances. This is accomplished by applying the AEFIE Eq. (14) or the MFIE Eq. (15) separately to the appropriate component. (This division is commonly made when solving infinite cylinder problems using the original integral equations as well.)

3.2 Derivation of the Augmented Integral Equations from Extended Integral Equations

The augmented Eqs. (14) and (15) can be derived directly from the electric and magnetic extended integral equations. The following form of these extended equations convenient for this derivation is obtained directly by applying the Stratton-Chu formulas to the exterior surface currents ($\overline{K}_s(\overline{r}) = \hat{n} \times \overline{H}(\overline{r}')$, $\overline{r}' \rightarrow S$ from outside S) and mathematically choosing the observation point $\overline{r}$ inside S:

45. Eisenhart, L. P. (1909) A Treatise on the Differential Geometry of Curves and Surfaces, Ginn, Boston.

$$\overline{E}_{inc}(\overline{r}) = \frac{1}{4\pi i\omega\epsilon_o} \int_S [k^2\overline{K}_s(\overline{r}')\phi - (\nabla_s' \cdot \overline{K}_s)\nabla'\phi]\, dS' \tag{17}$$

$$-\overline{H}_{inc}(\overline{r}) = \frac{1}{4\pi} \int_S \overline{K}_s(\overline{r}') \times \nabla'\phi \, dS' \quad . \tag{18}$$

Allowing $\overline{r}$ to approach S in Eqs. (17) and (18), and converting the surface integrations to circular principal area integrations via Eq. (6), immediately produces the augmented surface integral Eqs. (14) and (15).

Waterman's technique[35] involves applying a version of Eqs. (17) or (18) to a subvolume within S. Copley[39] raises the question of whether the acoustic equations corresponding to Eqs. (17) or (18) would produce a unique solution when satisfied only over a surface S_o lying inside S. In Section 3.1 we answered that question for the electromagnetic case when the chosen surface S_o lying inside S approaches the surface S itself.

The analysis of 3.1 can also be applied to the extended integral Eqs. (17) and (18) satisfied on a surface S_o inside S. That is, Eqs. (17) and (18) need only be applied for $\overline{r}$ on a surface S_o inside S in order to yield the unique solution for the surface current density $\overline{K}_s$ excited on the surface S of the scatterer (provided, as explained in Section 3.1, that S_o is not a sphere, and that for bodies of revolution and for infinite cylinders the appropriate integral equation be applied separately to E_z and H_z incident fields). No attempt has been made to program Eqs. (17) or (18) for $\overline{r}$ on any surface S_o other than S, where Eqs. (17) and (18) become equal to the augmented Eqs. (14) and (15), respectively. Intuitively, it is expected that accurate numerical results would be obtained from Eqs. (17) or (18) applied to $\overline{r}$ on S_o only if S_o were chosen fairly close to S.

Lastly, we point out that Kisliuk and Gozani[2] have recently derived an alternate and equivalent version of the AMFIE Eq. (15) which holds for $\overline{r}$ just outside S.

3.3 Efficient Solution of the Augmented Integral Equations

The same numerical techniques used to solve Maue's original integral Eqs. (1) and (2) could be applied directly to the augmented Eqs. (14) and (15) if it were not for one difficulty. The augmented Eqs. (14) and (15) are overdetermined. There are three scalar components to each of the equations but only two unknown components of surface current $\overline{K}_s$. Fortunately, because we have been able to prove that the augmented integral equations poss ɜs a unique solution at all frequencies (under the above stated conditions), an elegant theorem of linear algebra[46] allows us to solve Eqs. (14) or (15) by simply multiplying the set of equations by the Hermitian conjugate of its coefficient matrix, and then solving the resulting evendetermined Hermitian set of equations. This procedure is a standard means for finding the least squares solution to an overdetermined set of equations. It becomes an especially powerful tool for our purposes because of the theorem[46] which says that when the overdetermined set can be proven to have a unique solution, the least squares solution becomes identical to this unique solution.

Direct inversion of the resulting Hermitian matrix does have the drawback that its conditioning is worse than the original matrix.[47] Alternative solution techniques to avoid this degradation in matrix conditioning have been devised,[48] but no attempt was made nor was it found necessary to incorporate them in the straightforward, unsophisticated numerical solutions used to test the theory of the present report.

Specifically, suppose a 3-D surface S is divided into p patches and Eqs. (14) or (15) has been reduced by the method of moments or some similar numerical solution technique to the set of linear equations,

$$C_{mn}X_n = b_m \tag{19}$$

where n = 1, 2, ... 2p, m = 1, 2, ... 3p and summation over repeated indices is understood. (The original integral Eqs. (1) and (2) are contained in Eq. (19) for n, m = 1, 2, ... 2p.) Multiplying Eq. (19) by the Hermitian conjugate of the coefficient matrix converts the overdetermined $3p \times 2p$ set of equations to the evendetermined $2p \times 2p$ set,

$$[C^*_{m\ell}\, C_{mn}]\, X_n = C^*_{m\ell}\, b_m \quad . \tag{20}$$

Equation (20) can then be solved by any one of the readily available matrix inversion or solution algorithms. Equivalently, Eq. (19) can be solved directly by any one of the many least squares solution algorithms that are also readily available.

Equations (19) and (20) reveal that the solving of the augmented integral equations require two additions to the solving of the original Eqs. (1) and (2): 1) the $2p^2$ extra elements of the coefficient matrix C_{mn} for n = 2p + 1 to 3 p must be

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46. Mirsky, L. (1955) An Introduction to Linear Algebra, Theorem 5.5.4 and Sec. 5.5.5, Oxford University Press.

47. Osborne, E.E. (1961) On least squares solutions of linear equations, J. Assoc. Comp. Mach. **8**:628-636.

48. Golub, G. (1965) Numerical methods for solving linear least squares problems, Num. Math. **7**:206-216.

computed, and 2) the Hermitian conjugate multiplication must be performed. This means that matrix fill time and matrix storage requirements will increase by about 50 percent; and there will be a small increase in computer run time needed to perform the Hermitian conjugate multiplication. The multiplication of the Hermitian conjugate of the matrix by itself can be performed efficiently with little increase in computer storage because the resulting matrix is Hermitian and thus the original matrix array can be used to store successively computed elements of the product matrix without ever requiring a second $2p \times 2p$ dimensioned array. Also, the extra computer storage required by the augmented coefficient matrix can be supplied, if necessary, from outside the central memory core because the final matrix inversion of Eq. (20) requires the same in-core storage as the coefficient matrix of the original integral equations.

Generally, matrix inversion time rapidly dominates matrix fill time as the electrical size of the scatterer becomes larger than a small fraction of a wavelength.[4] However, if matrix fill time forms a significant portion of total computer run time, the extra matrix fill time introduced by the augmented integral equations can also be reduced by invoking the augmented integral equations only at frequencies where $\hat{n} \cdot \overline{H}$ (for the MFIE) and $\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_o$ (for the EFIE) exceed a present threshold. In short, the 3-D augmented integral equations require an insignificant increase in computer programming, run time, and central memory requirements over that of Maue's original integral equations.

4. AUGMENTED INTEGRAL EQUATIONS APPLIED TO INFINITE CYLINDERS AND BODIES OF REVOLUTION

Appendix A shows that the AEFIE and AMFIE remove the spurious resonances from all scatterers except at certain resonant frequencies of helicoids. For practical scattering problems this is true, except at 1) the E_z 2-D cavity mode frequencies of infinite cylinders and 2) the frequencies of the rotationally symmetric TE (with respect to the surface normals) cavity modes for bodies of revolution (including the infinite circular cylinder and the sphere). This section shows how the augmented equations can be applied to also eliminate these remaining resonances from infinite cylinders and bodies of revolution, except the sphere.

4.1 Infinite Cylinders (2-D Problems)

The 2-D EFIE corresponding to Eq. (1) was derived in 1950 by Papas[49] for an incident E_z-wave. Later Mei and Van Bladel[7] also derived the 2-D MFIE

49. Papas, C.H. (1950) Diffraction by a cylindrical obstacle, J. Appl. Phys. 21(4):318-325.

corresponding to Eq. (2) for an incident H_z-wave. Both the 2-D EFIE and MFIE for an arbitrary 2-D incident field emerge directly from the transverse components of the 3-D integral equations by substituting into Eqs. (1) and (2) the integral representation for the Hankel function:

$$\int_{-\infty}^{\infty} \phi(\bar{r}', \bar{r})\, dz' = \pi i H_0^{(1)}(k|\bar{t}' - \bar{t}|) \quad , \quad \bar{t}' \neq \bar{t} \quad , \quad (\bar{r} = \bar{t} + z\,\hat{e}_z \quad , \quad \bar{r}' = \bar{t}' + z'\hat{e}_z) \quad ,$$

after first showing that for 2-D geometries the circular principal area can be converted to an infinitely long narrow slit (with the singularity at the center of the slit) without altering the form of Eqs. (1) and (2). Specifically, for a 2-D body (infinite cylinder) with z-axis and cross-sectional boundary curve C,

$$E^z_{inc}(\bar{t}) = \frac{k^2}{4\omega\epsilon_0} \int_C K_z(\bar{t}')H_0^{(1)}(k|t' - \bar{t}|)\, dc' \tag{21a}$$

$$-H^z_{inc} = \frac{1}{2} K_c + \frac{i}{4} \int_C K_c \frac{\partial H_0^{(1)}}{\partial n'}\, dc' \quad . \tag{21b}$$

(The line integrations here and below are performed by approaching symmetrically the singularity of the integrals.) A valuable review of the 2-D electromagnetic work through 1973, including the spurious resonances, is contained in the thesis by Seidel.[14]

For 2-D problems the augmentations Eqs. (13a) and (13b) can be expressed as simply the circumferential derivative of the 2-D EFIE Eq. (21a) and the 2-D MFIE Eq. (21b), respectively; thus by performing the circumferential integration analytically and setting the arbitrary constant to zero, the 2-D augmentations formed from Eq. (13) reduce to Eq. (21).

For an incident E_z-wave, the remaining component of the 2-D AMFIE formed from Eq. (15) is:

$$H^c_{inc} = \frac{1}{2} K_z + \frac{i}{4} \int_C K_z \frac{\partial H_0^{(1)}}{\partial n}\, dc' \quad . \tag{22a}$$

Similarly, for an incident H_z-wave, the remaining component of the 2-D AEFIE formed from Eq. (14) is:

$$E^c_{inc} = \frac{1}{4\pi\epsilon_o} \oint_C \left[k^2 K_c (\hat{e}'_c \cdot \hat{e}_c) H_o^{(1)} + \frac{\partial K_c}{\partial c'} \frac{\partial H_o^{(1)}}{\partial c} \right] dc' \quad . \tag{22b}$$

Appendix A shows that, except of the circular cylinder, the spurious surface currents for the 2-D AEFIE and 2-D AMFIE with arbitrary incident fields lie in the z and c-directions, respectively; thus, Eqs. (21a) and (22a), and (21b) and (22b), will yield unique solutions except possibly for the circular cylinder. But since the circumferential integrations of the 2-D augmentations formed from Eq. (13) have been performed to reduce them to the original 2-D Eqs. (21a) and (21b), all boundary conditions are satisfied, and thus the 2-D AMFIE Eqs. (21a) and (22a), and the 2-D AEFIE Eqs. (21b) and (22b), also yield unique solutions for scattering from the circular cylinder.

In summary, the two pairs of scalar equations, Eqs. (21a) and (22a), and (21b) and (22b), comprise the 2-D AMFIE and the 2-D AEFIE for incident E_z and H_z-waves, respectively; and each of these 2-D augmented integral equations eliminate entirely the spurious resonances for scattering from perfectly conducting infinite cylinders. This is confirmed in the numerical results of Section 6. The 2-D augmented equations, like the 3-D ones, are overdetermined, but can be solved efficiently using the Hermitian conjugate multiplication or alternative solution techniques discussed in Section 3.3.

The 2-D augmented integral equations presented here are identical mathematically to the 2-D form of the augmented integral equations proposed by Kleinman and Roach,[23] who also prove their uniqueness of solution. If the components Eqs. (21a) and (22a) or Eqs. (21b) and (22b) of the 2-D augmented integral equations are added together, the 2-D combined field integral equations result.

4.2 Finite Bodies of Revolution

For finite rotationally symmetric bodies other than the sphere, we can also eliminate the remaining spurious resonances by dividing the incident field into E_z and H_z components. Appendix A shows that the spurious currents for finite bodies of revolution (except the sphere) do not vary with rotation angle ϕ, and are in the $\hat{e}_\phi$-direction for the AEFIE Eq. (14) and in the $\hat{\tau} = \hat{n} \times \hat{e}_\phi$ direction for the AMFIE Eq. (15).

Now if an incident field has only an E_z or H_z component (where z is the axis of the body of revolution) there will be excited no rationally symmetric component of K_ϕ or K_τ respectively. Thus the augmented integral Eqs. (14) and (15) also eliminate the spurious resonances from all finite bodies of revolution except the sphere by dividing the incident field into E_z and H_z components, and applying the AEFIE Eq. (14) (with the rotationally symmetric part of K_ϕ set to zero) and the AMFIE Eq. (15) (with the rotationally symmetric part of K_τ set to zero), respectively. Of course, for certain incident fields such as a plane wave with direction of propagation normal to the z-axis of rotation, no rotationally symmetric component of current is excited and either augmented Eqs. (14) or (15) alone will eliminate the spurious resonances.

5. THE ORIGINAL AND AUGMENTED INTEGRAL EQUATIONS EXPRESSED FOR ARBITRARY PRINCIPAL AREAS

The original and augmented integral Eqs. (1), (2), (14), and (15) assumed that a circular "principal area" excludes the singularity of the freespace Green's function at $\bar{r}' = \bar{r}$ in their surface integrals. The integrations are performed centering the circular principal area of diameter δ on the singular point and taking the limit as δ approaches zero. In solving the integral equations numerically, however, it may be desirable to choose principal areas, that is, self patches, other than the circular one.* Fortunately, it becomes a simple matter to generalize the integral equations to arbitrary principal areas once Eq. (6) is generalized. To generalize Eq. (6) to an arbitrary principal area, note that we can write as an identity,

$$\oint_S \nabla'\phi \, dS' = \oint_S \nabla'\phi \, dS' - \oint_{\Delta S} \nabla'\phi \, dS' \quad . \tag{23}$$

The surface integral on the left hand side of Eq. (23) excludes the singularity by the arbitrarily shaped principal area, the first surface integral on the right excludes the singularity by a centered circular principal area, and the second integral on the right is an integration over the limiting surface ΔS between the arbitrary and inscribed circular principal area. The geometry (shape, position, and orientation) of the principal area with respect to the singular point $\bar{r}$ is specified and maintained as its size shrinks to zero. Because the integral over ΔS does not include the singularity and because $\Delta S \to 0$ in the limit as the principal area shrinks to zero, this integral can be evaluated with the help of an integral formula contained in Van Bladel[50] Appendix 2, Eq. (43):

*In numerical work one must also use a finite sized self patch and either determine how small the self patch need be to produce negligible errors in the solution or compensate analytically for its finite size. This topic of finite principal areas will not be discussed in this report.

50. Van Bladel, J. (1964) Electromagnetic Fields, McGraw-Hill, New York.

$$\oint_{\Delta S} \nabla'\phi \, dS' = \oint_{\Delta S} \nabla'_s \phi \, dS' = \oint_C \frac{\hat{u}_\perp}{R} \, dc - \oint_C \frac{\hat{u}_\perp}{R} \, dc \quad . \tag{24}$$

The second integral on the right of Eq. (24) is zero and thus Eqs. (24) and (23) substituted into Eq. (6) yields,

$$\int_{S\,(\overline{r}\to S)} \nabla'\phi \, dS' = \oint_S \nabla'\phi \, dS' \pm 2\pi\hat{n} + 2\pi\overline{T} \quad , \tag{25a}$$

where

$$\overline{T} \equiv \frac{1}{2\pi} \oint_C \frac{\hat{u}_\perp}{R} \, dc \quad . \tag{25b}$$

As depicted in Figure 1 the unit vector $\hat{u}_\perp$ is in the plane of the principal area and perpendicular to its boundary C. Thus $\overline{T}$ is tangent to the surface S, and vanishes for a circular or regular-polygonal principal area centered on the singularity.

Applying the identity Eq. (25a) instead of Eq. (6) to the derivation of the original and augmented integral equations reveals that the form of the MFIE remains independent of the principal area (as Maue[1] proved), but the form of the EFIE, AEFIE and AMFIE change to, respectively

$$\hat{n} \times \overline{E}_{inc} = \frac{\hat{n}}{4\pi i\omega\epsilon_o} \times \oint_S [k^2 \overline{K}_s \phi - (\nabla'_s \cdot \overline{K}_s)\nabla'\phi] \, dS' - \frac{(\nabla_s \cdot \overline{K}_s)\,\hat{n} \times T}{2i\omega\epsilon_o} \tag{26}$$

$$\overline{E}_{inc} = \frac{1}{4\pi i\omega\epsilon_o} \oint_S [k^2 \overline{K}_s \phi - (\nabla'_s \cdot \overline{K}_s)\nabla'\phi] \, dS' + \frac{(\nabla_s \cdot \overline{K}_s)(\hat{n} - \overline{T})}{2i\omega\epsilon_o} \tag{27}$$

$$-\overline{H}_{inc} = \frac{1}{2}(\hat{n} - \overline{T}) \times \overline{K}_s + \frac{1}{4\pi} \oint_S \overline{K}_s \times \nabla\psi \, dS' \quad . \tag{28}$$

Both augmented integral Eqs. AEFIE (27) and AMFIE (28) and the original EFIE Eq. (26) (as Maue[1] stated) are "conditionally" convergent upon the shape of the principal area (self patch) chosen to exclude the singularity of the freespace Green's function. The strength of this dependence on principal area is determined

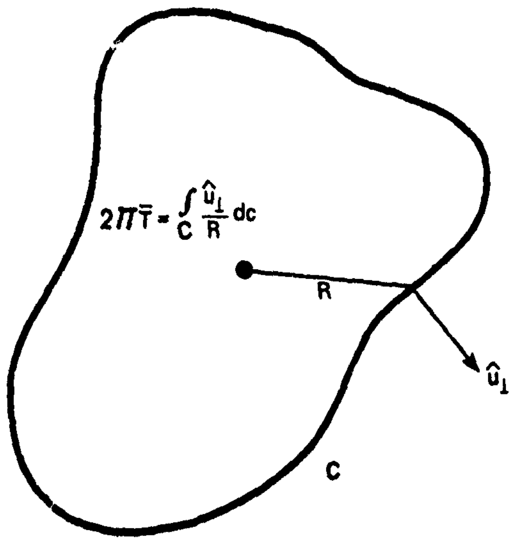

Figure 1. Definition of Tangential Principal Area Vector $\overline{T}$

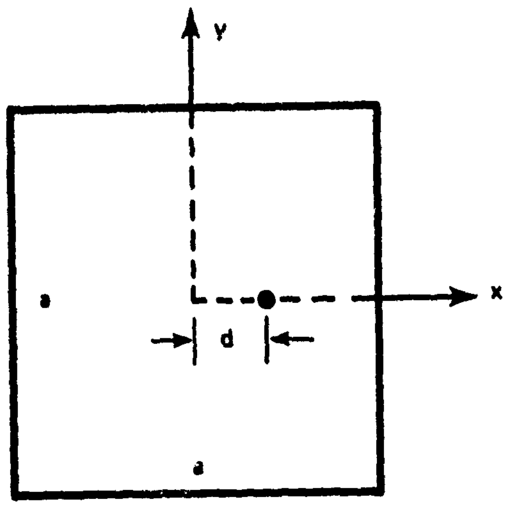

Figure 2. Square Principal Area of S de "a" Excluding the Singularity a Distance "d" Off-Center in the x-Direction

by $\overline{T}$ defined in Eq. (25b) and Figure 1. $\overline{T}$ is either zero or negligible unless the chosen principal area exhibits significant asymmetry with respect to the singular point. The value of $\overline{T}$ for a square principal area excluding the singularity a distance ratio d/a off center in the x-direction (Figure 2) was evaluated using Eq. (25b):

$$\overline{T} = -\frac{1}{\pi}\,\hat{e}_x \ln\left[\frac{\left(1-\frac{2d}{a}\right)\left(1+\sqrt{1+\left(1+\frac{2d}{a}\right)^2}\right)}{\left(1+\frac{2d}{a}\right)\left(1+\sqrt{1+\left(1-\frac{2d}{a}\right)^2}\right)}\right] \quad . \tag{29}$$

In the numerical work reported in this paper $\overline{T}$ was either zero or assumed negligible.

6. ALTERNATE FORMS OF THE INTEGRAL EQUATIONS WITH INTEGRALS INDEPENDENT OF PRINCIPAL AREA

In this section alternate forms of the integral Eqs. EFIE (26), AEFIE (27), and AMFIE (28) are derived that do not involve integrals which depend on the geometry of the principal area. We accomplish this derivation with the aid of the identity,

$$\oint_S \hat{n} \times (\hat{n}' \times \nabla'\phi)\, dS' = 2\pi\overline{T} \quad . \tag{30}$$

Proper use of Eq. (30) in the augmented integral Eqs. (26), (27), and (28) converts them to the following form independent of the geometry of the principal area used to exclude the singularity:

$$\hat{n} \times \overline{E}_{inc} = \frac{\hat{n}}{4\pi i\omega\epsilon_o} \times \oint_S [k^2\overline{K}_s(\overline{r}')\phi - \nabla_s' \cdot \overline{K}_s(\overline{r}')\nabla'\phi - \nabla_s \cdot \overline{K}_s(\overline{r})\hat{n} \times (\hat{n}' \times \nabla'\phi)]\, dS' \tag{31}$$

$$\overline{E}_{inc} = \frac{1}{4\pi i\omega\epsilon_o} \oint_S [k^2\overline{K}_s(\overline{r}')\phi - \nabla_s' \cdot \overline{K}_s(\overline{r}')\nabla'\phi - \nabla_s \cdot \overline{K}_s(\overline{r})\hat{n} \times (\hat{n}' \times \nabla'\phi)]\, dS'$$

$$+ \frac{\nabla_s \cdot \overline{K}_s}{2i\omega\epsilon_o}\,\hat{n} \tag{32}$$

$$-\overline{H}_{inc} = \frac{1}{2}\,\hat{n} \times \overline{K}_s + \frac{1}{4\pi} \oint_S [\overline{K}_s(\overline{r}') \times \nabla'\phi + \overline{K}_s(\overline{r}) \times (\hat{n} \times (\hat{n}' \times \nabla'\phi))]\, dS' \quad . \tag{33}$$

Moreover, the identity Eq. (30) and its use in Eqs. (31), (32), and (33) remains valid when ϕ is replaced by the static Green's function $1/|\overline{r}' - \overline{r}|$.

As mentioned previously, the integral in the original MFIE Eq. (2) is already independent of principal area, although a circular principal area was initially chosen in Eq. (2) to clearly define the surface integration procedure. The independence of the integrals in Eq. (31), (32), and (33) on principal area (self patch) make them attractive for numerical work; however, the author has not experimented with their use in the numerical work of this paper.

7. NUMERICAL RESULTS

Although the main intent of this paper is to report the theory leading to the augmented integral equations, numerical results have been obtained by applying the original and augmented integral equations to the problems of scattering from a cube, sphere, and infinite circular cylinder. In each case the predictions of the theory were confirmed.

Figure 3 plots the normalized backscattering cross section from a cube computed using the original MFIE and the AMFIE. Each face of the cube was divided into nine equal area square patches, simple pulse basis, and delta weighting functions were used with the $\nabla\phi$ value taken constant at its center value of each patch, and reduction of the matrix based on the symmetry of the cube was purposely avoided. Figure 3 shows that the augmented integral equation eliminated entirely the spurious resonances which were introduced by the original integral equation near the cavity resonances of $4s/\lambda = 2.8$, 4.5, and 4.9. (The spurious solution at $4s/\lambda = 3.5$ evidently does not contribute to the back direction.) Comparison with previous numerical results and with experimental results obtained out to about $4s/\lambda = 3.5$ show close agreement.[51] The programs were run well beyond the size parameter of 5 shown in Figure 3, with continued elimination of the spurious resonances by the augmented equation.

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51. Tsai, L.L., Dudley, D.G., and Wilton, D.R. (1974) Electromagnetic scattering by a three-dimensional conducting rectangular box, J. Appl. Phys. 45(10):4393-4400.

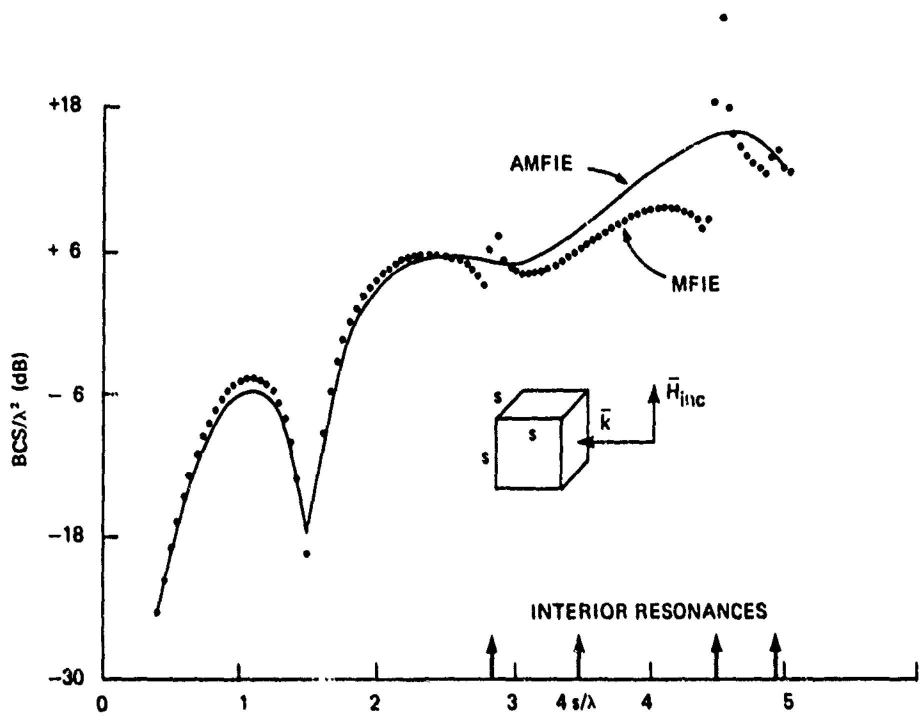

Figure 3. Back Scattering Cross Section vs. Size Parameters for a Cube Using the MFIE and AMFIE

Figure 4 contains the normalized backscattering cross section for a sphere computed with the MFIE and AMFIE. In addition, the average over the surface of the sphere of the absolute value of $\hat{n} \cdot \overline{H}$ is plotted. The theory predicted that the sphere is the one scatterer that retains spurious solutions (at the TE mode frequencies) where $\hat{n} \cdot \overline{H}$ is zero and thus they will not be eliminated by the augmented integral equation. Figure 4 confirms this prediction numerically by showing that $\hat{n} \cdot \overline{H}$ becomes nonnegligible at the TM (but not the TE) cavity mode frequencies when applying the MFIE, and thus the spurious solutions at the TM frequencies disappear using the AMFIE. However, the spurious solution at the one TE cavity mode frequency between ka equal to 0 and 5 remains in the AMFIE solution. Similar results not shown were obtained for backscattering from the sphere using the EFIE and AEFIE, plotting the average of the absolute value of $\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_o$ instead of $\hat{n} \cdot \overline{H}$.

Finally, backscattering from an infinite circular cylinder under planewave incidence was also computed using the original Eqs. (21) and augmented integral Eqs. (21) and (22), with comparison being made with the exact eigenfunction solution. As specified in Section 4.1, the AEFIE Eqs. (21b) and (22b) were used for

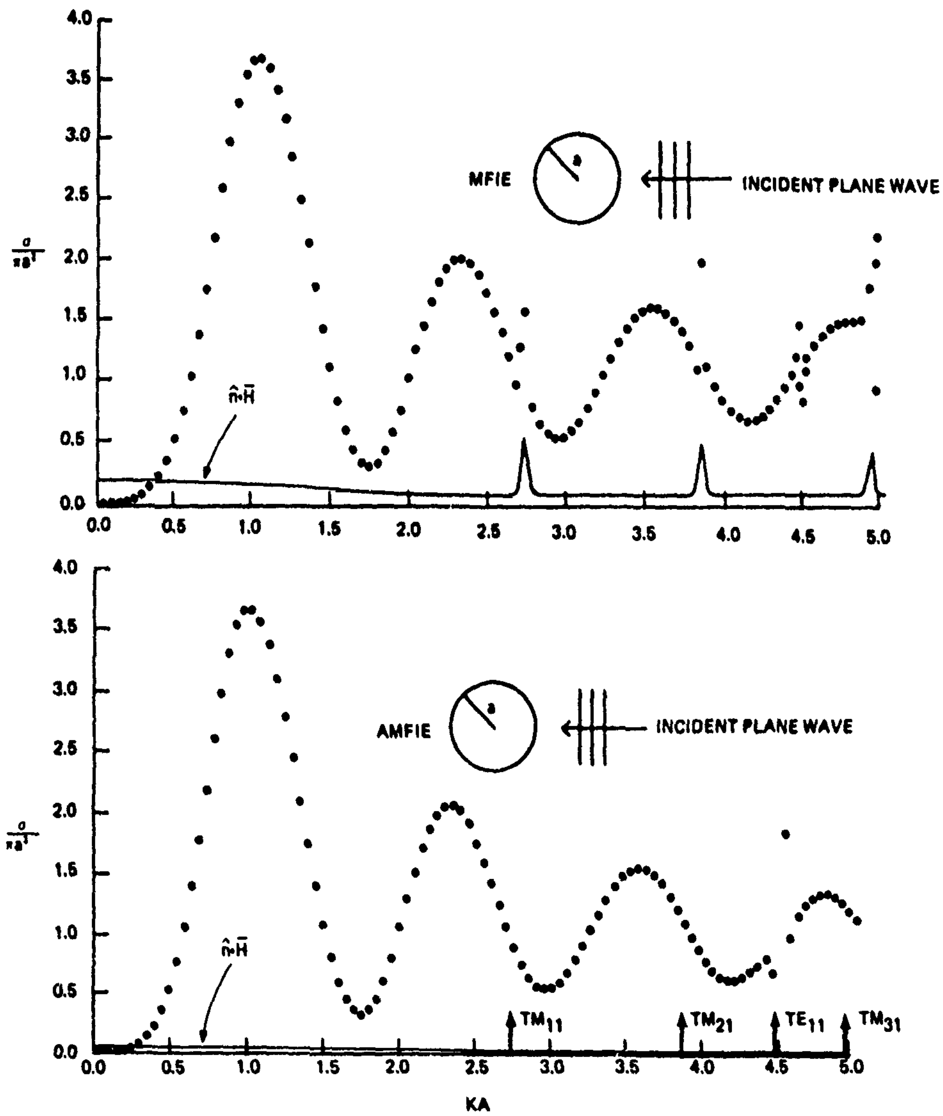

Figure 4. Back Scattering Cross Section vs. ka for a Sphere Using Original and Augmented Integral Equations

H_z incidence, and the AMFIE Eqs. (21a) and (22a) for E_z incidence. As the theory predicts, Figure 5 shows that the augmented integral equation indeed eliminates the spurious resonances introduced by the original equations. (The integral equation solutions deviate further from the exact one for larger ka because the cylinder was divided into the same number of line segments throughout the range of ka.)

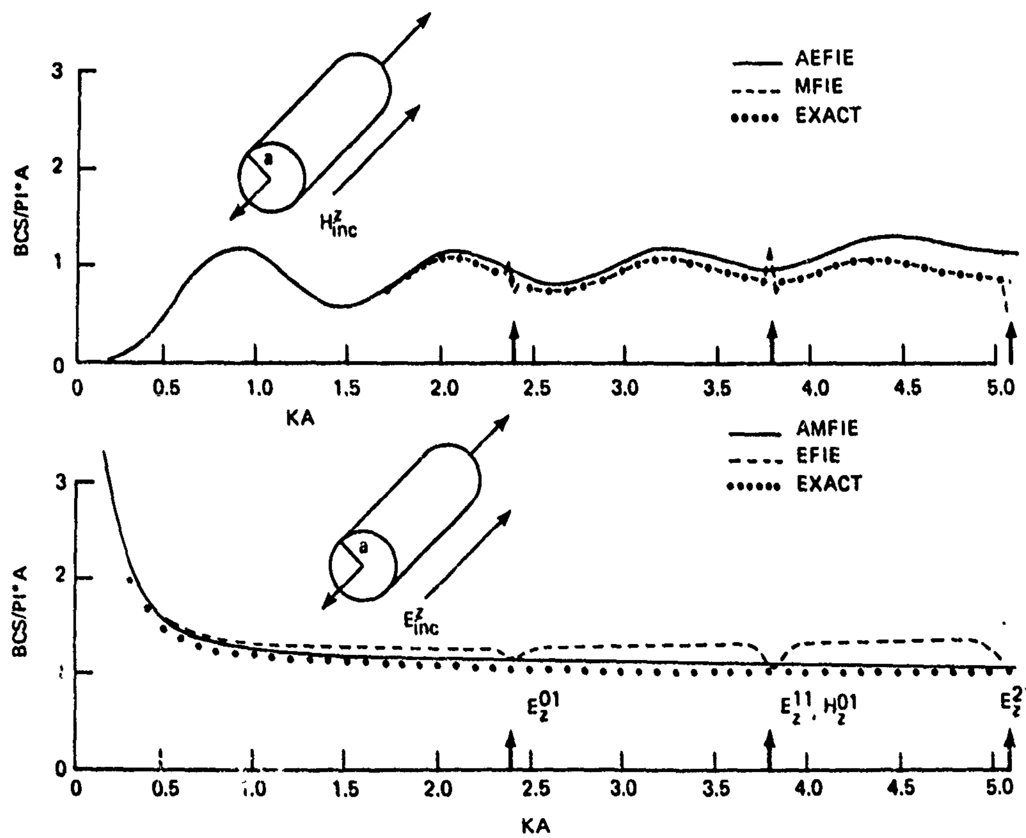

Figure 5. Back Scattering Cross Section vs. ka for an Infinite Circular Cylinder Using Original and Augmented 2-D Intergral Equations

8. CONCLUSION

In theory and numerical practice we have shown that the augmented electric and magnetic field integral equations remove the spurious resonances from Maue's original integral equations for all geometries except the sphere, while preserving the simplicity, solution capability, and basic electric and magnetic field character of the original integral equations. Effort was made to include the important derivations and to present the equations and techniques necessary for solving both 3-D and 2-D problems using the augmented integral equations. Included was an alternative derivation of the augmented integral equations from the extended boundary equations.

The principal area (self patch) dependence of the original EFIE and the augmented integral equations was determined explicitly for an arbitrarily shaped limiting principal area, and alternative forms for the integral equations were presented that avoid integrals dependent upon the geometry of the principal area.

Implicit in the theory leading to the augmented equations is a method for determining in numerical practice if the original integral equations are encountering a spurious solution; namely by monitoring the average $|\hat{n} \cdot \overline{H}|$ for the MFIE and $|\hat{n} \cdot \overline{E} - \nabla_s \cdot \overline{K}_s / i\omega\epsilon_0|$ for the EFIE on the surface of the scatterer, and noting if they exceed a present threshold.

Some closing suggestions for continued work may be in order. In Section 4.2 we specified how the augmented integral equations should be applied, in general, to bodies of revolution. To date this procedure enabling the avoidance of certain spurious rotationally symmetric solutions has not been tested numerically. Nor has numerical experimentation with the augmented equations been done to decide how close the shape of the scatterer can approach the sphere before the TE spherical mode spurious resonances appreciably contaminate the solution in numerical practice.

The integral Eqs. (26), (27), and (28) holding for arbitrary principal areas and the alternate Eqs. (31), (32), and (33) independent of principal area hold promise for more accurate numerical solutions. They also remain to be programmed.

Surface integral equations applied to homogeneous dielectric problems have also encountered spurious solutions.[52] Although it seems likely that a similar augmentation could be applied to these dielectric integral equations to eliminate their spurious resonances as well, we have not performed such an investigation.

Finally, in the time domain electric and magnetic field integral equations,[2] the spurious solutions are also present, since an assumed exp $(-i\omega_n t)$ time dependence converts the time domain solution to the frequency domain solution with its accompanying homogeneous solutions or spurious resonances in the exterior region at the discrete cavity mode frequencies ω_n. Fortunately, most time domain scattering solutions also demand zero fields before a finite initial time, and this initial condition reduces the effect of the discrete spurious resonant frequency spectrum to zero. A simple proof of this result consists in dividing the exterior

52. Wu, T.K. and Tsai, L.L. (1977) Scattering by arbitrarily cross-sectioned layered lossy dielectric cylinders, IEEE Trans. Antennas Propagat. AP-25(4):518-524.

solution to the time-domain integral equations into a particular solution, which does not involve the spurious exp $(\pm i\omega_n t)$ solutions, plus an infinite sum of the homogeneous exp $(\pm i\omega_n t)$ solutions. If the total and particular solutions are specified zero before a given time, the homogeneous solution must also be zero before this given time. Since the individual exp $(\pm i\omega_n t)$ homogeneous solutions are linearly independent, this implies that their coefficients must all be zero. For the interior region the same argument applies except that in the interior cavity the particular time-domain solution is also expressible in terms of exp $(\pm i\omega_n t)$ cavity mode solutions for time greater than the given initial time. In brief, then, for the time-domain surface integral equations, causality implies uniqueness of solution for both the interior and exterior regions, and no augmentation is necessary.

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Appendix A

Uniqueness Theorem for the Augmented Integral Equations

The exterior augmented electric and magnetic field integral Eqs. (14) and (15) have spurious solutions if and only if the interior boundary value problem Eq. (16) has nontrivial solutions. We investigate this boundary value problem by choosing an orthogonal curvilinear coordinate system with one coordinate (n) chosen along the normals to S and the remaining two tantential coordinates (v_1, v_2) chosen along the lines of curvature [contained in Van Bladel,[50] App. 2]. The coordinates $(v_1, v_2, v_3 = n)$ have metrical coefficients h_1, h_2, and $h_3 = h_n = 1$. The vector function $\bar{F}$ satisfying Eq. (16) has components $F_1\hat{u}_1$, $F_2\hat{u}_2$, and $F_n\hat{n}$ in this coordinate system.

First, we show that Eq. (16) implies F_n will be zero everywhere within the cavity not just on the surface S. Represent the surface S as the limit of many flat surface patches. In the narrow cylinder formed by the projection of each patch along its normal, F_n will equal a rectangular component of field (called F_x) satisfying Eq. (16a) and will thus satisfy the scalar wave equation

$$\nabla^2 F_x + k^2 F_x = 0 \quad . \tag{A1}$$

Moreover Eq. (16a) demands $F_x = 0$ on this patch; and because $\nabla \cdot \bar{F}$ equals zero, $\frac{\partial F_x}{\partial x} = 0$ on this patch as well.

Solving Eq. (A1) by separation of variables in the narrow cylinder about the normal axis produces the following 1-dimensional problem

$$\frac{d^2 f}{dx^2} + \alpha^2 f = 0 \quad , \tag{A2a}$$

$$f \; , \; \frac{df}{dx} = 0 \quad \text{for } x = 0 \quad , \quad \text{that is on S} \quad , \tag{A2b, c}$$

which has only the trivial solution $f = 0$; that is, $F_x = F_n = 0$ everywhere along the normal direction $x = n$. Since x can be the normal to any small surface patch, Eq. (16) implies $F_n = 0$ throughout the cavity. Thus the boundary value problem Eq. (16) is reduced to

$$\nabla \times \nabla \times \overline{F}_s - k^2 \overline{F}_s = 0 \tag{A3a}$$

$$\overline{F}_s = 0 \quad \text{on S} \quad . \tag{A3b}$$

The normal component of Eq. (A3a) can be written formally as $(\nabla \times \nabla \times \overline{F}_s)_n = 0$. With the help of Eqs. (18), (19), (21), (22), and (16) of Van Bladel (Appendix 2),[50] and $\nabla_s \cdot \overline{F}_s = 0$, this normal component converts to

$$(\nabla \times \nabla \times \overline{F}_s)_n = \nabla_s \cdot [D(F_1 \hat{u}_1 - F_2 \hat{u}_2)] = 0$$

or simply

$$D(\nabla_s \cdot \overline{A}_s) + \nabla_s D \cdot \overline{A}_s = 0 \tag{A4a}$$

where

$$\overline{A}_s \equiv F_1 \hat{u}_1 - F_2 \hat{u}_2 \quad , \tag{A4b}$$

and D is the difference of the reciprocals of the principle radii of curvature; that is,

$$D = \frac{1}{R_2} - \frac{1}{R_1} \quad . \tag{A4c}$$

Of course, since $\nabla_s \cdot \overline{F}_s = 0$, F_1 and F_2 are related further by

$$\nabla_s \cdot (F_1 \hat{u}_1 + F_2 \hat{u}_2) = 0 \quad . \tag{A4d}$$

Equation (A4a) is quite a stringent restriction, which emerges from the property that $F_n = 0$, and must hold throughout the cavity. The quantities $\overline{A}_s$ and $\nabla_s \cdot \overline{A}_s$ in Eq. (A4a) depend only on the fields and thus have functional dependence determined chiefly by the Maxwellian partial differential Eq. (A3a); whereas D and $\nabla_s D$ in Eq. (A4a) depend solely on the geometry of S. Consequently, we conclude that Eq. (A4a) can hold throughout the cavity only if each term is zero separately. In view of Eqs. (A4b) and (A4d), each term in Eq. (A4a) is zero separately only if

1. F_1 and F_2 are both nonzero and D is zero,
2. F_2 is zero and $\frac{\partial D}{\partial v_1}$ is zero.

(The case, F_1 and $\frac{\partial D}{\partial v_2}$ being zero, is redundant because whichever tangential curvilinear coordinate is labelled v_1 and v_2 is irrelevant.)

Case 1 above holds only if the principal radii of curvature are equal over the surface S, and case 2, only if R_1 and R_2 do not depend on v_1, that is, depend only on one tangential curvilinear coordinate v_2 of the surface. The former case holds only for a sphere (Eisenhart[45]), and we prove next that the latter case implies also that the four fundamental magnitudes of the surface S also depend only on this one tangential curvilinear coordinate.

Consider the Mainardi-Codazzi relations of differential geometry describing the surface S [Eisenhart, p. 157, Eqs. (14)]:[45]

$$\frac{\partial L}{\partial v_2} = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)\frac{\partial E}{\partial v_2} \tag{A5a}$$

$$\frac{\partial N}{\partial v_1} = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)\frac{\partial G}{\partial v_1} \tag{A5b}$$

where $E = h_1^2$, $G = h_2^2$, $\frac{E}{L} = R_1$, and $\frac{G}{N} = R_2$. R_1 and R_2 are not functions of v_1, so $\frac{\partial}{\partial v_1}\left(\frac{N}{G}\right) = 0$, or

$$\frac{\partial N}{\partial v_1} - \frac{N}{G}\frac{\partial G}{\partial v_1} = 0 \quad . \tag{A6}$$

Eqs. (A6) and (A5b) are compatible if either $E/L = G/N$, or if G and thus N are independent of v_1. The first condition ($R_1 = R_2$) holds only for a sphere which has already been discovered.

Taking $\partial/\partial v_2$ of the equation $L/E = 1/R_1$ gives

$$\frac{\partial L}{\partial v_2} = \frac{L}{E}\frac{\partial E}{\partial v_2} + E\frac{\partial}{\partial v_2}\left(\frac{1}{R_1}\right) \tag{A7}$$

which combines with Eq. (A5a) to give in turn,

$$\frac{\partial E}{\partial v_2} = a_1(v_2)\,E \tag{A8a}$$

or

$$\ln E = a_2(v_2) + b_1(v_1) \tag{A8b}$$

or

$$E = a_4(v_2)b_2(v_1) \tag{A9a}$$

and thus

$$L = \frac{a_4(v_2)}{R_1(v_2)}\,b_2(v_1) \quad . \tag{A9b}$$

Because E and L have the same functional dependence on v_1, we can merely redefine the v_1 curvilinear coordinate as

$$v_1' = \int \sqrt{b_2(v_1)}\,dv_1 \tag{A10}$$

to make E' and L' independent of v_1'.

In other words, we have proven that if the principal radii of curvature of a surface are independent of one tangential curvilinear coordinate, then the four fundamental magnetides defining the surface are (or can be made) independent of the same curvilinear coordinate. That is, the only geometries for which Eq. (16) has nontrivial solutions are the sphere and surfaces described by one curvilinear coordinate. As evidenced by Section 2 and Eq. (A4a), the fields satisfying Eq. (16) on these latter surfaces must also depend only on this one curvilinear coordinate. Koshlyakov et al (p. 591)[A1] also remark that these are the only conditions under which the electromagnetic field can be represented by means of two scalar functions; for example, divide into TM and TE modes with respect to the normal direction inside a cavity.

It will enable our application of the augmented integral equations to bodies of revolution and infinite cylinders to write down the cavity fields when they and the geometry depend on one curvilinear coordinate. In that case Eq. (16), or more specifically, Eq. (A3) simplify greatly to the following expressions for the electromagnetic field inside the perfectly electrically conducting cavity. Associating $\overline{F}$ in Eq. (A3) with the electric field $\overline{E}$, and letting $q = h_1 E_1$, yields for the other components of field

$$E_2 = 0 \quad , \quad E_n = 0 \quad , \quad H_1 = 0 \quad ,$$

$$H_2 = \frac{h_2}{i\omega\mu_o h_1} \frac{\partial q}{\partial n} \quad , \quad H_n = \frac{-1}{i\omega\mu_o h_1 h_2} \frac{\partial q}{\partial v_2} \quad , \tag{A11}$$

with q satisfying the partial differential equation

$$\frac{\partial}{\partial v_2}\left(\frac{1}{h_1 h_2}\frac{\partial q}{\partial v_2}\right) + \frac{\partial}{\partial n}\left(\frac{h_2}{h_1}\frac{\partial q}{\partial n}\right) + k^2 \frac{h_2}{h_1} q = 0 \quad . \tag{A12}$$

Note that for infinite cylinders, $h_1 = 1$, and Eq. (A12) reduces to the familiar 2-D scalar wave equation

$$\nabla_t^2 q + k^2 q = 0 \quad . \tag{A13}$$

We still must answer the question of what shape surfaces have their fundamental magnitudes dependent on just one tangential curvilinear coordinate. Fortunately, there is a theorem of differential geometry (Eisenhart; problem 23, p. 188 and p. 475[45]), which says that all such surfaces must be helicoids, surfaces generated by a curve which is rotated about a fixed straight line as axis, and at the same time translated in the direction of the axis with a velocity proportional to the velocity of rotation. (The theorem is proven using the Mainardi-Codazzi and Gauss equations.) Bodies of revolution are the only finite helicoids, and infinite cylinders are the only infinite helicoids which are solvable numerically by the surface integral equations. (Infinite helicoids in the shape of periodic

A1. Koshlyakov, N.S., Smirnov, M.M., and Gliner, E.B. (1964) Differential Equations of Mathematical Physics, Ch. 24, North-Holland, Amsterdam (Interscience, New York).

twisted ropes or tubes have currents which vary to infinity and thus are not solvable numerically by the surface integral equations unless special techniques are used to accommodate the variation to infinity.)

In summary then, both augmented integral Eqs. (14) and (15) eliminate the spurious resonances from all geometries except bodies of revolution and infinite cylinders. For finite bodies of revolution including the sphere (because its resonant frequencies are independent of the ϕ variation of the modes) spurious solutions remain only at the frequencies where Eq. (A11) has solutions with v_1 equal to ϕ, the axial rotation angle. For infinite cylinders except the circular one, spurious solutions remain only at the frequencies where Eq. (A11) has solutions with $v_1 = z$, the axis of the cylinder. For the infinite circular cylinder, which is also a body of revolution, spurious solutions remain with $v_1 = \phi$ as well. Moreover, Eq. (A11) along with the results of Section 2, and remembering the special case of the sphere, reveal that the spurious $\overline{K}_s$ in the AEFIE (or $\hat{n} \times \overline{E}$ on S in the AMFIE) must be in the z-direction for infinite cylinders (except the circular one), and in the $\hat{e}_\phi$-direction but not varying with ϕ for finite bodies of revolution (except the sphere). This observation allows us in Section 4 to eliminate the remaining spurious resonances from all infinite cylinders and bodies of revolution, except the sphere, using the augmented integral equations.

Appendix B

Sufficiency of Eq. (9)

Section 2 showed that the MFIE Eq. (2) and associated fields Eqs. (3a) and (3b) in an exterior region implied Eq. (9). Here we want to show the converse: All solutions $(\hat{n} \times \overline{E})$ on S to Eq. (9) are valid solutions to Eq. (2a) with $\overline{K}_s$ from Eq. (2). Equivalently we want to prove that any nontrivial solutions to Eq. (9) and homogeneous solutions Eq. (2) obey Eq. (3a); or specifically that $(\hat{n} \times \overline{E})$ on S obtained from Eq. (9) plus $\overline{K}_s{}^h$ obtained from

$$0 = \frac{1}{2}\overline{K}_s{}^h - \frac{1}{4\pi}\hat{n} \times \oint_S \overline{K}_s{}^h \times \nabla'\phi \, dS' \tag{B1}$$

satisfies Eq. (3a) on S.

Define a vector $\overline{A}$ by the integral

$$\overline{A} = \frac{1}{4\pi}\int_S \overline{K}_s{}^h \times \nabla'\phi \, dS' \quad , \quad \overline{r} \text{ inside S} \tag{B2}$$

and note that

$$\nabla \times \nabla \times \overline{A} - k^2\overline{A} = 0 \tag{B3a}$$

and from Eqs. (B1) and (6)

$$\hat{n} \times \overline{A} = 0 \quad , \quad \overline{r} \rightarrow S \quad . \tag{B3b}$$

Now the $(\hat{n} \times \overline{E})$ on S in Eq. (9) was shown to be equivalent to the $(\hat{n} \times \overline{E}_m)$ on S in Eqs. (11). Consequently, comparison of Eqs. (B3) and (11) shows that $\overline{F} = \overline{H}_m$ (except for arbitrary multiplicative constants). Thus from Eq. (11b)

$$(\hat{n} \times \overline{E})_{\text{on S}} = (\hat{n} \times \overline{E}_m)_{\text{on S}} = -\frac{\hat{n} \times (\nabla \times \overline{A})}{i\omega\epsilon_o} \tag{B4}$$

or, upon substitution of $\overline{A}$ from Eq. (B2) into Eq. (B4), and letting $\overline{r} \rightarrow S$,

$$(\hat{n} \times \overline{E})_{\text{on S}} = \frac{-\hat{n}}{4\pi i\omega\epsilon_c} \times \oint_S [k^2\overline{K}_s\phi - (\nabla'_s \cdot \overline{K}_s)\nabla'\phi]\, dS' \quad . \tag{B5}$$

Since Eq. (B5) is identical to Eq. (3a) when $\overline{r} \rightarrow S$, we have proven that all solutions to Eq. (9) satisfy Eq. (3) for the surface currents $\overline{K}_s$ obtained from the exterior MFIE. The converse was proven in Section 2.1.2.

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