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13. ABSTRACT

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FOREWORD

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Abstract

A body of literature has evolved for multivariable systems which is concerned with the placement of closed-loop eigenvalues and/or the question of decoupling. Attention is turned to the broader question of realizing specified rational transfer matrices with a standard feedback configuration for linear, time-invariant, finite-dimensional, real, multivariable, dynamical plants in this paper. A complete and precise realization theory for asymptotically-stable plants is developed. Unstable plants with asymptotically-stable hidden modes are also extensively treated.

Introduction

A body of literature has evolved which is concerned with the placement of closed-loop eigenvalues and/or the question of decoupling in multivariable systems [1]-[11]. Attention is turned in this paper to the broader question of realizing specified rational transfer matrices. A standard feedback configuration is considered and attention is restricted to linear, time-invariant, finite-dimensional, real, dynamical plants. Specifically, the standard feedback configuration shown in Fig. 1 is studied. It is assumed

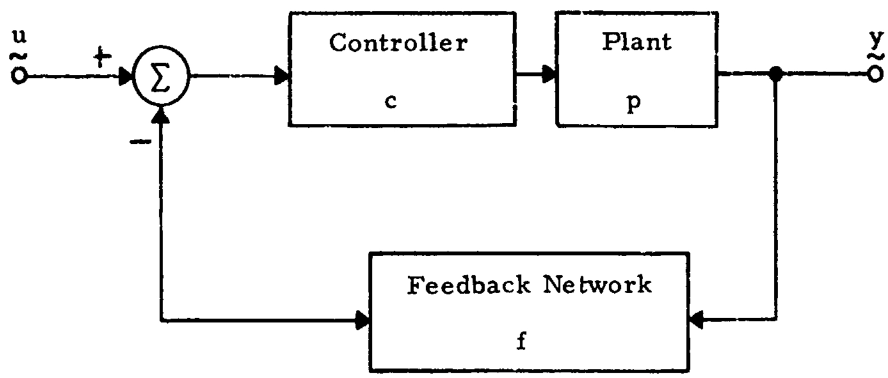

Fig. 1. Standard Feedback Configuration

that the plant, controller, and feedback network possess, respectively, the real state-variable descriptions

$$\dot{\underset{\sim}{x}}_p = F_p \underset{\sim}{x}_p + G_p \underset{\sim}{u}_p \tag{1}$$

$$\underset{\sim}{y}_p = H_p \underset{\sim}{x}_p + J_p \underset{\sim}{u}_p \quad , \tag{2}$$

$$\dot{\underset{\sim}{x}}_c = F_c \underset{\sim}{x}_c + G_c \underset{\sim}{u}_c \tag{3}$$

$$\underset{\sim}{y}_c = H_c \underset{\sim}{x}_c + J_c \underset{\sim}{u}_c \quad , \tag{4}$$

and

$$\dot{\underset{\sim}{x}}_f = F_f \underset{\sim}{x}_f + G_f \underset{\sim}{u}_f \tag{5}$$

$$\underset{\sim}{y}_f = H_f \underset{\sim}{x}_f + J_f \underset{\sim}{u}_f \quad . \tag{6}$$

As a result of the interconnection of plant, controller, and feedback network in the configuration of Fig. 1 it follows that

$$\underset{\sim}{u}_c = \underset{\sim}{u} - \underset{\sim}{y}_f \quad , \tag{7}$$

$$\underset{\sim}{u}_p = \underset{\sim}{y}_c \quad , \tag{8}$$

and

$$\underset{\sim}{u}_f = \underset{\sim}{y}_p = \underset{\sim}{y} \tag{9}$$

The sizes of the matrices in (1) through (6) are determined by the dimensions of the vectors $\underset{\sim}{x}_p$, $\underset{\sim}{u}_p$, $\underset{\sim}{y}_p$, $\underset{\sim}{x}_c$, etc. The symbols used to denote these dimensions are:

$$\nu_p = \dim \underset{\sim}{x}_p \quad , \tag{10}$$

$$\nu_c = \dim \underset{\sim}{x}_c \quad , \tag{11}$$

$$\nu_f = \dim \underset{\sim}{x}_f \quad , \tag{12}$$

$$n = \dim \underset{\sim}{y}_p = \dim \underset{\sim}{y} = \dim \underset{\sim}{u}_f \quad , \tag{13}$$

$$m = \dim \underset{\sim}{u}_p = \dim \underset{\sim}{y}_c \quad , \tag{14}$$

$$r = \dim \underset{\sim}{u} = \dim \underset{\sim}{u}_c = \dim \underset{\sim}{y}_f \quad . \tag{15}$$

It follows from (1) through (6) that the n x m plant transfer matrix P(s), the m x r controller transfer matrix C(s), and the r x n feedback network transfer matrix F(s) are given by

$$P(s) = H_p(s1_{\nu_p} - F_p)^{-1} G_p + J_p \quad , \tag{16}$$

$$C(s) = H_c(s1_{\nu_c} - F_c)^{-1} G_c + J_c \quad , \tag{17}$$

and

$$F(s) = H_f(s1_{\nu_f} - F_f)^{-1} G_f + J_f \quad . \tag{18}$$

[It is evident from (16)-(18) that the symbol used in this paper for the k x k identity matrix is 1_k.] The transfer matrix relating the transform of the output, $\underset{\sim}{Y}(s)$, to the transform of the input, $\underset{\sim}{U}(s)$, is easily shown to be

$$\begin{aligned} T(s) &= P(s)C(s)[1_r + F(s)P(s)C(s)]^{-1} \\ &= [1_n + P(s)C(s)F(s)]^{-1} P(s)C(s) \quad . \end{aligned} \tag{19}$$

Clearly, (19) is meaningful if and only if (hereafter denoted iff)

$$\det[1_r + F(s)P(s)C(s)] = \det[1_n + P(s)C(s)F(s)] \not\equiv 0 \quad . \tag{20}$$

Moreover, it is desired that the basic feedback configuration be dynamical. It is shown in [12] that this is the case iff

$$\lim_{s \to \infty} T(s) = \text{finite matrix} \tag{21}$$

or, equivalently,

$$\det[1_r + F(\infty)P(\infty)C(\infty)] = \det[1_r + J_f J_p J_c] \neq 0 \quad . \tag{21}$$

Practical arguments are also given in [12] which justify the limitation of the developments presented here in accordance with the following four restrictions:

R_1. The number of plant inputs m equals or exceeds the number of plant outputs n; i.e., $m \geq n$.

R_2. The number of system inputs r equals the number of system outputs n: i.e., $r = n$.

R_3. The normal rank* of $P(s)$ is equal to the number of its rows: i.e., normal rank $P(s) = n$.

R_4. The $m \times n$ controller matrix $C(s)$ is chosen so that the square $n \times n$ matrix $P(s)C(s)$ has normal rank n: i.e., $\det[P(s)C(s)] \not\equiv 0$.

The first significant contribution of the present paper is best described with the aid of the following definition.

Definition 1: An $n \times n$ rational matrix $T(s)$ is said to be realizable for $P(s)$ if for some choice of asymptotically-stable dynamical controller and feedback network the standard feedback configuration of Fig. 1 is a dynamical asymptotically-stable system possessing the transfer matrix $T(s)$.

The necessary and sufficient conditions which $T(s)$ must satisfy in order that it be realizable for $P(s)$ are derived here for the case in which the plant is asymptotically stable and rank $P(j\omega) = n$ for all ω infinity included. It is shown for this case that the limitations on the realizable $T(s)$ are due to the nonminimum phase properties of the plant. These properties are completely characterized for asymptotically-stable plants by the plant structure matrix which is introduced in the sequel.

The second significant contribution is the treatment of unstable plants whose uncontrollable and/or unobservable modes ("hidden modes"), are asymptotically stable.

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* The normal rank of a rational matrix is the order of the largest minor which is not identically zero.

It is shown that any unstable plant with asymptotically-stable hidden modes can be stabilized with a modified dynamical observer of the Luenberger type [2], [13]-[19]. Moreover, the structure matrices of the original and modified plant are shown to be strictly equivalent and the implications of this fact are thoroughly discussed.

The notation used in this paper is now summarized for easy reference, and some basic notions associated with a matrix function of a complex variable "s" are defined. For an arbitrary matrix A the transpose, the complex conjugate, the complex conjugate transpose, the inverse, the trace and the determinant of A are denoted by A', $\overline{A}$, A^*, A^{-1}, tr[A], and det A, respectively. A diagonal matrix Λ with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ is written as $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Column vectors are represented by $\underset{\sim}{x}$, $\underset{\sim}{y}$, etc., or in the alternative fashion $\underset{\sim}{x} = [x_1\ x_2 \cdots x_n]'$ whenever it is desirable to indicate the components explicitly. The n x n identity matrix, the n x n null matrix, the n-dimensional zero vector, and the n x m null matrix are denoted by 1_n, O_n, $\underset{\sim}{o}_n$, and $O_{n,m}$, respectively. The n-dimensional column vector with unity element in the i'th row and all other elements equal to zero is denoted by $\underset{\sim}{e}_i^{(n)}$ or simply $\underset{\sim}{e}_i$ when no confusion is likely to result. The right inverse of a p x q matrix A is the q x p matrix A^{-1} which has the property $A A^{-1} = 1_p$.

A matrix A(s) is rational when each of its elements is a rational function of s. When every element of a rational matrix is finite at infinity it is called a proper matrix. The matrix A(s) is analytic in a region of the complex s-plane when each element of the matrix is analytic in the region. A point s_0 is a pole of A(s) when some element of A(s) has a pole at $s = s_0$. A(s) is said to be real if $\overline{A}(s) = A(\overline{s})$. When the order of the largest minor of A(s) not identically zero is ν, then A(s) is said to have normal rank equal to ν. Finally, the notation

$$A_*(s) = A^*(-\overline{s}) \tag{23}$$

is used which for real matrices – the only kind of interest here – reduces to

$$A_*(s) = A'(-s) \quad . \tag{24}$$

Stability of the Standard Feedback Configuration

The basic requirements imposed on the overall system in Fig. 1 are that it be dynamical and asymptotically stable. Conditions for the former to be true are stated in the introduction. The latter requirement is discussed here. The first careful treatment of the stability question for multivariable feedback control systems is due to Chen [20]. Applications and extensions of Chen's results are given by Youla [12]. Youla established for the standard feedback configuration of Fig. 1 the following theorem.

Theorem 1: When (22) is satisfied, the standard feedback configuration is asymptotically stable iff the scalar function

$$\Delta(s) = \Delta_c(s)\Delta_p(s)\Delta_f(s)\det[1_n + P(s)C(s)F(s)] \tag{25}$$

is free of zeros in $\mathrm{Re}\, s \geq 0$. In (25),

$$\Delta_c(s) = \det(s1_{\nu_c} - F_c) \quad , \tag{26}$$

$$\Delta_p(s) = \det(s1_{\nu_p} - F_p) \quad , \tag{27}$$

and

$$\Delta_f(s) = \det(s1_{\nu_f} - F_f) \quad . \tag{28}$$

Theorem 1 indicates that in general one cannot determine stability of the standard feedback configuration solely from knowledge of the transfer matrices C(s), P(s), and F(s). One must in addition have knowledge of $\Delta_c(s)$, $\Delta_p(s)$, and $\Delta_f(s)$ which depend on the internal structure of the individual system components. Fortunately, however, practical considerations permit simplifications. Firstly, the controller and feedback network are in accordance with Definition 1 to be asymptotically stable. Thus, both $\Delta_c(s)$ and $\Delta_f(s)$ are free of zeros in $\mathrm{Re}\, s \geq 0$. Secondly, it is shown below that one can write

$$\Delta_p(s) = h_p(s)\psi_p(s) \tag{29}$$

where $h_p(s)$ is a polynomial whose zeros are associated with the hidden modes of the plant and $\psi_p(s)$ is the characteristic denominator of P(s): i.e., $\psi_p(s)$ is the monic least common multiple of the denominators of all the minors of P(s) when these minors are expressed as the ratio of two relatively prime polynomials. Obviously, for every practical plant $h_p(s)$ is free of zeros in $\mathrm{Re}\, s \geq 0$. Otherwise, it is not possible for the overall system to be asymptotically stable. It now immediately follows from Theorem 1 that

Theorem 2: When (22) is satisfied, when the hidden modes of the plant are asymptotically stable, and when the controller and feedback networks are asymptotically stable, then the standard feedback configuration is asymptotically stable iff

$$\Delta_0(s) = \psi_p(s)\det[1_n + P(s)C(s)F(s)] \tag{30}$$

is free of zeros in Re $s \geq 0$.

Theorem 2 is significant in that the test for stability embodied in it can be carried out solely from knowledge of the transfer matrices C(s), P(s), and F(s).

It is now established that (29) is a valid decomposition. The result follows from the fact (see [21] and Theorem 5-19 of [22]) that there exists a real nonsingular matrix K and square matrices $\hat{F}_p$, F_{22}, and F_{33} such that

$$F_p = K^{-1}\left[\begin{array}{c|c|c} \hat{F}_p & O_{\nu_{p_1},\nu_{p_2}} & F_{13} \\ \hline F_{21} & F_{22} & F_{23} \\ \hline O_{\nu_{p_3},\nu_{p_1}} & O_{\nu_{p_3},\nu_{p_2}} & F_{33} \end{array}\right] K \quad , \tag{31}$$

$$H_p = \left[\hat{H}_p \mid O_{n,\nu_{p_2}} \mid H_a\right] K \quad , \tag{32}$$

and

$$G_p = K^{-1}\left[\begin{array}{c} \hat{G}_p \\ \hline G_a \\ \hline O_{\nu_{p_3},m} \end{array}\right] \quad , \tag{33}$$

where $\hat{H}_p$ is $n \times \nu_{p_1}$, $\hat{G}_p$ is $\nu_{p_1} \times m$, and

$$\nu_{p_1} + \nu_{p_2} + \nu_{p_3} = \nu_p \quad . \tag{34}$$

Moreover, $\{\hat{F}_p, \hat{G}_p\}$ is a completely controllable pair, $\{\hat{F}_p, \hat{H}_p\}$ is a completely observable pair, and

$$P(s) = \hat{H}_p(s1_{\nu_{p_1}} - \hat{F}_p)^{-1}\hat{G}_p + J_p \quad . \tag{35}$$

In addition,

$$\psi_p(s) = \det(s1_{\nu_{p_1}} - \hat{F}_p) \tag{36}$$

and one can easily establish from (31) that

$$\Delta_p(s) = \psi_p(s) \det(s1_{\nu_{p_2}} - F_{22}) \det(s1_{\nu_{p_3}} - F_{33}) \quad , \tag{37}$$

or

$$h_p(s) = \det(s1_{\nu_{p_2}} - F_{22}) \det(s1_{\nu_{p_3}} - F_{33}) \quad . \tag{38}$$

Nonminimum Phase Properties

The objective in this section is the establishment of those properties of the given plant which prevent the realization with $F(s) = O_n$ of an arbitrarily specified rational transfer matrix $T(s)$. These properties are referred to as the nonminimum phase properties of the plant. With $F(s) = O_n$ it immediately follows from Theorem 2 that the overall system cannot be asymptotically stable unless $\psi_p(s)$ is free of zeros in $\mathrm{Re}\, s \geq 0$. Hence, closed-right-half-plane zeros of $\psi_p(s)$ contribute to the nonminimum phase properties of the plant.

When $F(s) = O_n$ then

$$T(s) = P(s)C(s) \quad . \tag{39}$$

A necessary condition for $T(s) = 1_n$ to be realizable for $P(s)$ is, therefore, that

$$C(s) = P^{-1}(s) \quad , \tag{40}$$

where $P^{-1}(s)$ is the right inverse of $P(s)$. Since $C(s)$ must be analytic in $\mathrm{Re}\, s \geq 0$, equation (40) indicates that $P^{-1}(s)$ must be analytic in $\mathrm{Re}\, s \geq 0$. This is never possible when the rank of $P(s)$ is less than n, the normal rank of $P(s)$, in $\mathrm{Re}\, s \geq 0$. Thus, in $\mathrm{Re}\, s \geq 0$ any decrease in the rank of $P(s)$ from its normal rank also contributes to the nonminimum phase properties of the plant. The nonminimum phase properties cited above are shown in this paper to be those properties of the plant which restrict the class of transfer matrices that can be realized.

When $\psi_p(s)$ is free of zeros in $\mathrm{Re}\, s \geq 0$, the nonminimum phase properties of the plant are completely characterized by the plant structure matrix

$$\Gamma(s) = \left[\begin{array}{c|c} J_p & \hat{H}_p \\ \hline \hat{G}_p & \hat{F}_p - s1_{\nu_{p_1}} \end{array}\right] \quad . \tag{41}$$

That this is the case follows from the identity

$$\left[\begin{array}{c|c} 1_n & -\hat{H}_p(\hat{F}_p - s1_{\nu_{p_1}})^{-1} \\ \hline O_{\nu_{p_1}, n} & 1_{\nu_{p_1}} \end{array}\right] \cdot \Gamma(s) = \left[\begin{array}{c|c} P(s) & O_{n, \nu_{p_1}} \\ \hline \hat{G}_p & \hat{F}_p - s1_{\nu_{p_1}} \end{array}\right] \quad . \tag{42}$$

It is clear from (36) that the inverse in (42) exists for $\mathrm{Re}\, s \geq 0$ when $\psi_p(s)$ is free of zeros in $\mathrm{Re}\, s \geq 0$. In this case, then, it immediately follows from (42) that

$$\mathrm{rank}\, \Gamma(s) = \nu_{p_1} + \mathrm{rank}\, P(s), \ \mathrm{Re}\, s \geq 0 \quad . \tag{43}$$

Plant structure matrices are utilized in the sequel when the unstable plant is considered.

It is possible to factor any plant transfer matrix into the product of two matrices, one of which accounts for the nonminimum phase properties of the plant. This factorization and its properties are now discussed. Given any n x m plant transfer matrix P(s) of normal rank n one can always write provided the rank of P(jω) is n for all finite ω that

$$P(s) = V(s)P_0(s) \quad , \tag{44}$$

where the n x m matrix $P_0(s)$ together with its right inverse $P_0^{-1}(s)$ are analytic in Re s $\geq$ 0 and the n x n matrix V(s) satisfies

$$V_*(s)V(s) = 1_n \quad . \tag{45}$$

The above stated results follow easily when Theorem 2 of [23] is applied to achieve the spectral factorization

$$G(s) = P_*(s)P(s) = P_{0*}(s)P_0(s) \quad . \tag{46}$$

It is not difficult to verify that the m x m matrix G(s) has normal rank n and that the rank of G(jω) is n for all finite ω. A computer program for factoring G(s) is available [24].

The paraconjugate unitary matrix V(s) accounts for the nonminimum phase properties of the plant. Any zeros of $\psi_p(s)$ in Re s $\geq$ 0 are zeros of the characteristic denominator for V(s) and the rank of V(s) decreases in Re s $\geq$ 0 where rank P(s) does. It is also useful to note that since

$$V(s) = P(s)P_0^{-1}(s) \quad , \tag{47}$$

V(s) is analytic in Re s $\geq \sigma$, $\sigma \geq 0$, when P(s) is analytic in the same region. Moreover, since P(s) is a real matrix V(s) is a real matrix, and it follows from (45) that

$$V_*(j\omega)V(j\omega) = V'(-j\omega)V(j\omega) = V^*(j\omega)V(j\omega) = 1_n \quad . \tag{48}$$

It is easy to infer from (48) that

$$\lim_{s \to \infty} V(s) \equiv V(\infty) = \text{finite matrix} \quad . \tag{49}$$

In addition to the properties already cited for V(s) one has from (47) and the fact that $P_0(s)$ is unique to within a constant real orthogonal matrix multiplier on the left that V(s) is unique to within a constant real orthogonal matrix multiplier on the right.

From (44), (45), and (49) it follows that

$$\lim_{s\to\infty} P_0(s) = \lim_{s\to\infty} V^{-1}(s)P(s) = \lim_{s\to\infty} V_*(s)P(s) = \text{finite matrix} \; ; \tag{50}$$

The rational matrix $P_0(s)$ is, therefore, analytic at infinity. Moreover, rank $P_0(s) = n$ for all Re $s \geq 0$. This last property is a consequence of

$$P_0(s)P_0^{-1}(s) = 1_n \quad . \tag{51}$$

For if rank $P_0(s) < n$ for some $s = s_0$, Re $s_0 \geq 0$, then (51) requires that $P_0^{-1}(s)$ have a pole at $s = s_0$. But this contradicts the analyticity of $P_0^{-1}(s)$ in Re $s \geq 0$. The arguments just given apply equally well at infinity provided the additional restriction rank $P(j\omega) = n$ for infinite ω is imposed. In summary, $P_0(s)$ is analytic and rank $P_0(s) = n$ in Re $s \geq 0$ infinity included when rank $P(j\omega) = n$ for all ω infinity included.

A method for constructing a right inverse for $P_0(s)$ satisfying

$$\lim_{s\to\infty} P_0^{-1}(s) = \text{finite matrix} \tag{52}$$

is now given. The construction is accomplished by introducing the change of variable

$$s = \frac{1+z}{1-z} \quad . \tag{53}$$

This transformation maps the region Re $s \geq 0$ of the complex s-plane into the region $|z| \leq 1$ in the complex z-plane. Clearly, $z = 1 + j0$ is the mapping of all points in the s-plane infinitely far from the s-plane origin. The matrix

$$W(z) \equiv P_0(s)\Big|_{s=\frac{1+z}{1-z}} \tag{54}$$

is next considered. In view of the properties of $P_0(s)$, it follows that $W(z)$ is analytic in $|z| \leq 1$ and rank $W(z) = n$ for $|z| \leq 1$. The matrix $W(z)$ therefore has the Smith-McMillan representation [25], [26]

$$W(z) = M(z)[\Lambda(z) | O_{n,\, m-n}] N(z) \quad , \tag{55}$$

where $M(z)$ and $N(z)$ are elementary polynomial matrices of appropriate size and

$$\Lambda(z) = \text{diag}[\lambda_1(z), \lambda_2(z), \ldots, \lambda_n(z)] \quad . \tag{56}$$

The rational functions $\lambda_i(z)$ are all analytic in $|z| \leq 1$. Moreover, $\lambda_i(z) \neq 0$ for any z satisfying $|z| \leq 1$. For if the contrary is true then rank $\Lambda(z) < n$ for some z satisfying $|z| \leq 1$ which contradicts rank $W(z) = n$ in the region $|z| \leq 1$. Hence, for any real rational matrix $K(z)$ analytic in $|z| \leq 1$

$$W^{-1}(z) = N^{-1}(z)\left[\frac{\Lambda^{-1}(z)}{K(z)}\right] M^{-1}(z) \tag{57}$$

is a right inverse of $W(z)$ which is analytic in $|z| \le 1$. It immediately follows that

$$P_0^{-1}(s) = W^{-1}(z)\Big|_{z = \frac{s-1}{s+1}} \tag{58}$$

is a right inverse of $P_0(s)$ analytic in Re $s \geq 0$ infinity included. The above results are summarized in

<u>Lemma 1:</u> A real rational proper $n \times m$ matrix $P(s)$ of normal rank n satisfying rank $P(j\omega) = n$ for all ω infinity included is expressible as $P(s) = V(s)P_0(s)$ where the $n \times n$ matrix $V(s)$ and the $n \times m$ matrix $P_0(s)$ are both real rational proper matrices having the properties:

a) $V_*(s)V(s) = 1_n$.

b) When $P(s)$ is analytic in Re $s \geq \sigma$, $\sigma \geq 0$, then $V(s)$ is analytic in the same region.

c) All zeros of the characteristic denominator of $P(s)$ in Re $s \geq 0$ are zeros of the characteristic denominator of $V(s)$.

d) The rank of $V(s)$ decreases in Re $s \geq 0$ wherever the rank of $P(s)$ does.

e) Both $P_0(s)$ and $P_0^{-1}(s)$ are analytic in Re $s \geq 0$ infinity included.

f) $P_0(s)$ is unique to within a real constant orthogonal matrix multiplier Q on the left and $V(s)$ is unique to within the matrix multiplier Q' on the right.

The Main Theorem on Realizability of T(s)

For the class of asymptotically-stable dynamical plants satisfying rank $P(j\omega) = n$ for all ω infinity included one can always choose $F(s) = O_n$ and

$$C(s) = P_0^{-1}(s)L(s) \quad , \tag{59}$$

where the n x n real rational proper matrix $L(s)$ is analytic in $\mathrm{Re}\, s \geq 0$ but is otherwise arbitrary. For this choice of controller

$$\lim_{s \to 0} C(s) = P_0^{-1}(\infty)L(\infty) = \text{finite matrix} \quad , \tag{60}$$

and any minimal realization (completely controllable and completely observable realization) of $C(s)$ is asymptotically-stable. Moreover,

$$T(s) = P(s)C(s) = V(s)P_0(s)P_0^{-1}(s)L(s) = V(s)L(s) \tag{61}$$

is realized. The transfer matrix $T(s)$ is real, rational, and proper; the overall system – plant with controller – is, therefore, dynamical. Moreover, $V(s)$ is analytic in $\mathrm{Re}\, s \geq 0$ since $P(s)$ is. Thus, $T(s)$ is analytic in $\mathrm{Re}\, s \geq 0$ and the system is asymptotically stable.

The above observations show that a sufficient condition for $T(s)$ to be realizable for $P(s)$ when the plant is asymptotically stable and rank $P(j\omega) = n$ for all ω infinity included is that $T(s) = V(s)L(s)$, where $V(s)$ and $L(s)$ are as previously defined. It is now established that this structure for $T(s)$ is also necessary.

Multiplying (19) on the left by $F(s)$ one obtains

$$F(s)T(s) = [1_r + F(s)P(s)C(s) - 1_r][1_r + F(s)P(s)C(s)]^{-1} \tag{62}$$

or

$$F(s)T(s) = 1_r - [1_r + F(s)P(s)C(s)]^{-1} \quad . \tag{63}$$

Since $F(s)$ and $T(s)$ must both be analytic in $\mathrm{Re}\, s \geq 0$, it follows from (63) that $[1_r + F(s)P(s)C(s)]^{-1}$ must also be. Thus,

$$T(s) = V(s)P_0(s)C(s)[1_r + F(s)P(s)C(s)]^{-1} \tag{64}$$

and

$$L(s) = P_0(s)C(s)[1_r + F(s)P(s)C(s)]^{-1} \tag{65}$$

is analytic in $\mathrm{Re}\, s \geq 0$. Moreover, $L(s)$ is real and rational, and one can establish with the aid of (22) that $L(s)$ is proper as well when $F(s)$, $P(s)$, and $C(s)$ are real, rational, proper matrices.

The above results are summarized in the following theorem:

Theorem 3: **Given a dynamical plant with asymptotically-stable hidden modes and a real, rational, proper, n x m transfer matrix P(s) having the properties:**

a) **Normal rank of P(s) is $n \leq m$,**

b) **P(s) is analytic in Re $s \geq 0$ infinity included,**

c) **Rank $P(j\omega) = n$ for all ω infinity included, then the necessary and sufficient condition for T(s) to be realizable for P(s) is that $T(s) = V(s)L(s)$ where**

d) **L(s) is any real, rational, proper, n x n matrix analytic in Re $s \geq 0$,**

e) **V(s) is determined by the factorization $P(s) = V(s)P_0(s)$ described in Lemma 1.**

Theorem 3 is the main theorem on the realizability of T(s). It is restricted to plants which satisfy rank $P(j\omega) = n$ for all ω infinity included. For plants with transfer matrices whose rank is less than n at points on the imaginary axis it is shown in the appendix that it is possible to factor the plant transfer matrix so as to obtain

$$P(s) = V_\pi(s)P_q(s) \tag{66}$$

where the n x m matrix $P_q(s)$ is analytic in Re $s \geq 0$ infinity included, rank $P_q(j\omega) = n$ for all ω infinity included, and $V_\pi(s)$ is analytic in Re $s \geq 0$ infinity included. The transfer matrix $P_q(s)$ can be factored in accordance with Lemma 1 to obtain

$$P_q(s) = V(s)P_0(s) \quad , \tag{67}$$

where V(s), $P_0(s)$, and $P_0^{-1}(s)$ are analytic in Re $s \geq 0$ infinity included. Combining (66) and (67) yields

$$P(s) = V_\pi(s)V(s)P_0(s) \quad . \tag{68}$$

It immediately follows for the choice $F(s) = O_n$ and $C(s) = P_0^{-1}(s)L(s)$, where L(s) is any real, rational, proper matrix analytic in Re $s \geq 0$, that

$$T(s) = V_\pi(s)V(s)L(s) \quad . \tag{69}$$

A sufficient condition for T(s) to be realizable for P(s) is, therefore, that it be factorable in accordance with (69). This condition is not necessary, however. Other methods for factoring P(s) exist and the representation (68) is not unique.

Unstable Plants

The preceding developments establish for a large class of asymptotically-stable plants the transfer matrices T(s) which can be realized with the standard feedback configuration. The class of plants for which P(s) is not analytic in $\mathrm{Re}\, s \geq 0$ is treated in this section. Attention is restricted to those plants with asymptotically-stable hidden modes: only plants of this type are practical. Before proceeding, it is important to establish certain facts which justify the procedure introduced in the sequel.

When P(s) is not analytic in $\mathrm{Re}\, s \geq 0$, the characteristic denominator $\psi_p(s)$ contains zeros in $\mathrm{Re}\, s \geq 0$. It immediately follows from Theorem 2, then, that the standard feedback configuration cannot be asymptotically stable with $F(s) = O_n$. This fact prevents the extension of Theorem 3 to unstable plants.

A more striking difficulty with the standard feedback configuration is that the class of T(s) realizable for P(s) can be empty for some unstable plants. A simple example is the single-input-output plant whose transfer function is $(s-2)/(s-1)(s-3)$. It is not difficult to establish that there exists no asymptotically-stable controller and feedback network which yields an asymptotically-stable standard feedback configuration for this plant. This result suggests the need for additional elements to first stabilize the plant before including it in a standard feedback configuration. In order to handle all cases, the additional elements should be sufficiently general so that they permit the stabilization of any unstable plant with asymptotically-stable hidden modes. It is shown below that any plant of the type just described can be stabilized using a modified Luenberger observer [2], [13]-[19].

In view of (31) thru (34), there is no loss in generality in assuming that the plant has the state variable description (1) and (2) in which F_p, G_p, and H_p are given, respectively, by (31) thru (33) with $K = 1_{\nu_p}$. The design of the modified Luenberger observer begins with the formation of the modified plant output vector

$$\hat{\underset{\sim}{y}}_p = E(\underset{\sim}{y}_p - J_p \underset{\sim}{u}_p) = EH_p \underset{\sim}{x}_p \quad , \tag{70}$$

where

$$E = \left[\underset{\sim}{e}_{i_1}\, \underset{\sim}{e}_{i_2} \cdots \underset{\sim}{e}_{i_h}\right]' \tag{71}$$

and $i_1, i_2, \ldots, i_h$ are the numbers of the h linearly independent rows of $\hat{H}_p$. When $J_p = O_{n,m}$, it follows from the assumption that normal rank of P(s) is n that h = n. In general, however, rank $\hat{H}_p = h < n$ is possible.

The plant state vector can be written as

$$\underset{\sim}{x}_p = \left[\underset{\sim}{x}'_{p_1}\, \underset{\sim}{x}'_{p_2}\, \underset{\sim}{x}'_{p_3}\right]' \quad , \tag{72}$$

where

$$\dim \underset{\sim}{x}_{p_i} = \nu_{p_i} \quad , \quad i = 1, 2, 3 \quad . \tag{73}$$

The objective is the design of an observer with state vector

$$\underset{\sim}{z} = T\underset{\sim}{x}_{p_1} + \underset{\sim}{e} \quad , \tag{74}$$

where the error vector $\underset{\sim}{e}$ is exponentially asymptotically stable: i.e., $\|\underset{\sim}{e}\| = \sqrt{\underset{\sim}{e}^* \underset{\sim}{e}} \leq c\epsilon^{-\lambda(t-t_0)}$ for real constants $c > 0$ and $\lambda > 0$ for all initial error vectors at $t = t_0$. The observer dimension is given by

$$\nu_0 = \dim \underset{\sim}{z} = \nu_{p_1} - h \quad . \tag{75}$$

That $\nu_0 \geq 0$ is an immediate consequence of the fact that the rank of the $n \times \nu_{p_1}$ matrix $\hat{H}_p$ is at most ν_{p_1}. The choice of the $\nu_0 \times \nu_{p_1}$ matrix T is discussed in the following paragraphs. Before proceeding it is first noted that

$$\hat{\underset{\sim}{y}}_p = [\tilde{H}_p | O_{h,\nu_{p_2}} | \tilde{H}_a] \underset{\sim}{x}_p = \tilde{H}_p \underset{\sim}{x}_{p_1} + \tilde{H}_a \underset{\sim}{x}_{p_3} \quad , \tag{76}$$

where

$$\tilde{H}_p = E\hat{H}_p \tag{77}$$

and

$$\tilde{H}_a = EH_a \quad . \tag{78}$$

When T is chosen so that the $\nu_{p_1} \times \nu_{p_1}$ matrix $[\tilde{H}_p' | T']$ is nonsingular, then

$$\hat{\underset{\sim}{x}}_{p_1} = \left[\frac{\tilde{H}_p}{T}\right]^{-1} \left[\frac{\hat{\underset{\sim}{y}}_p}{\underset{\sim}{z}}\right] = \underset{\sim}{x}_{p_1} + \underset{\sim}{e}_1 \quad , \tag{79}$$

where

$$\underset{\sim}{e}_1 = \left[\frac{\tilde{H}_p}{T}\right]^{-1} \left[\frac{\tilde{H}_a \underset{\sim}{x}_{p_3}}{\underset{\sim}{e}}\right] \quad , \tag{80}$$

is an asymptotic estimate

$$\left(\lim_{t\to\infty} \hat{\underset{\sim}{x}}_{p_1} = \lim_{t\to\infty} \underset{\sim}{x}_{p_1}\right)$$

of $\underset{\sim}{x}_{p_1}$ provided the error vector $\underset{\sim}{e}_1$ is asymptotically stable. The asymptotic stability of $\underset{\sim}{e}_1$ is determined by the behavior with time of $\underset{\sim}{e}$ and $\underset{\sim}{x}_{p_3}$. Using (1), (31), and (33) with $K = 1_{\nu_p}$ immediately yields

$$\dot{\underset{\sim}{x}}_{p_3} = F_{33}\underset{\sim}{x}_{p_3} \quad . \tag{81}$$

Since the hidden modes of the plant are asymptotically stable, the eigenvalues of F_{33} all have negative real parts. Thus, $\underset{\sim}{x}_{p_3}$ is exponentially asymptotically stable.

The determination of the behavior with time of $\underset{\sim}{e}$ requires more work. The dynamical part of the observer of interest is described by

$$\dot{\underset{\sim}{z}} = A\underset{\sim}{z} + B\hat{\underset{\sim}{y}}_p + C\underset{\sim}{u}_p \quad . \tag{82}$$

Substituting (74) and (76) into (82) and assuming that the matrix equations

$$\left.\begin{array}{c} T\hat{F}_p - AT = B\tilde{H}_p \\ T\hat{G}_p = C \end{array}\right\} \tag{83}$$

are satisfied yields

$$\dot{\underset{\sim}{e}} = A\underset{\sim}{e} + (B\tilde{H}_a - TF_{13})\underset{\sim}{x}_{p_3} \quad , \tag{84}$$

when it is recognized that

$$\dot{\underset{\sim}{x}}_{p_1} = \hat{F}_p\underset{\sim}{x}_{p_1} + F_{13}\underset{\sim}{x}_{p_3} + \hat{G}_p\underset{\sim}{u}_p \quad . \tag{85}$$

Since $\underset{\sim}{x}_{p_3}$ is exponentially asymptotically stable, it follows from (84) that $\underset{\sim}{e}$ is exponentially asymptotically stable whenever A has only eigenvalues with negative real parts. It immediately follows from the fact that $\{\hat{F}_p, \hat{H}_p\}$ is a completely-observable pair that $\{\hat{F}_p, \tilde{H}_p\}$ is also. Observer theory then guarantees that one can always find matrices A, B, C, and T which satisfy (83) and the requirement that $[\tilde{H}_p' \,|\, T']$ be nonsingular and A have only eigenvalues with negative real parts.

It is important to note that the design of the observer described above depends only on the matrices $\hat{F}_p$, $\hat{G}_p$, $\hat{H}_p$, and J_p. These matrices can be taken as the ones associated with any minimal realization of the plant transfer matrix. Fortunately, algorithms are available for generating minimal realizations starting with the plant transfer matrix (see [22], Chap. 6). This fact is important since it shows that the observer can be designed from knowledge of only the terminal properties of the plant.

The observer under consideration can be incorporated in a feedback loop around the original plant as shown in Fig. 2. In the figure,

$$[L_1 | L_2] = \begin{bmatrix} \tilde{H}_p \\ \hline T \end{bmatrix}^{-1} \tag{86}$$

and

$$\underset{\sim}{u}_0 = [\underset{\sim}{y}_p' \; \underset{\sim}{u}_p']' \quad . \tag{87}$$

This subsystem is referred to as the modified plant in the sequel.

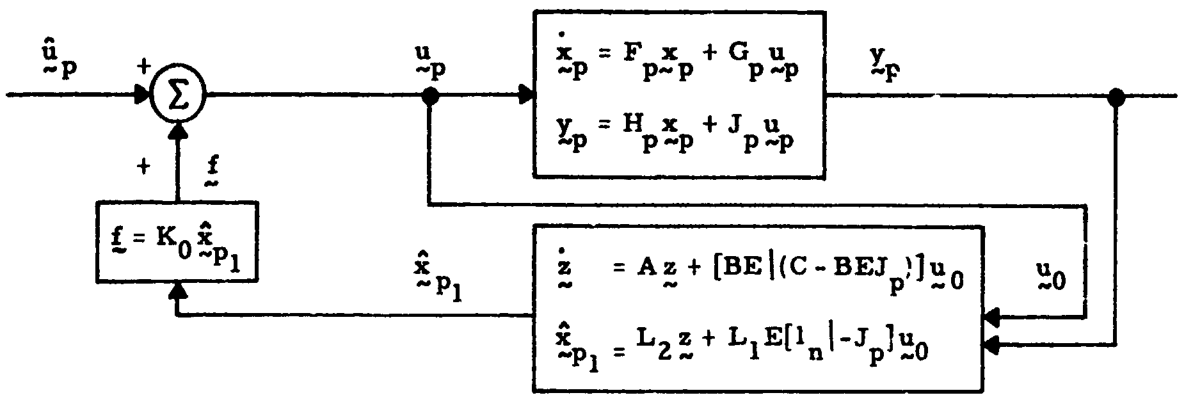

Fig. 2. Modified Plant

It is now established that one can always chose the feedback matrix K_0 so that the modified plant is asymptotically stable. With

$$\underset{\sim}{\eta} = [\underset{\sim}{x}_{p_1}' \; \underset{\sim}{x}_{p_2}' \; \underset{\sim}{x}_{p_3}' \; \underset{\sim}{e}']' \tag{88}$$

it is not difficult to verify that

$$\dot{\underset{\sim}{\eta}} = \left[\begin{array}{c|c|c|c} \hat{F}_p + \hat{C}_p K_0 & O_{\nu_{p_1}, \nu_{p_2}} & F_{13} + \hat{G}_p K_0 L_1 \tilde{H}_a & \hat{G}_p K_0 L_2 \\ \hline F_{12} + G_a K_0 & F_{22} & F_{23} + G_a K_0 L_1 \tilde{H}_a & G_a K_0 L_2 \\ \hline O_{\nu_{p_3}, \nu_{p_1}} & O_{\nu_{p_3}, \nu_{p_2}} & F_{33} & O_{\nu_{p_3}, \nu_0} \\ \hline O_{\nu_0, \nu_{p_1}} & O_{\nu_0, \nu_{p_2}} & B\tilde{H}_a - TF_{13} & A \end{array}\right] \underset{\sim}{\eta}$$

$$+ \left[\begin{array}{c} \hat{G}_p \\ \hline G_a \\ \hline O_{\nu_{p_3}, m} \\ \hline O_{\nu_0, m} \end{array}\right] \hat{u}_p \tag{89}$$

when (84) is recalled and it is recognized that for a properly designed observer

$$\underset{\sim}{u}_p = K_0 \hat{\underset{\sim}{x}}_{p_1} + \hat{\underset{\sim}{u}}_p = K_0(\underset{\sim}{x}_{p_1} + L_1 \tilde{H}_a \underset{\sim}{x}_{p_3} + L_2 \underset{\sim}{e}) + \hat{\underset{\sim}{u}}_p \quad . \tag{90}$$

The eigenvalues of the coefficient matrix of $\underset{\sim}{\eta}$ in (89) determine the stability of the system. It is not difficult to show that the eigenvalues of this matrix are the eigenvalues of A, F_{22}, F_{33}, and $\hat{F}_p + \hat{G}_p K_0$. The eigenvalues of A are the observer eigenvalues which are chosen to have negative real parts. The assumption that the hidden modes of the plant are asymptotically stable is equivalent to all the eigenvalues of F_{22} and F_{33} having negative real parts. Since $\{\hat{F}_p, \hat{G}_p\}$ is a completely-controllable pair, it follows that one can always choose a K_0 so that the eigenvalues of $\hat{F}_p + \hat{G}_p K_0$ all have negative real parts. An algorithm for choosing the matrix K_0 is described in [27]. Hence, it is always possible to make the modified plant asymptotically stable. The above is summarized in

> <u>Theorem 4:</u> Any real, linear, time-invariant, finite-dimensional, dynamical plant with asymptotically-stable hidden modes can be stabilized using a suitably designed dynamic observer.

Attention is now turned to the computation of the modified-plant transfer matrix. It follows from (90) that

$$\underset{\sim}{y}_p = [\hat{H}_p + J_p K_0 | O_{n,\nu_{p_2}} | H_a + J_p K_0 L_1 \tilde{H}_a | J_p K_0 L_2] \underset{\sim}{\eta} + J_p \hat{\underset{\sim}{u}}_p \quad . \tag{91}$$

Using the fact that the inverse of a block triangular matrix with two square blocks on the diagonal is also a block triangular matrix of the same form one readily deduces from (89) and (91) that the transfer matrix relating $\underset{\sim}{Y}_p(s)$ to $\hat{\underset{\sim}{U}}_p(s)$ is

$$\hat{P}(s) = (\hat{H}_p + J_p K_0)(s 1_{\nu_{p_1}} - \hat{F}_p - \hat{G}_p K_0)^{-1} \hat{G}_p + J_p \quad . \tag{92}$$

The matrix $\hat{P}(s)$ is the modified-plant transfer matrix.

It immediately follows from the identity

$$\left[\begin{array}{c|c} J_p & \hat{H}_p + J_p K_0 \\ \hline \hat{G}_p & \hat{F}_p + \hat{G}_p K_0 - s 1_{\nu_{p_1}} \end{array}\right] = \left[\begin{array}{c|c} J_p & \hat{H}_p \\ \hline \hat{G}_p & \hat{F}_p - s 1_{\nu_{p_1}} \end{array}\right] \left[\begin{array}{c|c} 1_m & K_0 \\ \hline O_{\nu_{p_1}, m} & 1_{\nu_{p_1}} \end{array}\right] \tag{93}$$

that the structure matrices for the original and modified plants are strictly equivalent:

$$\hat{\Gamma}(s) = \left[\begin{array}{c|c} J_p & \hat{H}_p + J_p K_0 \\ \hline \hat{G}_p & \hat{F}_p + \hat{G}_p K_0 - s1_{\nu_{p_1}} \end{array}\right] \tag{94}$$

is the structure matrix for the modified plant and

$$\Gamma(s) = \left[\begin{array}{c|c} J_p & \hat{H}_p \\ \hline \hat{G}_p & \hat{F}_p - s1_{\nu_{p_1}} \end{array}\right] \tag{95}$$

is the structure matrix for the original plant. The modified plant is designed to be asymptotically stable; its nonminimum phase properties, hence, are completely characterized by the structure matrix $\hat{\Gamma}(s)$. The same is not true for the original plant when P(s) is not analytic in Re s $\geq$ 0. The set of points in Re s $\geq$ 0 where P(s) has no poles is denoted by S. For all s ϵ S it readily follows from (42) and (93) th· u (95) that

$$\text{rank}\, P(s) = \text{rank}\, \Gamma(s) - \nu_{p_1} = \text{rank}\, \hat{\Gamma}(s) - \nu_{p_1} = \text{rank}\, \hat{P}(s) \quad . \tag{96}$$

Thus, for all s ϵ S rank $\hat{P}(s) < n$ where rank $P(s) < n$. It is also possible for rank $\hat{P}(s)$ to be less than n at the points in Re s $\geq$ 0 where P(s) is not analytic. The set of all points in Re s $\geq$ 0 associated with nonminimum phase properties of $\hat{P}(s)$ is, therefore a subset of the corresponding set for P(s).

The above results suggest that the nonmi 'mum phase properties of the modified plant are often equivalent to or less severe than those of the original plant. One is tempted to conclude from this fact that the class of T(s) realizable for the modified plant in the standard feedback configuration is equivalent to or larger than that realizable with the original plant even when it is possible to stabilize the standard feedback configuration without resorting to the use of the modified Luenberger observer. This point has not yet been rigorously established, however.

Some additional observation ncerning the modified plant are now made. It is not difficult to verify using well known matrix identities that (92) is equivalent to

$$\hat{P}(s) = (\hat{H}_p + J_p K_0)(s1_{\nu_{p_1}} - \hat{F}_p)^{-1}\hat{G}_p[1_{m_p} - K_0(s1_{\nu_{p_1}} - \hat{F}_p)^{-1}\hat{G}_p]^{-1} + J_p \quad . \tag{97}$$

When the bracketed inverse in (97) is factored to the right and (35) is recalled, one easily obtains the relationship

$$\hat{P}(s) = P(s)[1_{m_p} - K_0(s1_{\nu_{p_1}} - \hat{F}_p)^{-1}\hat{G}_p]^{-1} \quad . \tag{98}$$

Equation (98) clearly places in evidence the relationship between $\hat{P}(s)$ and $P(s)$. Since $\hat{P}(s)$ is analytic in Re s $\geq$ 0, the nonminimum phase properties of $\hat{P}(s)$ are determined by the points in Re s $\geq$ 0 where all n-order minors of $\hat{P}(s)$ are zero. The Binet-Cauchy formula leads to the fact* that each n-order minor of $\hat{P}(s)$ is the sum of products of n-order minors of P(s) and $[1_{m_p} - K_0(s1_{\nu_{p_1}} - \hat{F}_p)^{-1}\hat{G}_p]^{-1}$. It is not easy, therefore, to relate in precise fashion the nonminimum phase properties of $\hat{P}(s)$ and P(s) for nonsquare plants.

Considerable insight is obtainable for square plants. In this case $m_p = n_p$ and it follows from (98) that

$$\det \hat{P}(s) = \frac{\det P(s)}{\det[1_{n_p} - K_0(s1_{\nu_{p_1}} - \hat{F}_p)^{-1}\hat{G}_p]} , \tag{99}$$

or

$$\det \hat{P}(s) = \frac{\det P(s)}{\det(s1_{\nu_{p_1}} - \hat{F}_p)^{-1} \det(s1_{\nu_{p_1}} - \hat{F}_p - \hat{G}_p K_0)} . \tag{100}$$

Using (36) and

$$h(s) = \det(s1_{\nu_{p_1}} - \hat{F}_p - \hat{G}_p K_0) \tag{101}$$

in (100) gives the compact relationship

$$\det \hat{P}(s) = \frac{\psi_p(s) \det P(s)}{h(s)} . \tag{102}$$

By design the polynomial h(s) is free of zeros in Re s $\geq$ 0. Since P(s) is a rational matrix, it is also true that

$$\det P(s) = \frac{\alpha_p(s)}{\beta_p(s)} , \tag{103}$$

where $\alpha_p(s)$ and $\beta_p(s)$ are polynomials. Moreover, $\beta_p(s)$ divides the characteristic denominator $\psi_p(s)$ of P(s):

$$\phi_p(s) = \frac{\psi_p(s)}{\beta_p(s)} \tag{104}$$

is a polynomial in s. Substituting (103) and (104) into (102) yields

$$\det \hat{P}(s) = \frac{\phi_p(s)\,\alpha_p(s)}{h(s)} . \tag{105}$$

*See page 9 and equation (19) of reference [26].

The zeros of $\det \hat{P}(s)$ in $\text{Re}\, s \geq 0$ account for the nonminimum phase properties of the modified plant. Since $h(s)$ is strictly Hurwitz, it follows that the zeros of $\det \hat{P}(s)$ in $\text{Re}\, s \geq 0$ are the zeros of $\phi_p(s)$ and $\alpha_p(s)$ in $\text{Re}\, s \geq 0$. The zeros of $\alpha_p(s)$ in $\text{Re}\, s \geq 0$ are the points in $\text{Re}\, s \geq 0$ where rank $P(s) < n$. Any zeros of $\phi_p(s)$ in $\text{Re}\, s \geq 0$ are a result of the fact that $P(s)$ is not analytic in $\text{Re}\, s \geq 0$.

An example which demonstrates the generation of nonminimum properties in $\hat{P}(s)$ when $P(s)$ is unstable is easily generated. It is not difficult to verify for

$$P(s) = \begin{bmatrix} \frac{3}{s+1} & \frac{s-1}{s+2} \\ \frac{1}{s-1} & \frac{s+1}{s+2} \end{bmatrix} \tag{106}$$

that

$$\det P(s) = \frac{2}{s+2} = \frac{\alpha_p(s)}{\beta_p(s)} \neq 0, \quad \text{Re}\, s \geq 0 \quad . \tag{107}$$

Also,

$$\psi_p(s) = (s-1)(s+1)(s+2) \quad . \tag{108}$$

Thus,

$$\phi_p(s) = \frac{\psi_p(s)}{\beta_p(s)} = (s-1)(s+1) \tag{109}$$

and $\det \hat{P}(s) = 0$ at $s = +1$.

A special case of interest is the single-input-output plant: $n_p = m_p = 1$. In this case,

$$\det P(s) = P(s) = \frac{\alpha_p(s)}{\beta_p(s)} \tag{110}$$

and

$$\psi_p(s) = \beta_p(s) \quad . \tag{111}$$

Thus,

$$\det \hat{P}(s) = \hat{P}(s) = \frac{\alpha_p(s)}{h(s)} \tag{112}$$

and the nonminimum phase properties of the modified plant are completely determined by the zeros of the original plant in $\text{Re}\, s \geq 0$.

Conclusions

For the class of plants satisfying the conditions of Theorem 3 the transfer matrices T(s) realizable using the standard feedback configuration have been precisely defined. Much work remains for this class of plants, however. Fundamental questions are in need of answers. Given that T(s) is realizable for P(s) one can expect in general many combinations of C(s) and F(s) which yield the desired T(s). Which of these combinations is best? One possibility is to try to determine that combination which minimizes in some sense the sensitivity of the system to plant parameter variations and/or disturbance inputs. The ideas developed in [28] and [29] may prove useful in this regard. Another possibility is to select that controller and feedback network having the property that the sum of the dimensions of the state vectors for minimal realizations of both these elements is a minimum. Some preliminary results in this regard are contained in Chapter 9 of [22]. Another possibility of course is a compromise between the two already cited.

With regard to unstable plants the results reported here – although extensive – must be viewed as preliminary only. Much remains to be done. Suppose it is possible to stabilize a given P(s) using only the standard feedback configuration. What is the class of T(s) realizable in this case without resorting to the addition of a modified Luenberger observer? Does the addition of a modified Luenberger observer enlarge the class of T(s) realizable in this same case?

Finally, one can question the sacredness of the standard feedback configuration. This configuration is only a special case of the system shown in Fig. 3. This figure represents all possible plant compensation schemes. The connection network is

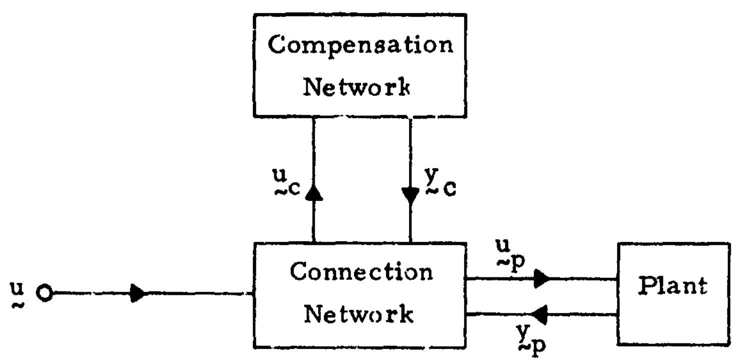

Fig. 3. General Plant Compensation Configuration

characterized by the real, constant connection matrix M which relates the interconnection of the system input $\underset{\sim}{u}$ and the inputs and outputs of the compensation and plant:

$$\left[\frac{\underset{\sim}{u}_p}{\underset{\sim}{u}_c}\right] = M\left[\begin{array}{c}\underset{\sim}{y}_p \\ \hline \underset{\sim}{y}_c \\ \hline \underset{\sim}{u}\end{array}\right] \quad .$$

The connection network can include operational amplifiers and this fact permits one – for all practical purposes – to assume that the elements of M can take on any real value. The compensation network includes all dynamical components of the system except the plant. Any real, linear, finite-dimensional, time-invariant, asymptotically-stable, dynamical system is a possible choice for the compensation network. One can now raise all the previous questions with regard to the admissable classes of connection matrices and compensation networks just described for the configuration shown in Fig. 3. Work directed toward answering these questions is presently under way.

Appendix

A method for achieving the plant factorization (66) is now described. It is assumed that P(s) is analytic in Re $s \geq 0$ and rank $P(j\omega_k) < n$, $\omega_k > 0$. Since P(s) is a real matrix, it follows that rank $P(-j\omega_k) < n$.

The matrix

$$P_k(s) = \left(1_n + \frac{A}{s-j\omega_k} + \frac{\overline{A}}{s+j\omega_k}\right)P(s) \tag{A.1}$$

is now considered where the matrix A is selected in accordance with the following considerations. From the fact that rank $P(j\omega_k) < n$, it follows that there exists a non-zero complex vector $\underset{\sim}{b}$ such that

$$\underset{\sim}{b}^* P(j\omega_k) = \underset{\sim}{o}'_m \quad . \tag{A.2}$$

The vector $\underset{\sim}{b}$ can be written as

$$\underset{\sim}{b} = \underset{\sim}{b}_1 + j\underset{\sim}{b}_2 \quad , \tag{A.3}$$

where $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are real vectors. Two possibilities exist: e ther the vectors $\underset{\sim}{b}_1$ and b_2 are linearly dependent or they are linearly independent. The former case is considered first.

When the real vectors $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are linearly dependent one can write

$$\left.\begin{array}{l} \underset{\sim}{b}_1 = c_1 \underset{\sim}{a} \\ \underset{\sim}{b}_2 = c_2 \underset{\sim}{a} \end{array}\right\} \quad , \tag{A.4}$$

where c_1 and c_2 are real scalars and the real vector $\underset{\sim}{a}$ satisfies

$$\|\underset{\sim}{a}\| \equiv \sqrt{\underset{\sim}{a}^* \underset{\sim}{a}} = \sqrt{\underset{\sim}{a}' \underset{\sim}{a}} = 1 \quad . \tag{A.5}$$

Then

$$\underset{\sim}{b} = (c_1 + jc_2)\underset{\sim}{a} = c\,\underset{\sim}{a} \quad , \quad c \neq 0 \tag{A.6}$$

and

$$\underset{\sim}{b}^* P(j\omega_k) = \overline{c}\,\underset{\sim}{a}' P(j\omega_k) = \underset{\sim}{o}'_m \tag{A.7}$$

nplies

$$\underset{\sim}{a}' P(j\omega_k) = \underset{\sim}{o}'_m \quad . \tag{A.8}$$

The choice

$$A = \overline{A} = \underset{\sim}{a}\,\underset{\sim}{a}' \tag{A.9}$$

is now considered. Equation (A.1) becomes

$$P_k(s) = V_k^{-1}(s)P(s) \quad , \tag{A.10}$$

where

$$V_k^{-1}(s) = \left(1_n + \frac{2s\,\underset{\sim}{a}\,\underset{\sim}{a}'}{s^2 + \omega_k^2}\right) \quad . \tag{A.11}$$

Clearly,

$$V_k^{-1}(\infty) = \lim_{s \to \infty} V_k^{-1}(s) = 1_n \tag{A.12}$$

and

$$P_k(\infty) = \lim_{s \to \infty} P_k(s) = P(\infty) \quad ; \tag{A.13}$$

thus, both $P_k(s)$ and $V_k^{-1}(s)$ are proper matrices. Using the fact that

$$\det(1_n + AB) = \det(1_m + BA) \tag{A.13}$$

for an arbitrary n x m matrix A and an arbitrary m x n matrix B, one easily obtains from (A.11) that

$$\det V_k^{-1}(s) = \det\left(1 + \frac{2s\,\underset{\sim}{a}'\underset{\sim}{a}}{s^2 + \omega_k^2}\right) = \frac{s^2 + 2s + \omega_k^2}{s^2 + \omega_k^2} \quad . \tag{A.14}$$

Equation (A.14) establishes that

$$\det V_k^{-1}(s) = \text{nonzero finite complex number}, \ \operatorname{Re} s \geq 0, \ s \neq \pm j\omega_k \quad . \tag{A.15}$$

Hence,

$$\operatorname{rank} P_k(s) = \operatorname{rank} P(s), \ \operatorname{Re} s \geq 0, \ s \neq \pm j\omega_k \quad . \tag{A.16}$$

Equation (A.16) is important since it shows that the set of points in $\operatorname{Re} s \geq 0$, $s \neq \pm j\omega_k$, where $\operatorname{rank} P_k(s) < n$ is the same set of points in $\operatorname{Re} s \geq 0$, $s \neq \pm j\omega_k$, where $\operatorname{rank} P(s) < n$.

The next point that one needs to make is that $P_k(s)$ is analytic at $s = \pm j\omega_k$. This fact follows immediately from

$$(s \mp j\omega_k)P_k(s)\Big|_{s = \pm j\omega_k} = (s \mp j\omega_k)V_k^{-1}(s)P(s)\Big|_{s = \pm j\omega_k} = 0 \quad . \tag{A.17}$$

It is also obvious from (A.10) and (A.11) that $P_k(s)$ is analytic in Re $s \geq 0$, $s \neq \pm j\omega_k$. Thus, $P_k(s)$ is analytic in Re $s \geq 0$ infinity included.

The final property of $P_k(s)$ which is investigated is rank $P_k(\pm j\omega_k)$. This is best done by first defining in lexicographic order all the corresponding n'th-order minors of $P_k(s)$ and $P(s)$. The j'th such minor of $P_k(s)$ and $P(s)$ is denoted by $\Delta_{P_k}^{(n,j)}$ and $\Delta_P^{(n,j)}$, respectively. It immediately follows from (A.10), (A.14), and the Binet-Cauchy formula (see [26], p. 12) that

$$\Delta_{P_k}^{(n,j)} = \left(\frac{s^2 + 2s + \omega_k^2}{s^2 + \omega_k^2}\right) \Delta_P^{(n,j)} \quad . \tag{A.18}$$

Since rank $P(\pm j\omega_k) < n$, it follows that for each j

$$\Delta_P^{(n,j)} = (s^2 + \omega_k^2)^{\nu_j} \hat{\Delta}_P^{(n,j)} \quad , \quad \nu_j \geq 1 \quad , \tag{A.19}$$

where $\hat{\Delta}_P^{(n,j)} \neq 0$ and is finite at $s = \pm j\omega_k$. Thus,

$$\Delta_{P_k}^{(n,j)} = (s^2 + 2s + \omega_k^2)(s^2 + \omega_k^2)^{\nu_j - 1} \hat{\Delta}_P^{(n,j)} \quad . \tag{A.20}$$

Clearly, if for any j it is true that $\nu_j = 1$ then there is one n'th-order minor of $P_k(s)$ which is not zero at $s = \pm j\omega_k$ and rank $P_k(\pm j\omega_k) = n$. When it is not true that $\nu_j = 1$ for any j, then although rank $P_k(\pm j\omega_k)$ is still less than n one has reduced the order of the factor $s^2 + \omega_k^2$ in each of the n'th-order minors of P(s) by one. The above process can then be repeated a finite number of time – provided each time that the new vectors $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are linearly dependent – until a matrix is obtained whose rank is n at $s = \pm j\omega_k$.

Before considering the case in which the vectors b_1 and b_2 are linearly independent, some additional observations are now made. First, the above developments are easily applied to the case rank $P(0) < n$. For this case, one has immediately that there exists a real vector $\underset{\sim}{a}$ satisfying (A.8) with $\omega_k = 0$ since P(s) is a real matrix. The final observation is in regard to the fact that (A.10) is not the ultimate relationship sought. One needs instead

$$P(s) = V_k(s)P_k(s) \quad . \tag{A.21}$$

It is easy to verify that

$$V_k(s) = 1_n - \frac{2s\,\underset{\sim}{a}\,\underset{\sim}{a}'}{s^2 + 2s + \omega_k^2} \tag{A.22}$$

satisfies $V_k(s)\,V_k^{-1}(s) = 1_n$. Equation (A.22) exposes the fact that $V_k(s)$ is a proper matrix analytic in Re $s \geq 0$.

Attention is now turned to the case where b_1 and b_2 are linearly independent. It is first noted that when the complex vector $\underset{\sim}{a} \neq \underset{\sim}{o}_n$ satisfies

$$\underset{\sim}{a}^* P(j\omega_k) = \underset{\sim}{o}'_m \quad , \tag{A.23}$$

then

$$\underset{\sim}{b} = \epsilon^{-j\theta} \underset{\sim}{a} \tag{A.24}$$

satisfies

$$\underset{\sim}{b}^* P(j\omega_k) = \underset{\sim}{o}'_m \quad . \tag{A.25}$$

Moreover, one can choose $\underset{\sim}{a}$ so that

$$\underset{\sim}{b}^* \underset{\sim}{b} = \underset{\sim}{a}^* \underset{\sim}{a} = \|\underset{\sim}{a}\|^2 = 1 \quad . \tag{A.26}$$

Also, θ can always be selected so that with μ a real scalar

$$\underset{\sim}{a}'\underset{\sim}{a} = \epsilon^{-2j\theta}\, \underset{\sim}{b}'\underset{\sim}{b} = \mu \geq 0 \quad . \tag{A.27}$$

Thus, writing

$$\underset{\sim}{a} = \underset{\sim}{a}_1 + j\underset{\sim}{a}_2 \quad ,$$

where $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ are real vectors, leads to

$$\underset{\sim}{a}'\underset{\sim}{a} = (\underset{\sim}{a}'_1\underset{\sim}{a}_1 - \underset{\sim}{a}'_2\underset{\sim}{a}_2) + j(\underset{\sim}{a}'_2\underset{\sim}{a}_1 + \underset{\sim}{a}'_1\underset{\sim}{a}_2) = \mu \quad . \tag{A.28}$$

Since μ is real, it immediately follows from (A.28) that

$$\underset{\sim}{a}'_2\underset{\sim}{a}_1 + \underset{\sim}{a}'_1\underset{\sim}{a}_2 = 2\underset{\sim}{a}'_1\underset{\sim}{a}_2 = 0 \tag{A.29}$$

and

$$\underset{\sim}{a}'_1\underset{\sim}{a}_1 - \underset{\sim}{a}'_2\underset{\sim}{a}_2 = \mu \quad . \tag{A.30}$$

It is also true that

$$\underset{\sim}{a}^*\underset{\sim}{a} = \underset{\sim}{a}'_1\underset{\sim}{a}_1 + \underset{\sim}{a}'_2\underset{\sim}{a}_2 = 1 \tag{A.31}$$

because of (A.29). Adding (A.30) and (A.31) yields

$$\underset{\sim}{a}'_1\underset{\sim}{a}_1 = \frac{1}{2}(1 + \mu) \quad . \tag{A.32}$$

Subtracting (A.30) from (A.31) gives

$$\underset{\sim}{a}_2'\underset{\sim}{a}_2 = \frac{1}{2}(1 - \mu) \quad . \tag{A.33}$$

Now

$$\mu = |\underset{\sim}{a}'\underset{\sim}{a}| = \left| \sum_{i=1}^{n} a_i^2 \right| \le \sum_{i=1}^{n} |a_i|^2 = \|\underset{\sim}{a}\|^2 = 1 \quad . \tag{A.34}$$

Hence,

$$0 \le \mu \le 1 \quad . \tag{A.35}$$

Since (A.33) shows that $\underset{\sim}{a}_2 = \underset{\sim}{o}_n$ when $\mu = 1$, it follows that $\underset{\sim}{a}$ is real in this case and the results already derived are applicable. It is, therefore, assumed in the sequel that

$$0 \le \mu < 1 \quad . \tag{A.36}$$

The choice

$$A = \underset{\sim}{a}\underset{\sim}{a}^* \tag{A.37}$$

in (A.1) is now considered. With this choice

$$P_k(s) = V_k^{-1}(s)P(s) \quad , \tag{A.38}$$

where

$$V_k^{-1}(s) = \left(1_n + \frac{\underset{\sim}{a}\underset{\sim}{a}^*}{s - j\omega_k} + \frac{\bar{\underset{\sim}{a}}\underset{\sim}{a}'}{s + j\omega_k} \right) \quad . \tag{A.39}$$

Now

$$\underset{\sim}{a}\underset{\sim}{a}^* = (\underset{\sim}{a}_1\underset{\sim}{a}_1' + \underset{\sim}{a}_2\underset{\sim}{a}_2') + j(\underset{\sim}{a}_2\underset{\sim}{a}_1' - \underset{\sim}{a}_1\underset{\sim}{a}_2') \tag{A.40}$$

and

$$\bar{\underset{\sim}{a}}\underset{\sim}{a}' = \overline{(\underset{\sim}{a}\underset{\sim}{a}^*)} \quad . \tag{A.41}$$

Thus, (A.39) becomes

$$V_k^{-1}(s) = \left[1_n + \frac{2s(\underset{\sim}{a}_1\underset{\sim}{a}_1' + \underset{\sim}{a}_2\underset{\sim}{a}_2')}{s^2 + \omega_k^2} + \frac{2\omega_k(\underset{\sim}{a}_1\underset{\sim}{a}_2' - \underset{\sim}{a}_2\underset{\sim}{a}_1')}{s^2 + \omega_k^2} \right] \quad . \tag{A.42}$$

Clearly,

$$V_k^{-1}(\infty) = \lim_{s \to \infty} V_k^{-1}(s) = 1_n \tag{A.43}$$

and

$$P_k(\infty) = \lim_{s \to \infty} P_k(s) = P(\infty) \quad ; \tag{A.44}$$

thus, both $P_k(s)$ and $V_k^{-1}(s)$ are again proper matrices.

As before, the determinant of $V_k^{-1}(s)$ is of interest. Now, however, more work is required in order to evaluate this quantity. The computation is facilitated by constructing the orthogonal matrix

$$Q = [\underset{\sim}{q}_1\ \underset{\sim}{q}_2 \cdots \underset{\sim}{q}_n] \quad , \tag{A.45}$$

where

$$\underset{\sim}{q}_1 = \frac{\underset{\sim}{a}_1}{\sqrt{\frac{1}{2}(1+\mu)}} \tag{A.46}$$

and

$$\underset{\sim}{q}_2 = \frac{\underset{\sim}{a}_2}{\sqrt{\frac{1}{2}(1-\mu)}} \quad . \tag{A.47}$$

That q_1 and q_2 are orthogonal is an immediate consequence of (A.29). It is also clear from (A.32) and (A.33) that

$$\|\underset{\sim}{q}_1\| = \|\underset{\sim}{q}_2\| = 1 \quad . \tag{A.48}$$

It now follows that

$$Q'V_k^{-1}(s)Q = 1_n + \left(\frac{2s}{s^2+\omega_k^2}\right)\begin{bmatrix} \underset{\sim}{q}_1' \\ \vdots \\ \underset{\sim}{q}_n' \end{bmatrix}\left[\left(\frac{1+\mu}{2}\right)\underset{\sim}{q}_1\underset{\sim}{q}_1' + \left(\frac{1-\mu}{2}\right)\underset{\sim}{q}_2\underset{\sim}{q}_2'\right][\underset{\sim}{q}_1 \cdots \underset{\sim}{q}_n]$$

$$+\left(\frac{2\omega_k}{s^2+\omega_k^2}\right)\begin{bmatrix} \underset{\sim}{q}_1' \\ \vdots \\ \underset{\sim}{q}_n' \end{bmatrix}\left[\left(\frac{\sqrt{1-\mu^2}}{2}\right)\underset{\sim}{q}_1\underset{\sim}{q}_2' - \left(\frac{\sqrt{1-\mu^2}}{2}\right)\underset{\sim}{q}_2\underset{\sim}{q}_1'\right][\underset{\sim}{q}_1 \cdots \underset{\sim}{q}_n] \quad , \tag{A.49}$$

or

$$Q'V_k^{-1}(s)Q = \left[\begin{array}{cc|c} 1 + \dfrac{(1+\mu)s}{s^2+\omega_k^2} & \dfrac{\omega_k\sqrt{1-\mu^2}}{s^2+\omega_k^2} & \\ & & O_{2,\,n-2} \\ \dfrac{-\omega_k\sqrt{1-\mu^2}}{s^2+\omega_k^2} & 1 + \dfrac{(1-\mu)s}{s^2+\omega_k^2} & \\ \hline \multicolumn{2}{c|}{O_{n-2,\,2}} & 1_{n-2} \end{array}\right] \quad . \tag{A.50}$$

Since

$$\det V_k^{-1}(s) = \det[Q' V_k^{-1}(s) Q] \quad , \tag{A.51}$$

one easily obtains from (A. 50) that

$$\det V_k^{-1}(s) = \frac{s^2 + 2s + \omega_k^2 + (1 - \mu^2)}{s^2 + \omega_k^2} \quad . \tag{A.52}$$

Now $0 \leq \mu < 1$ and it follows from (A. 52), therefore, that

$$\det V_k^{-1}(s) = \text{nonzero finite complex number}, \ \mathrm{Re}\, s \geq 0, \ s \neq \pm j\omega_k \ . \tag{A.53}$$

Hence,

$$\mathrm{rank}\ P_k(s) = \mathrm{rank}\ P(s), \ \mathrm{Re}\, s \geq 0, \ s \neq \pm j\omega_k \tag{A.54}$$

just as in the previous case considered.

From (A. 38) and (A. 42) it is clear that $P_k(s)$ is analytic in Re $s \geq 0$, $s \neq \pm j\omega_k$. It is now established that $P_k(s)$ is analytic at $s = \pm j\omega_k$ as well. One has from (A. 42) that

$$(s \mp j\omega_k) V_k^{-1}(s) = (s \mp j\omega_k) 1_n + \frac{2s(\underset{\sim}{a}_1 \underset{\sim}{a}_1' + \underset{\sim}{a}_2 \underset{\sim}{a}_2')}{(s \pm j\omega_k)} + \frac{2\omega_k(\underset{\sim}{a}_1 \underset{\sim}{a}_2' - \underset{\sim}{a}_2 \underset{\sim}{a}_1')}{(s \pm j\omega_k)} \quad . \tag{A.55}$$

Hence,

$$(s - j\omega_k) V_k^{-1}(s) \Big|_{s = +j\omega_k} = (\underset{\sim}{a}_1 \underset{\sim}{a}_1' + \underset{\sim}{a}_2 \underset{\sim}{a}_2') + j(\underset{\sim}{a}_2 \underset{\sim}{a}_1' - \underset{\sim}{a}_1 \underset{\sim}{a}_2') = \underset{\sim}{a}\, \underset{\sim}{a}^* \tag{A.56}$$

and

$$(s + j\omega_k) V_k^{-1}(s) \Big|_{s = -j\omega_k} = (\underset{\sim}{a}_1 \underset{\sim}{a}_1' + \underset{\sim}{a}_2 \underset{\sim}{a}_2') - j(\underset{\sim}{a}_2 \underset{\sim}{a}_1' - \underset{\sim}{a}_1 \underset{\sim}{a}_2') = \overline{(\underset{\sim}{a}\, \underset{\sim}{a}^*)} \quad . \tag{A.57}$$

Since (A. 23) also implies

$$\overline{\underset{\sim}{a}^* P(j\omega_k)} = \overline{\underset{\sim}{a}^*}' P(-j\omega_k) = \underset{\sim}{o}_m' \quad , \tag{A.58}$$

one readily concludes from (A. 56) and (A. 57) that

$$(s \mp j\omega_k) V_k^{-1}(s) P(s) \Big|_{s = \pm j\omega_k} = 0 \quad . \tag{A.59}$$

Thus, $P_k(s)$ is analytic in Re $s \geq 0$ infinity included.

The final property of $P_k(s)$ which is investigated is rank $P_k(\pm j\omega_k)$. Instead of (A.18) one now obtains

$$\Delta_{P_k}^{(n,j)} = \left(\frac{s^2+2s+\omega_k^2+1-\mu^2)}{s^2+\omega_k^2}\right)\Delta_P^{(n,j)} \quad . \tag{A.60}$$

Since $0 \leq \mu < 1$, all of the discussion following (A.18) is again applicable. Moreover, now when rank $P_k(\pm j\omega_n) < n$ one is assured that the process can be repeated.

The last item requiring consideration is the computation of $V_k(s)$. Clearly,

$$V_k(s) = Q[Q'V_k^{-1}(s)Q]^{-1}Q' \quad , \tag{A.61}$$

or from (A.50)

$$V_k(s) = Q\left[\begin{array}{cc|c} \dfrac{s^2+(1-\mu)s+\omega_k^2}{d(s)} & \dfrac{-\omega_k\sqrt{1-\mu^2}}{d(s)} & \\ & & O_{2,n-2} \\ \dfrac{\omega_k\sqrt{1-\mu^2}}{d(s)} & \dfrac{s^2+(1+\mu)s+\omega_k^2}{d(s)} & \\ \hline & O_{n-2,2} & 1_{n-2} \end{array}\right]Q' \quad , \tag{A.62}$$

where

$$d(s) = s^2+2s+\omega_k^2 + (1-\mu^2) \quad . \tag{A.63}$$

Since Q is a real finite orthogonal matrix and since $0 \leq \mu < 1$, one has that $V_k(s)$ is analytic in Re $s \geq 0$ infinity included.

One case remains to be considered. It is the one in which rank $P(j\infty) < n$. The matrix

$$P_\infty(s) = V_\infty^{-1}(s)\, P(s) \tag{A.64}$$

is considered where

$$V_\infty^{-1}(s) = (1_n + \underset{\sim}{a}\underset{\sim}{a}'s) \quad . \tag{A.65}$$

The real vector $\underset{\sim}{a}$ is one which satisfies $\|\underset{\sim}{a}\| = 1$ and

$$\underset{\sim}{a}'P(j\infty) = \underset{\sim}{a}'P(\infty) = \underset{\sim}{o}'_m \tag{A.66}$$

Clearly,

$$\lim_{s \to \infty} \frac{1}{s} P_\infty(s) = \lim_{s \to \infty} \underset{\sim}{a}\,\underset{\sim}{a}' P(s) = O_{n,m} \quad . \tag{A.67}$$

Thus, $P_\infty(s)$ is analytic at infinity or, equivalently, $P_\infty(\infty)$ is finite. That is, $P_\infty(s)$ is a proper matrix analytic in Re $s \geq 0$.

It readily follows from (A.65) that

$$\det V_\infty^{-1}(s) = 1 + s \quad . \tag{A.68}$$

Hence, for all finite $s \neq -1$ it follows that

$$\text{rank } P_k(s) = \text{rank } P(s) \quad . \tag{A.69}$$

Moreover, in place of (A.18) one now has

$$\Delta_{P_\infty}^{(n,j)} = (s+1)\, \Delta_P^{(n,j)} \quad . \tag{A.70}$$

The minor $\Delta_P^{(n,j)}$ is a rational function of s and the degree of the denominator polynomial is less than the degree of the numerator polynomial for all j. Because of (A.70) it is clear that the difference between the degrees of the denominator and numerator polynomials of $\Delta_{P_\infty}^{(n,j)}$ is one less than the same difference for $\Delta_P^{(n,j)}$. Hence, either rank $P_\infty(\infty) = n$ or the process indicated here can be repeated a finite number of times until a matrix with rank equal to n at infinity is obtained.

The final item requiring consideration is the computation of $V_\infty(s)$. It is easy to verify that

$$V_\infty(s) = 1_n - \frac{\underset{\sim}{a}\,\underset{\sim}{a}' s}{s+1} \quad . \tag{A.71}$$

Clearly, $V_\infty(s)$ is a proper matrix analytic in Re $s \geq 0$.

Given a plant transfer matrix $P(s)$ there is at most a finite number of points on the imaginary axis of the complex s-plane at which rank $P(s) < n$. This is so because $P(s)$ is rational and has normal rank n. Repeated applications of the factorizations described in this appendix then leads to

$$P(s) = \left[\prod_{k=1}^{q} V_k(s) \right] P_q(s) \quad , \tag{A.72}$$

where $P_q(s)$ and

$$V_\pi(s) = \prod_{k=1}^{q} V_k(s) \tag{A.73}$$

are both proper matrices analytic in Re $s \geq 0$ and rank $P_q(j\omega)$ is n for all ω infinity included.

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