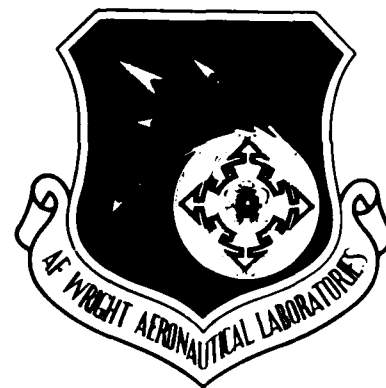


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**EXPERIMENTAL MODAL ANALYSIS AND
DYNAMIC COMPONENT SYNTHESIS**

VOL III - Modal Parameter Estimation

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<p>This report reviews and develop the theoretical basis for all of the modal parameter estimation algorithms in use today. Several new multiple reference methods are included. The methods reviewed include the Least Squares Complex Exponential, Polyreference Time Domain, Polyreference Frequency Domain, Multiple Reference Orthogonal Polynomial, Multiple Reference Ibrahim Time Domain, and the Multi MAC algorithms. In a comparison of these methods, the ability to estimate modal parameters is essentially the same. In general, the time domain methods are numerically more stable for large frequency bandwidths or for large numbers of modes. The frequency domain methods are superior when residual effects of modes above or below the frequency range of interest need to be included.</p>					
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SUMMARY

This report documents the area of modal parameter estimation in terms of reviewing efforts over the past twenty-five years, in developing several new multiple reference methods, and in attempting to provide a common basis and understanding for all of the modal parameter estimation methods developed to date. The review of modal parameter estimation includes a substantial literature review and the presentation of previous methods, such as the Least Squares Complex Exponential, as special cases of general methods, such as the Polyreference Time Domain method. Several new modal parameter estimation methods are developed and presented using consistent theory and nomenclature. The methods that are presented in this manner include: Polyreference Time Domain, Polyreference Frequency Domain, Multiple Reference Ibrahim Time Domain, Multiple Reference Orthogonal Polynomial, and Multi MAC. These methods, in terms of general characteristics, are also compared to other methods such as the Least Squares Complex Exponential, Ibrahim Time Domain, Eigensystem Realization Algorithm, and Direct Parameter Estimation methods. These methods are all similar in that the methods involve the decomposition of impulse response functions (time domain), frequency response functions (frequency domain), or forced response patterns (spatial domain) into characteristic functions in the appropriate domain. These characteristic functions are the single degree of freedom information in the respective domain.

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PREFACE

This volume is one of six Technical Reports that represent the final report on the work involved with United States Air Force Contract F33615-83-C-3218, Experimental Modal Analysis and Dynamic Component Synthesis. The reports that are part of the documented work include the following:

AFWAL-TR-87-3069

VOLUME I	Summary of Technical Work
VOLUME II	Measurement Techniques for Experimental Modal Analysis
VOLUME III	Modal Parameter Estimation
VOLUME IV	System Modeling Techniques
VOLUME V	Universal File Formats
VOLUME VI	Software User's Guide

For a complete understanding of the research conducted under this contract, all of the Technical Reports should be referenced.

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1. INTRODUCTION

Modal parameter estimation is the estimation of frequency, damping, and modal coefficients from the measured data which may be in: (1) relatively raw form in terms of force and response data in the time or frequency domain, or (2) in a processed form such as frequency response or impulse response functions. Most modal parameter estimation is based upon the measured data being the frequency response function; or the equivalent impulse response function, typically found by inverse Fourier transforming the frequency response function. Regardless of the form of the measured data, the modal parameter estimation techniques have traditionally been divided into two categories: (1) single degree-of-freedom (SDOF) approximations, and (2) multiple degree-of-freedom (MDOF) approximations. Since the single degree-of-freedom equations are simply special cases of the multiple degree-of-freedom equations, all theoretical discussion is made only in terms of the multiple degree-of-freedom case.

The current effort in the modal parameter estimation area is concerned with a unified theory that explains any previously conceived modal parameter estimation method as a subset of a general theory. This unified theory concept would eliminate the confusing nomenclature that currently exists and simplify the understanding of the strengths and weaknesses of each method. The modal parameter estimation methods that have been developed over the past several years involve multiple measurement, multiple reference concepts that can be viewed as an interaction between the temporal domains (time, frequency, etc.) and the spatial domains (physical coordinates, modal coordinates, etc.) in order to achieve the "best" estimate of the modal parameters.

1.1 Historical Overview

While engineers have tried to estimate the vibration characteristics of structures since the turn of the century, the actual history of experimental modal parameter estimation is normally linked to the work by Kennedy and Pancu^[1] in 1947. Until this time, the instrumentation that had been available was not sufficiently refined to allow for detailed study of experimental modes of vibration. As the instrumentation and analysis equipment has improved over the last forty years, major improvements in modal parameter estimation techniques have followed. Specifically, the development of accurate force and response transducers, the development of test equipment based upon digital computers and the development of the fast Fourier transform (FFT) have been the key advances that have initiated bursts of development in the area of modal parameter estimation.

During this time period, efforts in modal parameter estimation have involved two concepts. The first concept involved techniques oriented toward the forced normal mode approach to modal parameter estimation. This approach to the estimation of modal parameters involves exciting the system into a single mode of vibration by using a specific sinusoidal forcing vector. Since the success of this method is determined by the evaluation of the phase characteristics with respect to the characteristics occurring at resonance, this approach can be broadly classified as the phase resonance method. In order to refine the phase resonance method, particularly the force appropriation aspect of the method, efforts to estimate the modal information on a mode by mode basis using measured impedance, or frequency response, functions began. Much of the early work on this concept centered on using the phase information as a means of identifying the effects of separate modes of vibration in the measurement. For this reason, this concept has become known as the phase separation method. Most methods that are in use today can be classified as phase separation methods since no effort is made to excite only one mode of vibration at a time.

With respect to the phase resonance methods, the basic theory was first documented by Lewis and Wrisley^[2] in 1950. This theory was refined and presented in a more complete manner by F. de Veubeke^[3] in 1956. Significant advances in the approach to force appropriation were documented by Trail-Nash^[4] and Asher^[5] in 1958. Significant improvements and refinements in the phase

resonance methods have taken place in the last thirty years, particularly in the automation of the force appropriation and the use of digital computers, but the basic theoretical concept has not changed since 1950.

With respect to the phase separation methods, much effort has occurred over the last forty years and continues to the present day. Kennedy and Pancu^[1] in 1947 documented that the presence of two modes of vibration could be detected by observing the rate of change of the phase in the area of resonance. Since this method was developed based upon a plot of the real part of the impedance function versus the imaginary part of the impedance function, this method is referred to as a circle-fit method based upon these characteristics in the Argand plane. Broadbent^[6] applied this concept to flight flutter data in 1958. Since the data acquisition process was largely analog until 1970, most of the work until that time was oriented towards trying to fit a single degree of freedom model to portions of the analog data. The significant contributors during this time period began with Stahle^[7] in 1958, and continued with Bishop and Gladwell^[8] and Pendered and Bishop^[9-11] in 1963, and Mahalingham^[12] in 1967. Once data began to be collected and stored in a digital fashion, the phase separation methods migrated to multiple degrees of freedom approaches. The initial work involving multiple degree of freedom models was documented by Klosterman^[13] in 1971, Richardson and Potter^[14] in 1974 and Van Loon^[15] in 1974. While the work during this period involved the basic polynomial and partial fraction models that are the basis of modern experimental modal parameter estimation methods, the solution algorithms were basically unstable, iterative approaches to the solution for the unknowns. Also, these methods used only one measurement at a time in the estimation of the modal parameters. In 1978, Brown^[16] documented work on the Least Squares Complex Exponential method that was a two stage approach to the estimation of modal parameters using all of the available data. In the first stage, the frequency and damping values are estimated; in the second stage, the modal coefficients are estimated. Ibrahim^[17], also in 1977, documented the initial version of the Ibrahim Time Domain Method, which formulated the solution for the modal parameters into an eigenvalue-eigenvector solution approach. These last approaches represent conceptual approaches that have been extended today into similar methods involving multiple references. The significant advances in the multiple reference, or polyreference, methods used and being developed at the present time were first documented by Vold^[18] in 1982 with the Polyreference Time Domain method. Since that time, several other polyreference methods have been developed. Detailed documentation of the multiple reference methods is contained in later sections.

In summary, over the last forty years, many experimental modal parameter estimation methods have been developed that can be classified as either phase resonance or phase separation methods. Often, it seems that these methods are very different and unique. In reality, the methods all are derived from the same equation and are concerned with the decomposition of a composite function into its constituent parts. This decomposition may occur in the time domain in terms of damped complex exponentials, in the frequency domain in terms of single degree-of-freedom functions, or in the modal domain in terms of modal vectors. This decomposition may occur during the test, as in the phase resonance methods, or occur during analysis, as in the phase separation methods. The various modal parameter estimation methods are enumerated in the following list:

- Forced Normal Mode Method^[2-5,19,20]
- Quadrature Amplitude^[7,8,11]
- Kennedy-Pancu Circle Fit^[1,13,21-23]
- Single Degree-of-Freedom Polynomial^[14,21,22,24,25]
- Nonlinear Frequency Domain^[13,14,21,22]
- Complex Exponential^[26,27]

- Least-Squares Complex Exponential (LSCE) [16]
- Ibrahim Time Domain (ITD) [28-32]
- Eigensystem Realization Algorithm [33-35]
- Orthogonal Polynomial [36,37]
- Global Orthogonal Polynomial [38]
- Polyreference Time Domain [18,39,40]
- Polyreference Frequency Domain [40-43]
- Direct Parameter Identification: Time Domain [40]
- Autoregressive Moving Average (ARMA) [44-48]
- Direct Parameter Identification: Frequency Domain [40,49]

1.2 Multiple-Reference Terminology

1.2.1 Mathematical Models

The most general model that can be used is one in which the elements of the mass, damping, and stiffness matrices are estimated, based upon measured forces and responses. Thus, the model that is used is based upon a matrix differential equation transformed into the domain of interest.

Time domain:

$$[M] \{\ddot{x}\} + [C] \{\dot{x}\} + [K] \{x\} = \{f\} \quad (1)$$

Frequency domain:

$$-\omega^2 [M] \{X\} + j\omega [C] \{X\} + [K] \{X\} = \{F\} \quad (2)$$

Laplace domain:

$$s^2 [M] \{X\} + s [C] \{X\} + [K] \{X\} = \{F\} \quad (3)$$

If Eq. (1), (2), or (3) is used as the model for parameter estimation, the elements of the unknown matrices must first be estimated from the known force and response data measured in the time or frequency domain. Once the matrices have been estimated, the modal parameters can be found by the solution of the classic eigenvalue-eigenvector problem [38,42,49]. Due to truncation of the data in terms of frequency content, limited numbers of degrees-of-freedom, and measurement errors, the matrices found by Eq. (1), (2), or (3) are, in general, not directly comparable to matrices determined from a finite element approach. Instead, the matrices that are estimated simply yield valid input-output relationships and valid modal parameters. This is because there is an infinite number of sets of mass, damping, and stiffness matrices that yield the same modal parameters over a reduced frequency range limited to the dynamic range of the measurements. For this reason, Eqs. (1), (2), and (3) are often pre-multiplied by the inverse of the mass matrix so that the elements of the two matrices [D] and [E] are estimated:

Time domain:

$$[I] \{\ddot{x}\} + [D] \{\dot{x}\} + [E] \{x\} = \{f'\} \quad (4)$$

Frequency domain:

$$-\omega^2 [I] \{X\} + j\omega [D] \{X\} + [E] \{X\} = \{F'\} \quad (5)$$

Laplace domain:

$$s^2 [I] \{X\} + s [D] \{X\} + [E] \{X\} = \{F'\} \quad (6)$$

Existing modal parameter estimation methods used in commercial modal analysis systems most often employ a model based upon measured impulse response (time domain) or frequency response (frequency domain) functions. While the exact model used as the basis for modal parameter estimation varies, almost all models used in conjunction with frequency response function data can be described by a general model in the time domain, frequency domain, or Laplace domain. The general model in the time domain is a damped complex exponential model (often the impulse response function) while the general model in the frequency domain is the frequency response function. The general model in the Laplace domain is the transfer function. For general viscous damping, the mathematical models for each domain for a multiple degree degree-of-freedom mechanical system can be stated as:

Time Domain:

$$h_{pq}(t) = \sum_{r=1}^N A_{pqr} e^{\lambda_r t} + A_{pqr}^* e^{\lambda_r^* t} \quad (7)$$

Frequency Domain:

$$H_{pq}(\omega) = \sum_{r=1}^N \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A_{pqr}^*}{j\omega - \lambda_r^*} \quad (8)$$

Laplace Domain:

$$H_{pq}(s) = \sum_{r=1}^N \frac{A_{pqr}}{s - \lambda_r} + \frac{A_{pqr}^*}{s - \lambda_r^*} \quad (9)$$

where:

- s = Laplace variable
- s = $\sigma + j\omega$
- σ = angular damping variable (rad/sec)
- ω = angular frequency variable (rad/sec)
- p = measured degree-of-freedom (response)
- q = measured degree-of-freedom (input)
- r = modal vector number
- N = number of modal frequencies
- A_{pqr} = residue
- $A_{pqr} = Q_r \psi_{pr} \psi_{qr}$
- $A_{pqr} = \psi_{pr} L_{qr}$
- Q_r = complex modal scaling coefficient for mode r
- ψ_{pr} = modal coefficient for measured

L_{qr}	=	degree-of-freedom p and mode r
	=	modal participation factor for reference
	=	degree-of-freedom q and mode r
λ_r	=	system pole
λ_r	=	$\sigma_r + j\omega_r$

The models described in Eqs. (7) through (9) have many other equivalent forms based upon expansion of the terms under the summation. Also, the models take on slightly different forms under assumptions concerning specific physical damping mechanisms (hysteretic, etc.) [13,14,40]. Other forms of these models are also used where certain assumptions or mathematical relationships are utilized. For example, an equivalent model can be found when the common denominator of Eq. (8) is formed yielding a polynomial numerator and polynomial denominator of maximum order "2N" [13,14,22]. The denominator polynomial is then a function of the system poles. Often, an assumption is made concerning the modal vectors being normal (real) rather than complex. This reduces the number of unknowns that must be estimated by "N".

1.2.2 Sampled Data

The mathematical models described in the previous section are all developed based upon the concept that the data is continuous. In reality the data that is available must be thought of as sampled data in each domain. This restriction requires special consideration when applying the models developed in Eqs. (1) through (9). Differential equations must now be thought of as finite difference equations; continuous integral transforms are replaced by discrete transforms such as the Fast Fourier Transform (FFT) and the Z Transform. The concepts affecting the numerical processing of sampled data with respect to the continuous models represented in Eqs. (1) through (9) are exactly the same as the concepts that are the basis of the area of digital signal analysis with respect to the measurement of the data. The limitations of the frequency information creates special processing problems that are related to Shannon's Sampling Theorem; the limitations of the dynamic range of the measured data and of the computer precision yield special numerical problems with respect to the solution algorithm.

In general, the numerical considerations often determine which mathematical model will be most effective in the estimation of modal parameters. Time domain models tend to provide the best results when a large frequency range or large numbers of modes exist in the data. Frequency domain models tend to provide the best results when the frequency range of interest is limited and when the number of modes is small. While these are general considerations, the actual numerical implementation determines the ability of the algorithm to estimate modal parameters accurately and efficiently.

1.2.3 Consistent Data

Modal parameter estimation methods all assume that the system that is being investigated is linear and time invariant. While this is often nearly true, these assumptions are never exactly true. *Consistent data* refers to the situation where the data is acquired so as to best satisfy these two assumptions. Problems associated with linearity can be minimized by maintaining a prescribed force level and/or using excitation methods that give the best linear approximation to the nonlinear characteristic (random excitation). Problems associated with the time invariance constraint can be minimized by acquiring all of the data simultaneously using multiple excitations [50-54]. This reduces mass loading and boundary condition variations that can be caused by moving a transducer around the structure or by changing the location of the excitation.

1.2.4 Residuals

Continuous systems have an infinite number of degrees-of-freedom but, in general, only a finite number of modes can be used to describe the dynamic behavior of a system. The theoretical number of degrees-of-freedom can be reduced by using a finite frequency range (f_a, f_b) . Therefore, for example, the frequency response function can be broken up into three partial sums, each covering the modal contribution corresponding to modes located in the frequency ranges $(0, f_a)$, (f_a, f_b) , and (f_b, ∞) as shown in Figure 1.

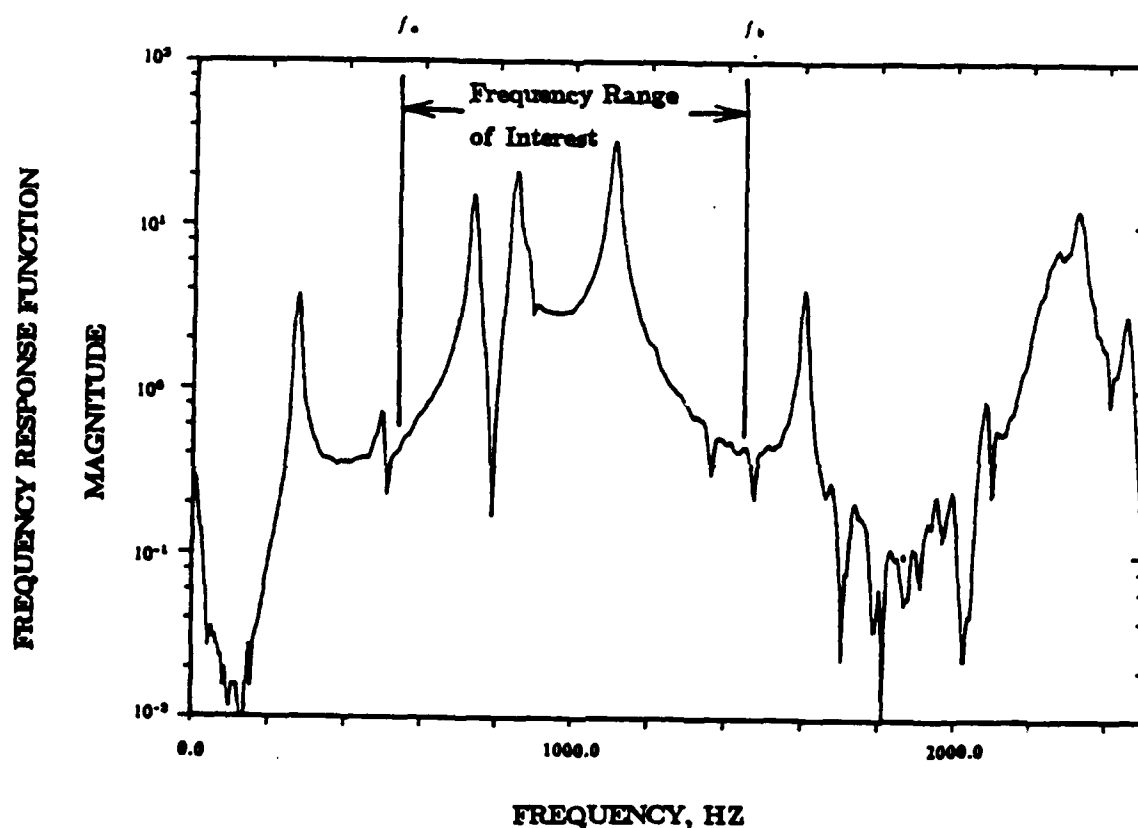


Figure 1. Frequency Range of Interest

In the frequency range of interest, the modal parameters can be estimated to be consistent with Eq. (8). In the lower and higher frequency ranges, residual terms can be included to handle modes in these ranges. In this case, the general frequency response function model can be stated:

$$H_{pq}(\omega) = R_{l_{pq}}(\omega) + \sum_{r=1}^N \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A_{pqr}^*}{j\omega - \lambda_r^*} + R_{u_{pq}} \quad (10)$$

where:

$R_{l_{pq}}(\omega)$ = residual effect of lower frequency modes
 $R_{u_{pq}}$ = residual effect of higher frequency modes (constant with ω)

In many cases the lower residual is called the *inertia restraint*, or *residual inertia*, and the upper residual is called the *residual flexibility* ^[13]. In this common formulation of residuals, both terms are real-valued quantities. The lower residual is a term reflecting the inertia or mass of the lower modes and is an inverse function of the frequency squared. The upper residual is a term reflecting the flexibility of the upper modes and is constant with frequency. Therefore, the form of the residual is based upon a physical concept of how the system poles below and above the frequency range of interest affects the data in the range of interest. As the system poles below and above the range of interest are located in the proximity of the boundaries of the frequency range of interest, these effects are not the simple real-valued quantities noted in Eq. (10). In these cases, residual modes may be included in the model to partially account for these effects. When this is done, the modal parameters that are associated with these residual poles have no physical significance, but may be required in order to compensate for strong dynamic influences from outside the frequency range of interest. Using the same argument, the lower and upper residuals can take on any mathematical form that is convenient as long as the lack of physical significance is understood. Power functions of frequency (zero, first, and second order) are commonly used within such a limitation. In general, the use of residuals is confined to frequency response function models. This is primarily due to the difficulty of formulating a reasonable mathematical model and solution procedure in the time domain for the general case that includes residuals.

1.2.5 Global Modal Parameters

Theoretically, modal parameters are considered to be unique based upon the assumption that the system is linear and time invariant. Therefore, the modal frequencies can be determined from any measurement and the modal vectors can be determined from any reference condition. If multiple measurements or reference conditions are utilized, the possibility of several, slightly different, answers for each modal parameter exists. The concept of *global modal parameters*, as it applies to modal parameter estimation, means that there is only one answer for each modal parameter and that the modal parameter estimation solution procedure enforces this constraint. Every frequency response or impulse response function measurement theoretically contains the information that is represented by the characteristic equation (modal frequencies and damping). If individual measurements are treated in the solution procedure independent of one another, there is no guarantee that a single set of modal frequencies and damping are generated. In a like manner, if more than one reference is measured in the data set, redundant estimates of the modal vectors can be estimated unless the solution procedure utilizes all references in the estimation process simultaneously. Most modal parameter estimation algorithms estimate the modal frequencies and damping in a global sense but few estimate the modal vectors in a global sense.

1.2.6 Modal Participation Factors

A *modal participation factor* is a complex-valued scale factor that is the ratio of the modal coefficient at one reference degree-of-freedom to the modal coefficient at another reference degree-of-freedom. A more general view of the modal participation factor is that it represents the relationship between the residue and the eigenvector coefficient as in the following equations:

$$A_{pp} = Q_r \psi_p \psi_p^* \quad (11)$$

$$L_q = Q_r \psi_q^* \quad (12)$$

$$A_{pp} = \psi_p L_q \quad (13)$$

where:

- p = measured degree-of-freedom (response)
- q = measured degree-of-freedom (reference)
- r = modal vector number
- A_{pp} = residue
- Q_r = complex modal scaling coefficient for mode r
- ψ_p = modal coefficient for measured degree-of-freedom p and mode r
- ψ_q = modal coefficient for reference degree-of-freedom q and mode r
- L_q = modal participation factor for reference degree-of-freedom q and mode r

From a mathematical standpoint, the modal participation matrix is equal to the left eigenvectors of next equation :

$$[H] = [\Psi] [\Lambda] [L]$$

where:

- $[H]$ = transfer function matrix
- $[\Psi]$ = complex modal vector matrix
- $[L]$ = diagonal matrix with poles

Note that for Eq. (12) the modal participation factor represents the product of a modal scaling coefficient and another term from the right eigenvector for reference degree-of-freedom q . This will always be true for reciprocal systems since the left and right eigenvectors for a given mode are equal. For non-reciprocal systems, the modal participation factor is the appropriate term from the left eigenvector. Note also that the modal participation factor, since it is related to the eigenvector, has no absolute value but is relative to the magnitudes of the other elements in the eigenvector.

Modal participation factors reflect the interaction of the spatial domain with the temporal domain (time or frequency). Modal participation factors can be computed anytime that multiple reference data are measured and such factors are used in multiple reference modal parameter estimation algorithms. Modal participation factors relate how well each modal vector is excited from each of the reference locations. This information is often used in a weighted least squares error solution procedure to estimate the modal vectors in the presence of multiple references. Theoretically, these modal participation factors should be in proportion to the modal coefficients of the reference degrees of freedom for each modal vector. Modal participation factors in a solution procedure enforce the constraint concerned with Maxwell's reciprocity between the reference degrees of freedom. Most multiple reference, modal parameter estimation methods estimate modal participation factors as part of the first stage estimation of global modal frequencies and damping.

1.2.7 Order of the Model

The *order* of the model equals the number of unknowns that must be estimated in the model. In the modal parameter estimation case, this refers to the frequency, damping, and complex modal coefficient for each mode of vibration at every measurement degree-of-freedom plus any residual terms that must be estimated. Therefore, the order of the model is directly dependent on the number of modal frequencies, "N", that are to be estimated. For example, for a system with "N" modes of vibration, assuming that no residuals were required, "4N" unknowns must be estimated. For cases involving measured data, the order of the model is extremely important. Estimates of modal parameters are affected by the order of the model. A problem arises from the inability to be certain that the correct order of the model has been chosen during the initial estimation phase. If the number of modes of vibration is more or less than "N", modes of vibration will be found that do not exist physically or modes of vibration will be missed that actually do occur. In addition, the values of frequency, damping, and complex modal coefficient for the actual modes of vibration will be affected.

The number of modes of vibration is normally chosen between one and an upper limit, dependent on the memory limitations of the computational hardware. The true number of system poles is a function of the frequency range of the measurements used to estimate the modal parameters. By observing the number of peaks in the frequency response function, the minimum number of system poles can be estimated. This estimate is normally low, based upon poles occurring at nearly the same frequency (psuedo-repeated roots), limits on dynamic range, and poorly excited modes. For these reasons, the estimate of the correct order of the model is often in error. When the order of the model is other than optimum, the estimate of the modal parameters will be in error.

Many of the parameter estimation techniques that are used assume that only one mode exists in a limited range of interest and all of the other modes appear as residual terms. For this case Eq. (10) can be rewritten as:

$$H_{pq}(\omega) = R_{I_{pq}}(\omega) + \frac{A_{pq}}{j\omega - \lambda_1} + \frac{A_{pq}^*}{j\omega - \lambda_1^*} + R_{F_{pq}} \quad (14)$$

1.2.8 Solution Procedure

Equations (7) through (9) represent a nonlinear model in terms of the unknown modal parameters. This can be noted from the unknowns in the numerator and denominator of Eq. (8) and the unknowns as the argument of the transcendental functions of Eq. (7). The nonlinear aspect of the model must be treated in one of two ways: (1) by the use of an iterative solution procedure to solve the nonlinear estimation problem, allowing all modal parameters to vary according to a constraint relationship until an error criterion reaches an acceptably low value, or (2) by separating the nonlinear estimation problem into two linear estimation problems. For the case of structural dynamics, the common technique is to estimate "2N" frequencies and damping values in a first stage and then to estimate the "4N" modal coefficients plus any residuals in a second stage.

In the iterative technique in the solution of the nonlinear estimation approach, a set of starting values must be chosen to initiate the sequence. The number and value of these starting values affect the final result. Poor initial estimates can lead to problems of convergence, as a result of which, close operator supervision usually is required for a successful use of this technique.

An alternative method is to reformulate the nonlinear problem into a number of linear stages so that each stage is stable. The actual data that are used in the estimate of the modal parameters also affect the results. Based on the choice of the order of the model, "N", there are "4N" modal parameters to be estimated. If residuals, in one form or another, are also included, the number of modal parameters to be estimated will be slightly higher. The common method of solving for these

unknown modal parameters is to find an equation involving known information for every unknown that will be found. In this case, the frequency response function or impulse response function that has been measured provides the known information and Eq. (7) or Eq. (8) can be repeated for different frequencies or time values in order to obtain the sufficient number of equations. These equations, for the linear case, can then be solved simultaneously for the unknown modal parameters. As an illustration of this relationship, consider a common modal parameter situation in which the number of modes in the frequency range of interest is between 1 and 30. Assuming the highest ordered model means that slightly more than 120 modal parameters must be estimated. From a single frequency response measurement, 1024 known values of the function will be available (512 complex values at successive values of frequency). Many more equations, based on the known values of frequency response, can be formed than are needed to find the unknown modal parameters. An obvious solution is to choose enough equations to solve for the modal parameters. The problem arises in determining what part of the known information is to be involved in the solution. As different portions of the known data (data near a resonance compared to an anti-resonance, for example) are used in the solution, the estimates of modal parameters vary. As the quality of the data becomes marginal, this variance can be quite large. When the modal parameters that are estimated appear to be non-physical, this is often the reason.

To solve this problem, all or a large portion of the data can be used if a pseudo-inverse type of solution procedure is used. One procedure that is used is to formulate the problem so as to minimize the squared error between the data and the estimated model. This least-squares error method to the solution is the most commonly used technique in the area of modal parameter estimation. If there are many more known pieces of information than unknowns that must be estimated, many more equations can be formed than are needed to solve for the unknowns. The least-squares error method to the solution allows for all of these redundant data to be used to estimate the modal parameters in a computationally efficient manner. The least-squares error method usually can be derived directly from the linear equations using a normal equations approach. In general, this procedure does not increase the memory or computational requirements of the computational hardware significantly. Any solution procedure that can be used is only estimating a "best" solution based upon the choice of the model, the order of the model, and the known, measured data used in the model.

1.2.9 Characteristic Polynomial

The impetus of this section is to show that for discrete data, a difference characteristic equation can be formulated in order to solve for the poles of the system. Further, it will be shown that the difference equation can be formulated directly from the impulse response function data. By solving for the polynomial coefficients and the roots of the polynomial equation, the modal parameters, frequency and damping, are determined. The characteristic polynomial will be formulated for the continuous case, as a differential function, and then extended to the discrete case, as a difference function.

1.2.9.1 Differential Theory

The homogeneous differential equation for a single degree of freedom system is:

$$m \ddot{x}(t) + c \dot{x}(t) + k x(t) = 0 \quad (15)$$

In order to solve the differential equation assume a solution of the form $x(t) = X e^{st}$, where X is a scalar value. Substituting the appropriate derivatives of the assumed solution into Eq.(15):

$$(m s^2 + c s + k) X e^{st} = 0$$

Thus, the differential equation is transformed into an algebraic polynomial equation, called the

characteristic equation.

$$m s^2 + c s + k = 0 \quad (16)$$

The complex valued roots of the characteristic equation will yield the characteristic solutions, λ_1 and λ_2 . The real part is the damping and the imaginary part is the eigenfrequency, or damped natural frequency. Thus, the solution to the governing differential equation is:

$$x(t) = \sum_{r=1}^2 X_r e^{\lambda_r t}.$$

The scalar magnitudes, X_1 and X_2 , are determined from the system initial conditions. Note that *any* exponential function will satisfy the differential equation. One such function of particular interest, is the impulse response function.

$$h(t) = \sum_{r=1}^2 A_r e^{\lambda_r t}.$$

A system with N degrees-of-freedom can be described by a set of N , coupled, second order differential equations. The characteristic equation for this system is represented by the following polynomial:

$$a_{2N} s^{2N} + a_{2N-1} s^{2N-1} + a_{2N-2} s^{2N-2} + \dots + a_1 s + a_0 = 0 \quad (17)$$

Solving this polynomial equation will yield $2N$ complex valued roots, or, characteristic solutions λ_r . Then the solutions to the differential equations will be complex exponentials of the form:

$$x_{pq}(t) = \sum_{r=1}^{2N} X_r e^{\lambda_r t}$$

where:

- p = response location degree-of-freedom
- q = reference location degree-of-freedom.

Thus, impulse response functions,

$$h_{pq}(t) = \sum_{r=1}^{2N} A_{pq,r} e^{\lambda_r t}$$

will also satisfy the differential equations. Consider a few impulse response functions for various reference and response points.

$$h_{11}(t) = \sum_{r=1}^{2N} A_{11,r} e^{\lambda_r t}$$

$$h_{12}(t) = \sum_{r=1}^{2N} A_{12,r} e^{\lambda_r t}$$

$$h_{13}(t) = \sum_{r=1}^{2N} A_{13,r} e^{\lambda_r t}$$

$$h_{pq}(t) = \sum_{r=1}^{2N} A_{pq,r} e^{\lambda_r t}$$

The common characteristic in each of the above equations is that every impulse response function is a linear superposition of identical damped complex exponentials, $e^{\lambda_r t}$ for $r = 1 \rightarrow 2N$. That is, the roots of the characteristic polynomial are common to all reference and response locations. Thus, the characteristic solutions are global system parameters, since they are independent of reference or, response location. The important result is that since *each* $e^{\lambda_r t}$ is a characteristic solution to the homogeneous linear differential equation,

$$a_{2N} D^{2N} (A_{pr} e^{\lambda_r t}) + a_{2N-1} D^{2N-1} (A_{pr} e^{\lambda_r t}) + a_{2N-2} D^{2N-2} (A_{pr} e^{\lambda_r t}) + \dots + a_1 D (A_{pr} e^{\lambda_r t}) + a_0 = 0$$

where,

$$\bullet D^n [f(t)] = \frac{d^n [f(t)]}{dt^n} \quad (\text{differential operator})$$

that a linear superposition of characteristic solutions will also be a solution. That is, $h_{pr}(t)$ will also satisfy the differential equation. Actually, a set of N second order linear differential equations must be satisfied, but, a differential equation of order $2N$ can be found that will have the same roots as the set of N second order equations.

Substituting a few impulse response functions for various reference and response points, a number of differential equations are obtained.

$$a_{2N} D^{2N} (h_{11}(t)) + a_{2N-1} D^{2N-1} (h_{11}(t)) + a_{2N-2} D^{2N-2} (h_{11}(t)) + \dots + a_1 D (h_{11}(t)) + a_0 = 0$$

$$a_{2N} D^{2N} (h_{12}(t)) + a_{2N-1} D^{2N-1} (h_{12}(t)) + a_{2N-2} D^{2N-2} (h_{12}(t)) + \dots + a_1 D (h_{12}(t)) + a_0 = 0$$

$$a_{2N} D^{2N} (h_{13}(t)) + a_{2N-1} D^{2N-1} (h_{13}(t)) + a_{2N-2} D^{2N-2} (h_{13}(t)) + \dots + a_1 D (h_{13}(t)) + a_0 = 0$$

$$a_{2N} D^{2N} (h_{pr}(t)) + a_{2N-1} D^{2N-1} (h_{pr}(t)) + a_{2N-2} D^{2N-2} (h_{pr}(t)) + \dots + a_1 D (h_{pr}(t)) + a_0 = 0$$

Note that the coefficients a_0 to a_{2N} do not vary with reference or response location and thus, can be estimated from a combination of various number of reference and response points.

1.2.9.2 Difference Theory

From an experimental standpoint, the data are sampled, which means instead of continuous knowledge of the system, the values obtained are for distinct discrete time points. The impulse response functions are generally obtained by inverse Fourier transforming the frequency response functions. Thus, from the discrete impulse response functions the pole information, frequency and damping, is determined. The model for the discrete impulse response function is:

$$h_{pr}(t_k) = \sum_{r=1}^{2N} A_{pr} e^{\lambda_r t_k} = \sum_{r=1}^{2N} A_{pr} z_r^k \quad (18)$$

where,

- $t_k = k \Delta t$
- Δt is the sample interval
- $k = 1 \rightarrow \text{blocksize}$
- $z_r = e^{\lambda_r \Delta t}$

It should be noted, for discrete data, that the sample interval, Δt , limits the frequency for which valid information can be determined, whereas, in the analysis of continuous data, there are no frequency constraints. In other words, theoretically, characteristics can be determined to infinite frequency for continuous functions, but, the process of digitally sampling continuous data causes a maximum frequency for which characteristics can be determined. The frequencies above this maximum will alias back into the sampled bandwidth, and thus bias the results. For this reason, low-pass filters are used to exclude information above the maximum frequency.

Recall the characteristic equation for an n degree-of-freedom system:

$$a_0 s^0 + a_1 s^1 + a_2 s^2 + \dots + a_{2N-1} s^{2N-1} + a_{2N} s^{2N} = 0 \quad (19)$$

will have $2N$ characteristic solutions, λ_r , for $r = 1 \rightarrow 2N$. The characteristic polynomial is not unique in that, many polynomials can be constructed that will yield the same characteristic solutions, even though the coefficients will be different. For this reason, another polynomial can be formulated that will have characteristic solutions that are related to the characteristic solutions of Eq.(19). The polynomial has the form:

$$a'_0 z^0 + a'_1 z^1 + a'_2 z^2 + \dots + a'_{2N-1} z^{2N-1} + a'_{2N} z^{2N} = 0 \quad (20)$$

The relationship between z and s is $z = e^{s\Delta t}$. Analogous to Eq.(19), there are also $2N$ characteristic solutions of Eq.(20), z_r for $r = 1 \rightarrow 2N$. The roots of the two equations are related by $z_r = e^{\lambda_r \Delta t}$ where, z_r are precisely the values of z for which the characteristic equation, Eq.(20), is zero. Note that z_r is simply the sampled form of the continuous exponential solution in the differential case. Thus, by knowing the system characteristics, z_r , the desired parameters, λ_r , can be determined. If the coefficients are known, Eq.(20) could be solved, but, from an experimental aspect, both the coefficients and the system characteristics are unknown. Thus, in order to determine the system characteristics, the a' coefficients must be determined first. This is accomplished by substituting a characteristic solution of the system, z_r , into Eq.(20).

$$a'_0 A_{prr} z_r^0 + a'_1 A_{prr} z_r^1 + a'_2 A_{prr} z_r^2 + \dots + a'_{2N-1} A_{prr} z_r^{2N-1} + a'_{2N} A_{prr} z_r^{2N} = 0$$

Substituting $z_r = e^{\lambda_r \Delta t}$ into the above equation,

$$a'_0 A_{prr} (e^{\lambda_r \Delta t})^0 + a'_1 A_{prr} (e^{\lambda_r \Delta t})^1 + a'_2 A_{prr} (e^{\lambda_r \Delta t})^2 + \dots + a'_{2N} A_{prr} (e^{\lambda_r \Delta t})^{2N} = 0$$

or,

$$a'_0 A_{prr} e^0 + a'_1 A_{prr} e^{\lambda_r \Delta t} + a'_2 A_{prr} e^{\lambda_r 2 \Delta t} + \dots + a'_{2N} A_{prr} e^{\lambda_r 2N \Delta t} = 0 \quad (21)$$

The important result is that since *each* $e^{\lambda_r \Delta t}$ is a characteristic solution to the homogeneous linear difference equation, that a linear superposition of characteristic solutions will also be a solution of Eq.(21), which means that, in general, Eq.(18) can be substituted into Eq.(21). Once again, a set of N second order linear difference equations must be satisfied, but, a difference equation of order $2N$ can be found that will have the same roots as the set of N second order equations.

Consider a number of equations for various reference and response locations:

$$a'_0 h_{11}(t_0) + a'_1 h_{11}(t_1) + a'_2 h_{11}(t_2) + \dots + a'_{2N-1} h_{11}(t_{2N-1}) + a'_{2N} h_{11}(t_{2N}) = 0$$

$$a'_0 h_{12}(t_0) + a'_1 h_{12}(t_1) + a'_2 h_{12}(t_2) + \dots + a'_{2N-1} h_{12}(t_{2N-1}) + a'_{2N} h_{12}(t_{2N}) = 0$$

$$a'_0 h_{13}(t_0) + a'_1 h_{13}(t_1) + a'_2 h_{13}(t_2) + \dots + a'_{2N-1} h_{13}(t_{2N-1}) + a'_{2N} h_{13}(t_{2N}) = 0$$

$$a'_0 h_{pq}(t_0) + a'_1 h_{pq}(t_1) + a'_2 h_{pq}(t_2) + \dots + a'_{2N-1} h_{pq}(t_{2N-1}) + a'_{2N} h_{pq}(t_{2N}) = 0$$

Note that the coefficients a'_0 to a'_{2N} do not vary with reference or response location and thus, can be estimated from a combination of various number of reference and response points. Once the a' coefficients are estimated from a set of equations similar to the ones above, the poles, z_r , and hence λ_r , can be estimated from the $2N$ solutions of the characteristic equation,

$$a'_0 z^0 + a'_1 z^1 + a'_2 z^2 + \dots + a'_{2N-1} z^{2N-1} + a'_{2N} z^{2N} = 0$$

where:

$$\bullet z_r = e^{\lambda_r \Delta t}$$

In summary, a series of $2N$ linear difference equations with constant coefficients are formed from the sampled impulse response function data in order to solve for the common constant coefficients. These coefficients are then used in the characteristic equation to solve for the system characteristics, z_r , which contain the desired parameters, λ_r .

Note that the characteristic polynomial for the continuous, or discrete case, is of order $2N$, that is, twice the number of modes. This results in a time domain differential, or difference equation of order $2N$. For this reason, from a numerical analysis concept, for large numbers of modes, N , or large differences in modal frequency (λ_1 compared to λ_N), time domain methods are numerically better conditioned.

2. CHARACTERISTIC SPACE CONCEPTS

A new way of conceptualizing the area of parameter identification was developed during the course of performing the work defined under this contract. One of the objectives of this contract was to summarize existing modal parameter estimation methods and develop new ones. In the process of performing this task it became obvious that most of the current algorithms could be described conceptually in terms of a three-dimensional complex space of the system's characteristics. Modal parameter estimation is the process of deconvolving measurements defined by this space into the system's characteristics.

The frequency and/or unit impulse response function matrix which describes a system, can be expressed in terms of the convolution of three fundamental characteristic functions; two complex spatial, and one complex temporal. The spatial characteristics are a function of geometry and the temporal corresponds to either time or frequency. Mathematically the frequency response matrix and the impulse response matrix can be expressed as follows:

$$[H(\omega_k)] = [\Psi][\Lambda_k][L] \quad [h(t_k)] = [\Psi][e^{\lambda_k t_k}][L] \quad (22)$$

where:

- $[H(\omega_k)]_{N_o \times N_i}$ = frequency response matrix (element $H_{pq}(\omega_k)$)
- $[h(t_k)]_{N_o \times N_i}$ = unit impulse response matrix (element $h_{pq}(t_k)$)
- $[\Psi]_{N_o \times 2N}$ = modal vector matrix (function of spatial variable p , element ψ_{pr})
- $[L]_{2N \times N_i}$ = modal participation factor (function of spatial variable q , element L_{rq})
- $[\Lambda_k]_{2N \times 2N}$ = diagonal matrix of characteristic roots (element $\frac{1}{j\omega_k - \lambda_r}$)
- $[e^{\lambda_k t_k}]_{2N \times 2N}$ = diagonal matrix of characteristic roots (element $e^{\lambda_r t_k}$)
- ω_k = frequency temporal variable ($k = 1 \rightarrow \text{blocksize} / 2$, may be unequally spaced)
- t_k = time temporal variable ($t_k = k \Delta t$)
- p = response degree-of-freedom spatial variable
- q = reference degree-of-freedom spatial variable
- r = temporal degree-of-freedom variable
- N = number of modes (system poles, indexed by r)
- N_o = number of responses (indexed by p)
- N_i = number of references (indexed by q)

The frequency response function matrix consists of three-dimensional complex space, which for a real system is a continuous function of the three characteristic variables (p, q, ω). However, in terms of measurements the functions consist of sampled data where, p, q and ω_k are sampled characteristic variables. In other words, the frequency response function is measured at discrete input, or reference points (q), output response points (p), and discrete frequency (ω_k), or time points (t_k).

A summary of the characteristic vectors are:

- The response characteristic functions consist of a set of vectors which are proportional to the eigenvectors of the system. The eigenvectors are indexed by r and the elements of the vectors are indexed by p .

- The reference characteristic functions consist of a set of vectors which are proportional to the modal participation factors, which are in turn proportional to the system eigenvectors at the reference degrees-of-freedom. The modal participation vectors are indexed by r and the elements of the vectors are indexed by q .
- The temporal characteristic functions consist of vectors which are equivalent to sampled single degree-of-freedom frequency response functions, or unit impulse response functions. The index on the vector is r and the index on the sampled element of each vector is ω_k , or t_k .

The variable r is the index on the characteristic. For a given r there is a discrete characteristic space. The summation, or superposition with respect to r defines; the measured, or sampled frequency response, or impulse response matrix, or, in other words, the three-dimensional complex space.

This concept is difficult to visualize, since the matrix is represented by three-dimensional complex characteristic space. The easiest method is to describe the variation along lines parallel to axes of the space. Lines parallel to the temporal axis correspond to individual frequency response functions, or unit impulse response functions. These frequency response functions consist of a summation of the temporal characteristics, weighted by the two spatial characteristics, which define the other two axis of the characteristic space.

Lines parallel to the response axis correspond to forced modes of vibration. These forced modes consist of a summation of the system eigenvectors weighted by the input characteristic and the temporal characteristic.

Likewise, lines parallel to the input, or reference axis consist of a summation of the system eigenvectors weighted by the response characteristic and the temporal characteristic. The variation along these lines are referred to in the literature as the modal participation factors.

Modal parameter estimation is the process of deconvolving this sampled space into the basic characteristic functions which describe the space. In practice, there are many more measured, or sampled points in the space than there are elements in the three characteristic vectors, therefore, the parameter estimation process is over determined. As a result, one of the important steps in the process has been the reduction of the data to match the number of unknowns in the parameter identification process. This data reduction has historically been done by using a pseudo inverse, or a principal component method, with least squares being the most common pseudo inverse method.

The early single degree-of-freedom (SDOF) and multiple degree-of-freedom (MDOF) modal parameter estimation methods used subsets of the sampled data and extracted one of the characteristic functions at a time, normally the temporal characteristic. For example, the very early methods like the complex exponential were used to fit individual frequency response, or unit impulse response functions for the temporal characteristics (eigenvalue) and the residues (convolution of the response and input characteristics). For these cases, each frequency response measurement gave a different estimation of the system eigenvalues, or temporal characteristics. Since the measurements were taken one function at a time some of this variation was due to inconsistencies in the data base and the rest of the variation due to noise and distortion errors.

Later methods started to use either, least square, or principal component methods to condense the data over a number of sampled frequency response functions, into small subsets parallel to the temporal axis (for example the Least Square Complex Exponential and/or the Polyreference Time Domain methods). These methods then give global estimates of the eigenvalues, or temporal characteristic functions. The Least Squares Complex Exponential parameter estimation algorithm reduced the information to a single function parallel to the temporal axis and as a result, only estimated the temporal characteristic in a global sense. The Polyreference Time Domain algorithm estimates several functions parallel to the temporal axis at the input, or reference points. As a result

this method also gives global estimates of the input characteristic functions, or modal participation factors.

The more recent methods use larger subsets of the sampled data and utilize simultaneous data from all three axis resulting in global estimates of all three characteristics. In order to use these global methods, it is important that a consistent data base be measured.

3. THEORY OF MULTIPLE REFERENCE PARAMETER ESTIMATION TECHNIQUES

3.1 Polyreference Time Domain

3.1.1 Introduction

In the last decade modal analysis has become important in the design process. There are two reasons for this. First, there have been significant developments in computer hardware. More powerful computers have made the heavy computational load involved in modal analysis achievable within reasonable time periods. Furthermore, developments in electronics have resulted in larger, cheaper and more reliable data acquisition systems. In addition to developments in the hardware sector, there have been significant developments in software. New parameter estimation algorithms have been developed to provide more accurate results.

The subject of this section is one of the latest of the parameter estimation algorithms, namely the Polyreference Time Domain Technique^[18], as developed by H. Vold at S.D.R.C..

The Polyreference Time Domain Technique is a two stage parameter estimation technique. In the first stage, the damped natural frequency and damping values are extracted from time data using single or multiple references. The modal coefficients or residues are then calculated in the second stage. Although the first stage utilizes a time domain technique, calculation of the residues can be done in the time domain as well as in the frequency domain. When only data acquired from a single reference is used to estimate frequency, damping values and the modal coefficients, the Polyreference Time Domain Technique is equal to the Least Square Complex Exponential^[16]. Therefore, one can use the Polyreference Technique with a single reference or with multi-references.

3.1.2 General Equations

The Polyreference Time Domain Technique is used to extract modal parameters, namely the modal damping, frequency and residues, from free decay or impulse response function data. Instead of utilizing a frequency response function, like many frequency domain algorithms, the Polyreference Time Domain Technique is formulated from the impulse response function, $h(t)$. This $h(t)$ represents the response of the system due to a Dirac impulse. The impulse response function may be obtained by taking the inverse Fourier transform of the frequency response function.

$$h(t) = F^{-1}(H(\omega))$$

Where :

- $h(t)$ = impulse response function
- F^{-1} = inverse Fourier transform
- $H(\omega)$ = frequency response function

The impulse response function represents a decaying time history of the system and can be expressed as a function of the damped exponentials of the system.

$$h_{pr}(t) = \sum_{r=1}^N A_{pr} e^{\lambda_r t} + A_{pr}^* e^{\lambda_r^* t} \quad (23)$$

$$h_{pq}(t) = \sum_{r=1}^{2N} A_{pqr} e^{\lambda_r t}$$

Where :

- p = response point
- q = input point or reference point
- r = mode number
- A_{pqr} = residue for mode r at point p due to an input at point q
- A_{pqr}^* = complex conjugate of A_{pqr}
- λ_r = system pole
- $= \sigma_r + j\omega_r$
- σ_r = system damping for mode r (rad/sec)
- ω_r = damped natural frequency for mode r (rad/sec)
- λ_r^* = complex conjugate of λ_r
- N = number of modes in the frequency bandwidth of interest
- for $r=1,2,3,\dots,N$ $\left\{ \begin{array}{l} A_{pqr+N} = A_{pqr}^* \\ \lambda_{r+N} = \lambda_r^* \end{array} \right.$

Since sampled data is used, $h(t)$ is not a continuous function, but a discrete function.

$$h_{pq}(t_k) = \sum_{r=1}^{2N} A_{pqr} e^{\lambda_r t_k} \quad (24)$$

Where :

- $t_k = k \Delta t$
- Δt = sample time or time resolution
- k = integer value
- $e^{\lambda_r t_k} = e^{\lambda_r k \Delta t} = z_r^k$
- $z_r = e^{\lambda_r \Delta t}$

By substitution, Eq. (24) becomes :

$$\begin{aligned} h_{pq}(t_k) &= \sum_{r=1}^N A_{pqr} z_r^k + A_{pqr}^* z_r^{*k} \\ h_{pq}(t_k) &= \sum_{r=1}^{2N} A_{pqr} z_r^k \end{aligned} \quad (25)$$

Because there typically can be more than one reference, each input will generate an equation like Eq. (24). Therefore, for every response point, a set of N_i equations, of the form shown in Eq. (26), can be written.

$$\begin{cases} h_{p1}(t_k) = \sum_{r=1}^{2N} A_{p1r} e^{\lambda_r t_k} \\ h_{p2}(t_k) = \sum_{r=1}^{2N} A_{p2r} e^{\lambda_r t_k} \\ \vdots \\ h_{pN_i-1}(t_k) = \sum_{r=1}^{2N} A_{pN_i-1r} e^{\lambda_r t_k} \\ h_{pN_i}(t_k) = \sum_{r=1}^{2N} A_{pN_i r} e^{\lambda_r t_k} \end{cases} \quad (26)$$

Where : N_i = number of references or inputs

In basic modal analysis theory [58], a relationship has been established between the residues and the eigenvectors :

$$A_{pqr} = Q_r \psi_{pr} \psi_{qr} \quad r=1,2,\dots,2N$$

Where :

- Q_r = scaling factor for the r-th mode
- ψ_{pr} = scaled p-th response of a complex modal vector for mode r
- ψ_{qr} = scaled q-th response of a complex modal vector for mode r

This relationship enables us to express all residues as a function of the residues of one particular reference.

$$A_{pir} = \frac{\psi_{ir}}{\psi_{qr}} A_{pqr} \quad (27)$$

Writing :

$$\frac{\psi_{ir}}{\psi_{qr}} = L_{iqr} \quad (28)$$

where L_{iqr} is defined as the modal participation factor, sometimes referred to as the weighting or forcing factor, i is a reference location, q the reference or input location used as reference, and r the mode number.

Substituting Eqs. (27) and (28) into Eq. (26) results into Eq. (29).

$$\begin{cases} h_{p1}(t_k) = \sum_{r=1}^{2N} A_{p1r} e^{\lambda_r t_k} \\ h_{p2}(t_k) = \sum_{r=1}^{2N} L_{21r} A_{p1r} e^{\lambda_r t_k} \\ \vdots \\ h_{pN_i-1}(t_k) = \sum_{r=1}^{2N} L_{N_i-1,1r} A_{p1r} e^{\lambda_r t_k} \\ h_{pN_i}(t_k) = \sum_{r=1}^{2N} L_{N_i,1r} A_{p1r} e^{\lambda_r t_k} \end{cases} \quad (29)$$

These equations have been written as a function of the first reference. This is arbitrary and done for convenience. The following application of theory is independent of the reference used for the modal participation factors.

Using the concept developed in Eq. (25), Eq. (29) can be written in matrix notation :

$$\begin{Bmatrix} h_{p1}(t_k) \\ h_{p2}(t_k) \\ \vdots \\ h_{pN_i}(t_k) \end{Bmatrix} = \begin{bmatrix} \{I\} \\ [L_{i1}] \end{bmatrix} [Z]^k \{A_p\} \quad (30)$$

Where :

$$\bullet [Z]_{2N \times 2N} = \begin{bmatrix} z_1 & 0 & 0 & \dots & 0 \\ 0 & z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z_{2N} \end{bmatrix}$$

$$\bullet z_i = e^{\lambda_i \Delta t}$$

$$\bullet \{A_p\}_{2N \times 1} = \text{residue vector} = \begin{Bmatrix} A_{p1,1} \\ A_{p1,2} \\ \vdots \\ A_{p1,2N} \end{Bmatrix}_{2N \times 1}$$

$$\bullet [L_{i1}]_{(N_i-1) \times 2N} = \begin{bmatrix} L_{2,1,1} & L_{2,1,2} & \dots & L_{2,1,2N} \\ L_{3,1,1} & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ L_{N_i,1,1} & L_{N_i,1,2} & \dots & L_{N_i,1,2N} \end{bmatrix}$$

$$\bullet \{I\}_{1 \times 2N} = \text{a row vector for which all entries are 1}$$

$$\bullet [L]_{N_i \times 2N} = \begin{bmatrix} \{I\} \\ [L_{i1}] \end{bmatrix}_{N_i \times 2N} = \text{modal participation matrix}$$

$$\text{So } \begin{Bmatrix} h_{p1}(t_k) \\ h_{p2}(t_k) \\ \vdots \\ h_{pN_i}(t_k) \end{Bmatrix}_{N_i \times 1} = [L] [Z]^k \{A_p\}$$

To simplify the notation, the following convention will be used :

$$\left\{ h_{pk} \right\} = \begin{Bmatrix} h_{p1}(t_k) \\ h_{p2}(t_k) \\ \vdots \\ h_{pN_i}(t_k) \end{Bmatrix}_{N_i \times 1}$$

Equation (30) can now be written in a more concise form.

$$\left\{ h_{pk} \right\}_{N_i \times 1} = \left[L \right]_{N_i \times 2N} \left[Z \right]_{2N \times 2N}^k \left\{ A_p \right\}_{2N \times 1} \quad (31)$$

Note that only the left hand side of Eq. (31) is known, based upon measured data.

3.1.3 First Stage Solution: Poles and the Modal Participation Matrix

In this section a solution will be developed for the poles and the modal participation matrix. It will be shown that the poles are the eigenvalues of a matrix polynomial and that the columns of the modal participation matrix are the eigenvectors of this matrix polynomial. The matrix polynomial is the characteristic equation in the case of multiple references.

A matrix polynomial is a polynomial which has matrices as coefficients instead of scalars [59]. The matrix coefficients of the polynomial can be derived in different ways. In this section, the Prony-method is used to obtain these matrix coefficients. In Section 3.1.5 the matrix coefficients are derived from a relationship between the Laplace transform and the Z-transform.

Using a matrix polynomial instead of a scalar polynomial enables the Polyreference Time Domain Technique not only to process data from multiple reference locations simultaneously but also makes use of the fact that multiple reference data is available. The advantage of this is that the Polyreference Time Domain Technique is able to detect repeated roots equal to the number of references, or, to the dimension of the matrix coefficients. A proof of this is given in Section 3.1.6.

It should be mentioned that data from all references can be used for the pole calculation. However, certain references can be selected to emphasize certain properties of the system. In the previous section, N_i was defined as the number of references. In this section this definition is still valid, however N_i does not have to be equal to the number of references used during the test. Instead, N_i is equal to the number of selected references chosen to be included in the calculations.

The primary concern in any parameter estimation technique is the order of the model. For the Polyreference Time Domain Technique, this problem can be translated as the choice of $2N$ unknown poles and $2N$ unknown mode shapes, where $2N$ is the assumed number of degrees of freedom of the system. In the first stage of the calculations, a solution is sought only for $2N$ unknown poles.

As mentioned before, the unknown poles are found to be the eigenvalues of the matrix polynomial. The number of eigenvalues in a matrix polynomial is equal to the order of the polynomial times the dimension of the matrix coefficients. As defined before, the dimension of the matrix coefficient is equal to the number of references, while the number of unknown poles is $2N$. Therefore, the order of the matrix polynomial, n , has to be :

$$n \geq \frac{2N}{N_i} \quad (32)$$

Where :

- n = order of the matrix polynomial
- $2N$ = number of degrees of freedom
- N_i = number of references

Due to the discrete nature of the sampled data, it should be noted that n must be defined as the smallest integer that will satisfy Eq. (32). In the case $n > \frac{2N}{N_i}$ there will be $n \times N_i$ eigenvalues found. Since $n \times N_i > 2N$, there will be some computational poles.

A matrix polynomial that yields the poles and the modal participation matrix is sought. This polynomial will be shown to be of the form:

$$[a(0)] [L] [Z]^n + [a(1)] [L] [Z]^{n-1} + \dots + [a(n)] [L] [Z]^0 = [0] \quad (33)$$

Where : $[a(i)]_{N_i \times N_i}$ = i -th matrix coefficient

In complete matrix notation, Eq. (33) can be written as:

$$[Q]_{N_i \times (n+1)N_i} [G]_{(n+1)N_i \times 2N} = [0]_{N_i \times 2N} \quad (34)$$

Where :

$$[Q] = \begin{bmatrix} [a(n)] & [a(n-1)] & \dots & [a(0)] \end{bmatrix}_{N_i \times (n+1)N_i}$$

$$[G] = \begin{bmatrix} [L] [Z]^0 \\ \vdots \\ [L] [Z]^n \end{bmatrix}_{(n+1)N_i \times 2N}$$

One way to prove the validity of Eq. (33) is to start from the matrix $[G]$ in Eq. (34). This matrix has $2N$ columns and $(n+1)N_i$ rows, where :

$$(n+1) \times N_i = \left(\frac{2N}{N_i} + 1 \right) \times N_i = 2N + N_i$$

The columns are $2N$ vectors in a $2N + N_i$ dimensional space. These $2N$ vectors may be independent with respect to each other, but do not have to be. In case that the $2N$ vectors are independent, they form a base for a subspace of the space, of dimension $2N + N_i$. In other words, these $2N$ vectors span a $2N$ -dimensional space which is a subspace of the global $2N + N_i$ dimensional space. Since this global space is spanned by $2N + N_i$ independent vectors, the $2N$ independent vectors of matrix $[G]$ can be taken, expanded with N_i vectors, independent from these $2N$ vectors, such that a complete base is defined for the total space. It is possible to define the N_i additional vectors in such a way that they are orthogonal to all $2N$ vectors. In other words, the dot product of the N_i vectors with the $2N$ vectors is zero. This is stated by Eq. (34).

$$[Q]_{N_i \times (n+1)N_i} [G]_{(n+1)N_i \times 2N} = [0]_{N_i \times 2N}$$

Where :

- $[Q]$ = the transpose of the N_i vectors
- $[G]$ = the $2N$ vectors of the subspace

In the case that the vectors do not form an independent set of vectors, they do not span a $2N$ dimensional space, but a Y dimensional space, where Y is the number of independent vectors. For this case, there still exists a set of N_i vectors which are independent and orthogonal with respect to these Y vectors and Eq. (34) is still valid. In other words, if the $2N$ vectors are independent or not, N_i vectors can always be found such that Eq. (34) is satisfied.

Using this property of the $[G]$ matrix, an equation can be derived to determine the matrix coefficients $[a(i)]$. The method used is called Prony's algorithm [26]. This method makes use of the matrix coefficients $[a(i)]$ and Eq. (31) to derive a set of equations. Equation (31) is repeated here for convenience.

$$\{h_{pk}\} = [L] [Z]^k \{A_p\}$$

A first equation is obtained by taking Eq. (31) at time t_{0+n_a} (in other words, k equal to $0+n_a$), and multiplying this equation by the matrix coefficient $[a(n)]$. The next equation is obtained by shifting the time by one time interval (so that k is equal to n_a+1), and multiplying the equation by the matrix coefficient $[a(n-1)]$. This procedure is repeated another $n-1$ times so that the last equation is equal to Eq. (31) at time t_{n+n_a} , multiplied by the coefficient matrix $[a(0)]$. Summing this set of equations gives :

$$\begin{aligned} [a(n)] \{h_{p,n_a+0}\} &= [a(n)] [L] [Z]^{n_a+0} \{A_p\} \\ [a(n-1)] \{h_{p,n_a+1}\} &= [a(n-1)] [L] [Z]^{n_a+1} \{A_p\} \\ &\vdots \\ [a(1)] \{h_{p,n_a+n-1}\} &= [a(1)] [L] [Z]^{n_a+n-1} \{A_p\} \\ [a(0)] \{h_{p,n_a+n}\} &= [a(0)] [L] [Z]^{n_a+n} \{A_p\} \\ \hline \sum_{i=0}^n [a(i)] \{h_{p,n_a+i}\} &= \sum_{i=0}^n [a(i)] [L] [Z]^{n_a+i} \{A_p\} \end{aligned} \quad (35)$$

The index n_a , in Eq. (35), is used to indicate an arbitrary origin of the time history. This enables Eq. (35) to be written at different starting points for the same time history.

Note that the right hand side of Eq. (35) is Eq. (33) multiplied by $[Z]^{n_a} \{A_p\}$. Therefore, the right hand side of Eq. (35) equals zero.

$$\sum_{i=0}^n [a(i)] \{h_{p,n_a+i}\} = \{0\} \quad (36)$$

Equation (36) has the coefficient matrices $[a(i)]$ as unknowns. It should be emphasized that Eq. (36) is valid only for free decay functions or for impulse response functions because Eq. (36) has been derived from Eq. (31), which has the same limitations.

Since $[a(0)]$ is the matrix coefficient of the highest order in the matrix polynomial, $[a(0)]$ can be chosen as the identity matrix $[I]$, without a loss of generality [60]. Therefore Eq. (36) can be written as :

$$\sum_{i=1}^n [a(i)] \left\{ h_{p, n_0+n-i} \right\} = - \left\{ h_{p, n_0+n} \right\} \quad (37)$$

For each point p , a set of equations can be written by varying n_a from 0 to $(X-1)$, where X is an arbitrary large number. An identical set of equations can be obtained for every response location p . All of these sets of equations can be written as one equation in matrix notation, namely,

$$[B^n] \left[[\Upsilon_1^n] [\Upsilon_2^n] \dots [\Upsilon_{N_0}^n] \right] = - \left[[R_1^n] [R_2^n] \dots [R_{N_0}^n] \right] \quad (38)$$

Where :

- n = order of the polynomial
- N_0 = number of response locations
- $[R_p^n] = \left[\{h_{p,n}\}, \{h_{p,n+1}\}, \dots, \{h_{p,n+X-1}\} \right]_{N_i \times X}$
- $[B^n] = \left[[a(1)], [a(2)], \dots, [a(n)] \right]_{N_i \times nN_i}$
- $[\Upsilon_p^n]_{nN_i \times X} = \begin{bmatrix} \{h_{p,n-1}\} & \{h_{p,n}\} & \dots & \{h_{p,n+X-2}\} \\ \{h_{p,n-2}\} & \{h_{p,n-1}\} & \dots & \{h_{p,n+X-3}\} \\ \vdots & \vdots & \ddots & \vdots \\ \{h_{p,0}\} & \{h_{p,1}\} & \dots & \{h_{p,X-1}\} \end{bmatrix}$

- X = large number

Note that the last row in the $[\Upsilon_p^n]$ submatrix is the initial portion of the impulse response function, associated with each respective input location. The second to last row is the same impulse response function, shifted over one time sample (Δt), and so on, as shown in Figure 2.

The notation can be simplified by using the following definitions :

$$[R] = - \left[[R_1^n], [R_2^n], \dots, [R_{N_0}^n] \right]_{N_i \times N_0 X}$$

$$[B] = \left[[a(1)], [a(2)], \dots, [a(n)] \right]_{N_i \times N_i n}$$

$$[\Upsilon] = \left[[\Upsilon_1^n], [\Upsilon_2^n], \dots, [\Upsilon_{N_0}^n] \right]_{N_i n \times N_0 X}$$

Equation (38) can be written in a more concise form.

$$[B] [\Upsilon] = [R] \quad (39)$$

There are $N_i \times nN_i$ unknowns in Eq. (39). The number of equations is N_i (equals the number of rows

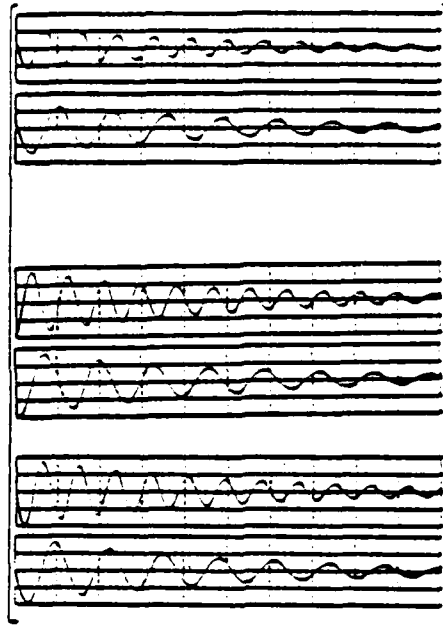


Figure 2. $[T]$ - Matrix

of $[B] \times (N_o X)$ (equals the number of columns of $[T]$). By taking X such that $N_o X$ is equal to nN_i , there are exactly as many equations as unknowns [60]. Therefore, there exists a unique solution for the coefficient matrices, as long as the equations are consistent.

However, it is advisable that X be determined such that $N_o X$ is greater than nN_i . This makes Eq. (39) an overdetermined system. Therefore the least squares method can be used to solve for the coefficient matrices. Using a least squares method provides an advantage in that it reduces random errors in the data [61].

However, an X which is too large has several disadvantages. Since the computational load is not linear with this variable, it is not advantageous to have X very large from a computational viewpoint. Second, the values in the time history decrease for large values of X , since the impulse response function is a decaying function. As a result, the signal to noise decreases with increasing time. Third, the truncation error, due to an inverse Fast Fourier Transform (F.F.T.), increases at the end of the data block (Figure 3). Note, this truncation error only exists for F.F.T. data.

The least squares solution of Eq. (39) can be found from the normal equation:

$$[B] [T] [T]^T = [R] [T]^T$$

$$[B] \left(\sum_{p=1}^{N_o} [T_p^*] [T_p^*]^T \right) = \sum_{p=1}^{N_o} [R_p] [T_p^*]^T \quad (40)$$

where the entities $[T_p^*] [T_p^*]^*$ and $[R_p^*] [T_p^*]^T$ can be shown to be asymptotically equal to the lagged auto- and cross-correlation matrices [62]. The unknown coefficient matrices can be solved from Eq. (40) by using a simultaneous equation solution technique such as Gaussian Elimination.

Once the coefficient matrices have been estimated, the natural frequency and damping values can be determined by Eq. (33).

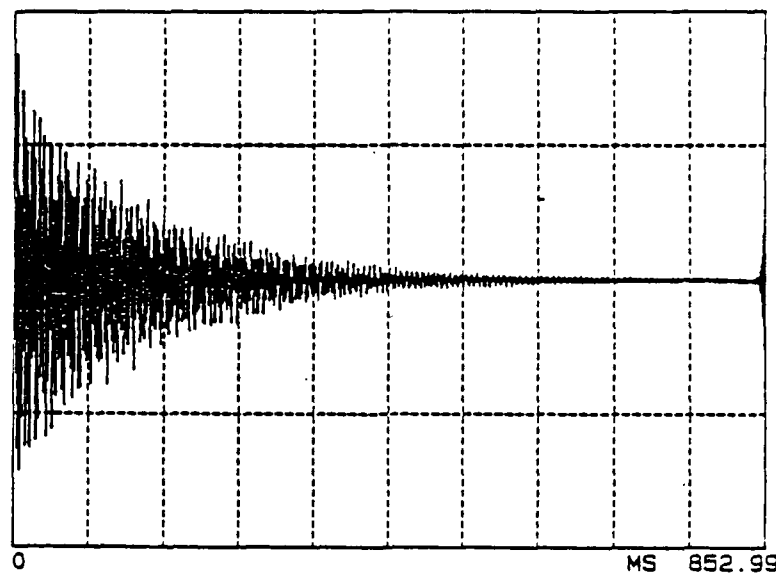


Figure 3. Impulse Response Function

$$\sum_{i=0}^n [a(i)] [L] [Z]^{n-i} = [0] \quad (41)$$

Both sides of Eq. (41) are postmultiplied by a unit vector ($2N \times 1$), composed of all zeros except for unity at the position corresponding to the system pole to be extracted.

$$\sum_{i=0}^n [a(i)] z_d^{n-i} \{L_d\} = \{0\} \quad (42)$$

Where :

- $z_d^{n-i} = (e^{\lambda_d \Delta t})^{n-i}$
- $\{L_d\}_{N_i \times 1}$ = column 'd' of $[L]$
- $\{0\}_{N_i \times 1}$ = zero vector

Since $\{L_d\}$ is a non zero vector, Eq. (42) can only be satisfied if :

$$\det \left[\sum_{i=0}^n [a(i)] z_d^{n-i} \right] = 0 \quad (43)$$

Therefore, the resonance frequency and damping values are obtained by solving for the poles of the matrix polynomial.

As mentioned earlier this matrix polynomial has $nN_i = (\frac{2N}{N_i} \times N_i) = 2N$ poles, or sometimes $(\frac{2N}{N_i} + 1) \times N_i = 2N + N_i$ poles.

One method of solving this system of equations is to formulate Eq. (42) as a standard eigenvalue problem. Recalling that $[a(0)] = [I]$, the following matrix polynomial equation is obtained from Eq. (43).

$$\left\{ [a(1)] z_d^{n-1} + [a(2)] z_d^{n-2} + \dots + [a(n)] \right\} \times \{L_d\} = \{L_d\} z_d^n \quad (44)$$

The roots of Eq. (44) can be found by using a companion matrix approach [63]. This approach uses (n-1) matrix equations expressing identity relationships. This method converts the polynomial into an eigenvalue problem of the form :

$$([E] - \beta [I]) \times \{X\} = \{0\} \quad (45)$$

Where :

- β = eigenvalue
- $\{X\}$ = eigenvector
- $[E]$ = the coefficient matrix

Before building the companion matrix, the following vectors are defined to enable (n-1) identity equations to be written:

$$\begin{cases} \{V_0\} = \alpha_d z_d^0 \{L_d\} \\ \{V_1\} = \alpha_d z_d^1 \{L_d\} = z_d \{V_0\} \\ \{V_2\} = \alpha_d z_d^2 \{L_d\} = z_d \{V_1\} \\ \vdots \\ \{V_{n-1}\} = \alpha_d z_d^{n-1} \{L_d\} = z_d \{V_{n-2}\} \end{cases} \quad (46)$$

where α_d is a proportionality constant

Using Eq. (44) and Eq. (46) the following matrix equation can be formulated as shown in Eq. (47).

$$\begin{bmatrix} -[a(1)] & -[a(2)] & \dots & -[a(n-1)] & -[a(n)] \\ [I] & [0] & \dots & [0] & [0] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [0] & [0] & \dots & [I] & [0] \end{bmatrix} \times \begin{Bmatrix} \{V_{n-1}\} \\ \{V_{n-2}\} \\ \vdots \\ \{V_0\} \end{Bmatrix}_{nN_1 \times 1} = z_d \begin{Bmatrix} \{V_{n-1}\} \\ \{V_{n-2}\} \\ \vdots \\ \{V_0\} \end{Bmatrix}_{nN_1 \times 1} \quad (47)$$

Equation (47) is the companion matrix equation for the matrix polynomial Eq. (44). It should be noted that the poles of Eq. (47) are the same as those for Eq. (44). The vector $\{L_d\}$ in Eq. (44) is the eigenvector associated with the pole z_d and also the modal participation factor. This vector $\{L_d\}$ is proportional to $\{V_0\}$, as defined in Eq. (46).

By using the appropriate solution algorithm [64], the eigenvalues and eigenvectors of Eq. (47) can be found. The resonance frequency and damping can be calculated by substitution of the following equations :

$$\begin{cases}
 z_r = z_{r,real} + z_{r,imag} \\
 \sigma_r = \text{damping} = \ln \frac{Mag}{\Delta t} \\
 Mag = (z_{r,real}^2 + z_{r,imag}^2)^{1/2} \\
 \omega_r = \text{resonance frequency} = \frac{\tan^{-1}(\frac{z_{r,imag}}{z_{r,real}})}{\Delta t} \\
 \Delta t = \text{sample time}
 \end{cases} \quad (48)$$

3.1.4 Second Stage Solution: Residues

Although the pole calculation for the Polyreference Time Domain is a purely time domain technique, the residue calculation can be performed in the time domain as well as in the frequency domain. The residue calculation in the time domain will be first discussed, followed by a brief discussion of the frequency domain calculation.

3.1.4.1 Time Domain Residue Calculation

To determine the residues, in the time domain, and the resulting modal coefficients, Eq. (31) is utilized.

$$\left\{ h_{pb} \right\}_{N \times 1} = [L]_{N \times 2N} [Z]_{2N \times 2N} \left\{ A_p \right\}_{2N \times 1} \quad (49)$$

Matrix $[Z]$ is a diagonal matrix with the poles on the diagonal, with every pole appearing with its complex conjugate. Therefore, matrix $[Z]$ can be rewritten as :

$$[Z] = \begin{bmatrix} [Y] & [0] \\ [0] & [Y^*] \end{bmatrix}_{2N \times 2N} \quad (50)$$

Where :

$$\bullet [Y]_{N \times N} = [Y^*]_{N \times N}^* = \begin{bmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_N \end{bmatrix}$$

Since the columns of the $[L]$ matrix were obtained as eigenvectors of an eigenvalue problem that had z_i as its eigenvalues, these vectors also appear in complex conjugate pairs. Therefore, the $[L]$ matrix can be written as :

$$[L] = [[E], [E^*]] \quad (51)$$

Where :

$$\begin{aligned}
 \bullet [E] &= [E^*]^* = [l_1, l_2, \dots, l_N] \\
 \bullet [L] &= [l_1, l_2, \dots, l_N, l_1^*, l_2^*, \dots, l_N^*]
 \end{aligned}$$

Concentrating on the matrix product between $[L]$ and $[Z]$, this matrix consists of two submatrices which are complex conjugates of one another.

$$\begin{aligned}
 [L] [Z]^k &= [E], [E^*] \begin{bmatrix} [Y]^k & [0] \\ [0] & [Y^*]^k \end{bmatrix} \\
 &= [E] [Y]^k, [E^*] [Y^*]^k \\
 &= [V]^k, [V^*]^k_{N_1 \times 2N}
 \end{aligned} \tag{52}$$

Where :

- $[V]^k_{N_1 \times N} = [E] [Y]^k$
- $k = \text{time point}$

The residues also appear in complex conjugate pairs, corresponding to complex conjugate poles. So vector $\{A_p\}$ in Eq. (49) can be rewritten as :

$$\{A_p\}_{2N \times 1} = \begin{bmatrix} \{U_p\} \\ \{U_p^*\} \end{bmatrix}_{2N \times 1} \tag{53}$$

Where :

$$\bullet \{U_p\} = \{U_p^*\} = \begin{bmatrix} A_{p11} \\ A_{p12} \\ \vdots \\ A_{p1N} \end{bmatrix}$$

Substituting Eqs. (52) and (53) into Eq. (49) results in :

$$\begin{aligned}
 \{h_{pk}\}_{N_1 \times 1} &= [V]^k, [V^*]^k_{N_1 \times 2N} \begin{bmatrix} \{U_p\} \\ \{U_p^*\} \end{bmatrix}_{2N \times 1} \\
 &= [V]^k \{U_p\} + [V^*]^k \{U_p^*\}
 \end{aligned}$$

$$\left\{ h_{pk} \right\}_{N_1 \times 1} = 2 \operatorname{Re} \left\{ [V]^k \left\{ U_p \right\} \right\} \quad (54)$$

Where : Re = real part of a complex number

Note that V and U are complex vectors.

$$\begin{aligned} [V]^k &= [V_{\text{real}}]^k + j [V_{\text{imag}}]^k \\ \left\{ U_p \right\} &= \left\{ U_{p,\text{real}} \right\} + j \left\{ U_{p,\text{imag}} \right\} \end{aligned}$$

Where :

- $\left\{ U_{p,\text{real}} \right\} = \operatorname{Re} \left\{ \left\{ U_p \right\} \right\}$
- $\left\{ U_{p,\text{imag}} \right\} = \operatorname{Im} \left\{ \left\{ U_p \right\} \right\}$
- $[V_{\text{real}}] = \operatorname{Re} \left\{ [V] \right\}$
- $[V_{\text{imag}}] = \operatorname{Im} \left\{ [V] \right\}$
- Re = real part of a complex number
- Im = imaginary part of a complex number

Equation (54) can be written as :

$$\left\{ h_{pk} \right\} = 2 \left[[V_{\text{real}}]^k, -[V_{\text{imag}}]^k \right] \times \begin{Bmatrix} \left\{ U_{p,\text{real}} \right\} \\ \left\{ U_{p,\text{imag}} \right\} \end{Bmatrix} \quad (55)$$

In certain cases, it is correct to assume that the modes are normal. In these cases, $\left\{ U_{p,\text{real}} \right\} = \{0\}$ [58], and Eq. (55) can be simplified as :

$$\left\{ h_{pk} \right\} = -2 [V_{\text{imag}}]^k \left\{ U_{p,\text{imag}} \right\} \quad (56)$$

A least squares estimate of the residues can also be formulated. By varying k in Eq. (49), the following equation is obtained:

$$\begin{Bmatrix} \left\{ h_{p0} \right\} \\ \left\{ h_{p1} \right\} \\ \vdots \\ \left\{ h_{pX} \right\} \end{Bmatrix}_{(X+1)N_1 \times 1} = \begin{Bmatrix} [L] [Z]^0 \\ [L] [Z]^1 \\ \vdots \\ [L] [Z]^X \end{Bmatrix}_{(X+1)N_1 \times 2N} \left\{ A_p \right\}_{2N \times 1} \quad (57)$$

Where : X = an arbitrary positive integer

Premultiplying both sides by

$$[B]_{2N \times (X+1)N_1} = \begin{bmatrix} [L^*] [Z^*]^0 \\ [L^*] [Z^*]^1 \\ \vdots \\ [L^*] [Z^*]^X \end{bmatrix}^T$$

results in the normal equation as follows :

$$[B] \{H_p\} = [C] \{A_p\} \quad (58)$$

Where :

$$\bullet [C]_{2N \times 2N} = [B] \times [B^*]^T$$

$$\bullet \{H_p\}_{(X+1)N_1 \times 1} = \begin{Bmatrix} \{h_{p0}\} \\ \{h_{p1}\} \\ \vdots \\ \{h_{pX}\} \end{Bmatrix}$$

From Eq. (58), the unknown residues can be determined by using a simultaneous linear equation solution technique.

A least squares solution can also be found for the normal mode assumption. By varying k in Eq. (56) a set of equations is obtained, similar to Eq. (57) :

$$\{H_p\}_{(X+1)N_1 \times 1} = -2 [D] \{U_{p,imag}\}_{N \times 1} \quad (59)$$

$$\text{Where : } [D]_{(X+1)N_1 \times N} = \begin{bmatrix} [V_{imag}]^0 \\ [V_{imag}]^1 \\ \vdots \\ [V_{imag}]^X \end{bmatrix}$$

By premultiplying both sides of the equation by the transpose of the coefficient matrix $[D]$, the normal equation is obtained, which can be solved again by a simultaneous linear equation solution technique.

$$[D]^T \{H_p\} = -2 [C] \{U_{p,imag}\} \quad (60)$$

$$\text{Where : } [C]_{N \times N} = [D]^T [D]$$

3.1.4.2 Frequency Domain Residue Calculation

As mentioned before, once the poles and modal participation matrix are known, the residues do not have to be calculated in the time domain. The advantage of calculating the residues in the frequency domain is that the effects of modes outside the frequency range of interest can be compensated by the use of residuals. The disadvantage of the frequency domain calculation is that for a wide frequency range (order of 2 difference between the lowest and highest frequency), this calculation is numerically not as stable as the time domain calculation.

In the basic modal analysis theory [58] has been proven that the frequency response function in the frequency domain, can be written as :

$$H_{pq}(\omega) = \sum_{r=1}^N \left(\frac{A_{pqr}}{j\omega + \lambda_r} + \frac{A_{pqr}^*}{j\omega + \lambda_r^*} \right) + R_F - R_I$$

where :

- R_F = residual flexibility
- R_I = residual inertia = $\frac{-\text{constant}}{\omega^2}$

for the case of a single reference. For multiple references, the previous equation becomes :

$$\left\{ H_p(\omega) \right\} = [L] [C(\omega)] \left\{ A_p \right\} + \left\{ R_F \right\} - \left\{ R_I \right\} \quad (61)$$

where :

- $[L]_{N \times 2N}$ = modal participation matrix
- $\{R_F\}_{N \times 1}$ = flexibility residue vector
- $\{R_I\}_{N \times 1}$ = inertia restraint vector = $\frac{-\{Ir\}}{\omega^2}$
- $[C(\omega)]_{2N \times 2N} = \begin{bmatrix} c_{1,1} & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & c_{N,N} & . & . & . \\ . & . & . & c_{N+1,N+1} & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & c_{2N,2N} \end{bmatrix}$

Where

- $c_{i,i} = \frac{1}{j\omega - \lambda_i}$ for $i = 1, 2, \dots, N$
- $c_{i,i} = \frac{1}{j\omega - \lambda_i^*}$ for $i = N+1, N+2, \dots, 2N$

$$\bullet \{H_p\}_{N_i \times 1} = \begin{Bmatrix} H_{p1}(\omega) \\ H_{p2}(\omega) \\ \vdots \\ H_{pN_i}(\omega) \end{Bmatrix}$$

Equation (61) can be simplified to the following Eq.:

$$\{H_p(\omega)\} = [D] [E(\omega)] \{R\} = [F(\omega)] \{R\} \quad (62)$$

Where :

$$\bullet [D]_{N_i \times 2(N+N_i)} = \begin{bmatrix} [L] & [I] & [I] \end{bmatrix} \text{ Where } [I] \text{ is the identity matrix}$$

$$\bullet [E]_{2(N+N_i) \times 2(N+N_i)} = \begin{bmatrix} [C] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & \begin{bmatrix} -1 \\ \omega^2 \end{bmatrix} \end{bmatrix}$$

$$\bullet [F(\omega)]_{N_i \times 2(N+N_i)} = [D] [E(\omega)]$$

$$\bullet \{R\}_{2(N+N_i) \times 1} = \begin{Bmatrix} \{A_p\} \\ \{R_f\} \\ \{I_r\} \end{Bmatrix}$$

Equation (62) is a set of N_i equations in $2(N+N_i)$ unknowns. In order to have a unique solution for these $2(N+N_i)$ unknowns, at least $2(N+N_i)$ equations are needed. If more than $2(N+N_i)$ equations are available, an overdetermined system is obtained and a least squares estimation can be used.

By using different values of ω in Eq. (62) an overdetermined system is obtained of the form :

$$\begin{Bmatrix} \{H_p(0 \Delta \omega)\} \\ \{H_p(1 \Delta \omega)\} \\ \vdots \\ \{H_p(X \Delta \omega)\} \end{Bmatrix}_{XN_i \times 1} = \begin{bmatrix} [F(0 \Delta \omega)] \\ [F(1 \Delta \omega)] \\ \vdots \\ [F(X \Delta \omega)] \end{bmatrix}_{XN_i \times 2(N+N_i)} \{R\}_{2(N+N_i) \times 1}$$

$$\{H_p\} = [F] \{R\} \quad (63)$$

where X is an arbitrary large integer which has to be smaller than the used blocksize during the measurements.

By premultiplying by $[F]^T$, the normal equation is obtained :

$$[F]^T \{H_p\} = [F]^T [F] \{R\} \quad (64)$$

This set of simultaneous equations can be solved, for example, by a Gauss elimination process.

It should be noted that the solution for Eq. (64) gives complex residues while the residuals are real numbers.

For the case where real modes are desired the same procedure is followed. In Eq. (61), the matrices $[L]$ and $[C]$ are replaced by $[L']$ and $[C']$ where :

$$\bullet [C']_{N \times N} = \begin{bmatrix} c_{1,1} & 0 & \dots & 0 \\ 0 & c_{2,2} & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & c_{N,N} \end{bmatrix}$$

Where :

$$\bullet c_{i,i} = \frac{1}{j\omega - \lambda_i} - \frac{1}{j\omega - \lambda_i^*}$$

$$\bullet [L']_{N \times N} = [l_1 \ l_2 \ \dots \ l_N]$$

$$[L]_{N_1 \times 2N} = [l_1 \ l_2 \ \dots \ l_N \ l_1^* \ l_2^* \ \dots \ l_N^*]$$

Performing the same substitutions, a similar equation to Eq. (64) is obtained. Solving this equation gives real modes.

3.1.5 Time, Frequency, Laplace and Z-Domain Relations

In this section the validity of Eq. (36) will be proven. The proof is based on the relationship between the Laplace domain and Z-domain. The main difference between these domains is that a Laplace transformation is used for continuous data, while the Z-transformation is used on sampled data. The Laplace Transformation transforms a differential equation, in the time domain, into a polynomial equation as a function of s in the Laplace domain. The Z-transformation converts a difference equation, in the time domain, into a polynomial equation as a function of z in the Z domain.

The differential equation for a one degree of freedom system with single input is :

$$m \ddot{x}(t) + c \dot{x}(t) + k x(t) = f(t) \quad (65)$$

The Laplace Transformation of Eq. (65), assuming zero initial conditions, is :

$$(m s^2 + c s + k) X(s) = F(s)$$

or

$$\frac{X(s)}{F(s)} = \frac{1}{m s^2 + c s + k} \quad (66)$$

Where :

- $m s^2 + c s + k = 0$ the characteristic equation
- $X(s)$ = Laplace transform of the response $x(t)$

- $F(s)$ = Laplace transform of the force $f(t)$

A system with N degrees of freedom and single input has a characteristic equation equal to :

$$b_{2N} s^{2N} + b_{2N-1} s^{2N-1} + b_{2N-2} s^{2N-2} + \dots + b_1 s + b_0 = 0 \quad (67)$$

The roots of this polynomial are called eigenvalues. The real part is the damping and the imaginary part is the eigenfrequency or damped natural frequency.

In the Z domain, z is equal to $e^{s \Delta t}$ with Δt the sample time. Therefore there is a relation between s , the Laplace domain variable and z , the Z domain variable. Based on this fact it can be shown that there exists a polynomial in the Z domain that has exactly the same roots as the characteristic equation in the Laplace domain, namely:

$$z_r = e^{\lambda_r \Delta t}$$

where λ_r is a root of Eq. (67). This new polynomial with different coefficients, has the following form:

$$a_{2N} z^{2N} + a_{2N-1} z^{2N-1} + \dots + a_1 z + a_0 = 0 \quad (68)$$

Assuming now that the different characteristic values for Eq. (68) are known and can be represented by z_r , where r refers to the root number. If z_r is substituted into Eq. (68), the equation is satisfied and can be multiplied by any arbitrary function without violating the equation. Taking $\frac{z}{z - e^{\lambda_r \Delta t}}$ as an arbitrary function, Eq. (68) becomes :

$$(a_{2N} z_r^{2N} + a_{2N-1} z_r^{2N-1} + \dots + a_1 z_r + a_0) \frac{z}{z - e^{\lambda_r \Delta t}} = 0 \quad (69)$$

Where :

- $t_k = k \Delta t$

Equation (69) represents a sum of constant coefficients multiplied by a function in z . Therefore, if an inverse Z -transform is performed on Eq. (69) the only factor that changes is the function in z and Eq. (69) becomes :

$$(a_{2N} z_r^{2N} + a_{2N-1} z_r^{2N-1} + \dots + a_1 z_r + a_0) e^{\lambda_r t_k} = 0 \quad (70)$$

or

$$\left(\sum_{i=0}^n a_i z_r^i \right) e^{\lambda_r t_k} = 0 \quad (71)$$

Substituting $z_r = e^{\lambda_r \Delta t}$ into Eq. (71) gives:

$$\sum_{i=0}^n a_i e^{\lambda_r (t_k + i \Delta t)} = 0 \quad (72)$$

Instead of $\frac{z}{z - e^{\lambda_r \Delta t}}$, the function $\frac{A_{per} z}{z - e^{\lambda_r \Delta t}}$ could have been taken as arbitrary function, without loss of generality. For this arbitrary function Eq. (72) becomes :

$$\sum_{i=0}^n a_i A_{pq} e^{\lambda_r (t_k + i \Delta t)} = 0 \quad (73)$$

Equation (73) is valid for all characteristic values of Eq. (67). In other words, each damped exponential, associated with each particular characteristic values, satisfied Eq. (73). Since Eq. (73) is a linear equation in e , any linear combination of the individual solutions will be a solution for Eq. (73). Recalling now the equation for the impulse response function, namely :

$$h_{pq}(t_k) = \sum_{i=0}^n A_{pq} e^{\lambda_r t_k}$$

This is a linear combination of the individual solutions of the linear Eq. (73), and is therefore also a solution. This means :

$$\sum_{i=0}^n a_i h_{pq}(t_k + i \Delta t) = 0 \quad (74)$$

Where :

- Δt = sample time
- $h(t_k)$ = impulse response function

What has been proven up till now in this section can be summarized as follows. Equation (65) represents the differential equation for a continuous function. By performing a Laplace transform, this differential equation is converted into a polynomial. Then by showing the relation between the Laplace domain variable s and the Z domain variable z one can prove that a polynomial in the Z domain exists with the same poles. This polynomial is the Z transform of a difference equation. This difference equation is the equivalent of the differential equation in Eq. (65). Namely a differential equation, which is only valid for continuous data, converts into a difference equation when the function is a discrete function.

It is known that the solution for the characteristic equation are damped exponentials in the time domain. For the case when discrete data is available, the characteristic equation is a difference equation in the time domain. Every part of the damped exponential satisfies the characteristic difference equation. This property has been proven between Eq. (68) and Eq. (73). Namely by multiplying by the function $\frac{z}{z - e^{\lambda_r \Delta t}}$ gives in the characteristic difference equation the term $e^{\lambda_r t_k}$. By taking different values for t , different portion of the damped exponential function are used and the characteristic difference equation is still satisfied. This proves that indeed any portion of the damped exponential function satisfies the characteristic difference equation as was stated previously.

In the case of multiple references, Eq. (67) becomes a matrix polynomial of the form :

$$[b(n)] s^n + [b(n-1)] s^{n-1} + [b(n-2)] s^{n-2} + \dots + [b(1)] s + [b(0)] = [0] \quad (75)$$

The dimensions of the matrix coefficients and the value for n , the order of the matrix polynomial, have been discussed in Section 3.2.

From Eq. (75) and using a similar procedure as used in a single input case, the following equation is obtained.

$$[a(0)] \{h(t_k)\} + [a(1)] \{h(t_k + \Delta t)\} + \dots + [a(n)] \{h(t_k + n \Delta t)\} = \{0\}$$

or

$$\sum_{i=0}^n [a(i)] \left\{ h(t_k + i \Delta t) \right\} = \{0\} \quad (76)$$

Where : $t_k = k \Delta t$

Restricting $k \geq 0$, Eq. (36) and Eq. (76) are identical.

3.1.6 Detecting Repeated Roots

The single reference algorithms, developed prior to the Polyreference Time Domain Technique, are theoretically not able to detect repeated roots in a data set. However, in practice, perfect multiple roots hardly ever occurs. This does not mean that very close coupled poles are not possible in a data set. Even for close coupled poles, the single reference algorithms may have trouble to detect all close coupled poles or calculating a proper estimate of the modal vectors. When a modal model is wanted for later applications, such as modification analysis for example, missing a pole or having a poor modal vector estimate in the modal model, makes the results of the further analysis very doubtful. The Polyreference Time Domain Technique has the ability to detect repeated roots.

This proof starts with Eq. (36) :

$$\sum_{i=0}^n [a(i)] \left\{ h_{p,n-i} \right\} = \{0\} \quad (77)$$

where n_a is taken zero for convenience.

As mentioned in Section 2, this equation is only valid for an impulse response function or for a free decay function.

Consider a free decay function of only one frequency, namely :

$$\left\{ h_{p,n-k} \right\} = \left\{ A_{p,r} \right\} e^{\lambda_r t_{n-k}} \quad (78)$$

Where :

$$\bullet \left\{ A_{p,r} \right\}_{N_1 \times 1} = \begin{Bmatrix} A_{p,1,r} \\ A_{p,2,r} \\ \vdots \\ A_{p,N_1,r} \end{Bmatrix}$$

- $A_{p,r}$ = residue for mode r at point p due to an input at point q
- λ_r = system pole = $\sigma_r + j \omega_r$
- σ_r = system damping for mode r (rad/sec)
- ω_r = damped natural frequency for mode r (rad/sec)

Since Eq. (78) satisfies the conditions for which Eq. (77) is true, Eq. (77), for this special case, becomes :

$$\sum_{i=0}^n [a(i)] \{A_{p,r}\} e^{\lambda_r t_{i+1}} = \{0\} \quad (79)$$

Assume now that the pole λ_r is a repeated root with multiplicity two. Further, let $\{A_{p,r}\}$ and $\{A_{p,r+1}\}$ be two distinctive residue vectors that are related to the two respective eigenvectors that are connected with this pole.

For both residue vectors, Eq. (79) becomes :

$$\begin{cases} \left(\sum_{i=0}^n [a(i)] e^{\lambda_r t_{i+1}} \right) \{A_{p,r}\} = \{0\} \\ \left(\sum_{i=0}^n [a(i)] e^{\lambda_r t_{i+1}} \right) \{A_{p,r+1}\} = \{0\} \end{cases} \quad (80)$$

Taking :

$$\left(\sum_{i=0}^n [a(i)]_{N_1 \times N_1} e^{\lambda_r t_{i+1}} \right) = [C(r)]_{N_1 \times N_1}$$

Equation (80) can be written as

$$[C(r)]_{N_1 \times N_1} \begin{bmatrix} \{A_{p,r}\} \\ \{A_{p,r+1}\} \end{bmatrix}_{N_1 \times 2} = [0]_{N_1 \times 2} \quad (81)$$

Therefore, the matrix $[C]$ transforms the $\{A_{p,r}\}$ residue vector as well as the $\{A_{p,r+1}\}$ residue vector into the zero vector. As a result, the nullspace of the transformation matrix is at least two, since $\{A_{p,r}\}$ and $\{A_{p,r+1}\}$ are independent.

If the nullspace has a maximum dimension of at least two, the rowspace has a dimension of maximum $N_1 - 2$ with N_1 the number of rows in the $[C]$ matrix (Figure 4).

Therefore, to determine if a pole has a multiplicity greater than one, the matrix $[C]$, for this particular pole, can be calculated. The rank of this matrix $[C]$ is equal to the dimension of the row space. The dimension of the nullspace is smaller than or equal to the multiplicity of the pole. If the rank of the $[C]$ matrix is different from zero, then the dimension of the nullspace is equal to the multiplicity of the pole. In the case in which the rank of the $[C]$ matrix is equal to zero, the multiplicity of the pole is at least L , the number of rows of the $[C]$ matrix. Only L eigenvectors connected with that pole can be detected, although there may be more.

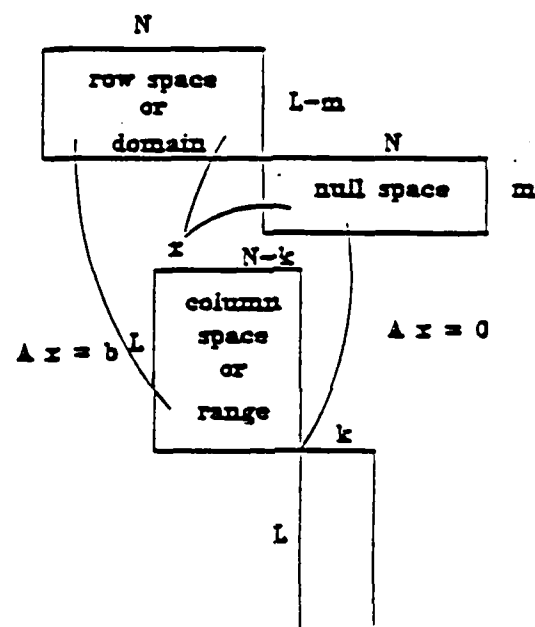


Figure 4. Row-Null Space Relation

3.2 Polyreference Frequency Domain

3.2.1 Introduction

Recent advances in frequency response function (FRF) estimation techniques clearly show the benefits of multiple input [50,52]. The redundancy in the measurement data allows the analyst to separate closely spaced modal frequencies, to identify repeated modal frequencies (repeated roots) or to distinguish between closely coupled modes. Noise can also effectively be averaged out by taking into account more data simultaneously.

In the last decade several new techniques have been developed to take advantage of multiple input measurements such as the Polyreference Time Domain Technique [18] and the Modified Ibrahim Technique [65]. These methods all apply to time domain impulse response functions. Since the actual measurements are commonly stored as frequency response functions, the measurements have to be transformed to the time domain by an inverse Fourier transform. This inversion gives some disadvantages to the time domain techniques:

- Truncation errors are introduced when converting data from frequency to the time domain by an inverse Fourier transform.
- In order to be able to perform an inverse fast Fourier transform (FFT), the data has to be taken with a constant frequency spacing.

In recent years, data acquisition systems have evolved from 4 channel systems to 64 channels or more. This development minimizes the disadvantage of sine testing, specifically, the time consuming nature of the sine test method. This is due to the fact that 64 measurements, or more, can be taken simultaneously. Since the sine technique may yield superior measurements compared to random excitation, due to the signal to noise ratios, there is a tendency to evaluate sine testing methods again. One of the advantages of sine testing is that the frequency spacing does not have to be constant. However, existing time domain modal parameter estimation techniques require equidistant frequency spacing in the data in order to use an inverse FFT. To take optimal advantage of sine testing, without resorting to a forced normal mode approach, a polyreference frequency domain method was necessary.

With this background in mind, a Polyreference Frequency Domain parameter estimation technique has been developed simultaneously at the University of Cincinnati and the Katholieke University of Leuven (Belgium) [42,43]. At the University of Leuven, a second order system was used as a starting point for the algorithm, while at the University of Cincinnati a first order system was used. This makes the algorithm developed at the University of Cincinnati basically a special case of the algorithm developed at the University of Leuven.

In this section, both Polyreference Frequency Domain Methods will be described. Both methods calculate global estimates for the system poles as well as the modal vectors.

3.2.2 General Concepts

When the force is equal to a unity Dirac impulse, the time response is equal to the impulse response function. This means that this time function can be expressed as a sum of complex exponentials each multiplied by a constant [58]. This constant is a function of the input point and the response point. For the case where there are N_i inputs and N_o responses, the impulse response matrix can be written as:

$$[h(t)] = [\Psi] e^{[A]t} [L] \quad (82)$$

where:

- $[h(t)]_{N_0 \times N_1}$ = impulse response matrix
- $[\Psi]_{N_0 \times 2N}$ = matrix with the modal vectors
- $[A]_{2N \times 2N}$ = diagonal matrix with the system poles
- $[L]_{2N \times N_1}$ = modal participation matrix
- N_0 = number of measurements
- N_1 = number of references
- N = number of system poles

$[L]$ is not identical to the modal participation matrix as it is defined for the Polyreference Time Domain. The scaling factors for the different modal vectors are included in this matrix. This makes the $[L]$ matrix and the $[\Psi]$ matrix together a unique set and their product is equal to the residue matrix. However, the same interpretation can be still given to the $[L]$ matrix as to the modal participation matrix as it is defined in the Polyreference Time Domain. Namely the modal participation matrix indicates how well a particular reference excites a particular mode. Therefore, the $[L]$ matrix will be simply referred as the modal participation matrix.

In order to obtain an expression for the displacement transfer function in the frequency domain, the Laplace transform of Eq. (82) can be taken, which gives:

$$[H_d(s)]_{N_0 \times N_1} = [\Psi] [s[I] - [A]]^{-1} [L] \quad (83)$$

where:

- s = Laplace variable
- $[H_d(s)]_{N_0 \times N_1}$ = transfer function matrix equal to the Laplace transform of the impulse response matrix $[h(t)]$
- $[s[I] - [A]]_{2N \times 2N}^{-1}$ = Laplace transform of $e^{[A]t}$
- $[I]$ = identity matrix

The subscript d on the $[H]$ matrix indicates that the transfer function matrix is in terms of displacement over force. By defining a $[T]$ matrix as:

$$[T]_{2N \times N_1} = [s[I] - [A]]^{-1} [L] \quad (84)$$

Equation (84) can be simplified to:

$$[H_d(s)]_{N_0 \times N_1} = [\Psi]_{N_0 \times 2N} [T]_{2N \times N_1} \quad (85)$$

In order to obtain an expression for the transfer function in terms of velocity and acceleration in the frequency domain, the single and double partial derivative with respect to time can be taken of Eq. (82) and a Laplace transform can be performed on the obtained equations. Another way to obtain these two equations is by making use of the Laplace transformation properties and apply these immediately to Eq. (85). The resulting two equations, obtained for the transfer function matrix with

respect to the velocity and the acceleration, are:

$$\left[H_v(s) \right]_{N_0 \times N_1} = s \left[H_d(s) \right] - \left[h(t)_{t=0} \right] \quad (86)$$

$$= s \left[H_d(s) \right] - \left[\Psi \right] \left[L \right] \quad (87)$$

$$= \left[\Psi \right] \left[\Lambda \right] \left[T(s) \right] \quad (88)$$

and

$$\left[H_a(s) \right]_{N_0 \times N_1} = s^2 \left[H_d(s) \right] - s \left[h(t)_{t=0} \right] - \left[\dot{h}(t)_{t=0} \right] \quad (89)$$

$$= s^2 \left[H_d(s) \right] - s \left[\Psi \right] \left[L \right] - \left[\Psi \right] \left[\Lambda \right] \left[L \right] \quad (90)$$

$$= \left[\Psi \right] \left[\Lambda \right]^2 \left[T(s) \right] \quad (91)$$

In order to compensate for the influence of the modes outside the frequency range of interest, an additional term can be added to the previous equations to compensate for these residuals effects. Eq. (85), (88) and (91) become:

$$\left[H_d(s) \right] = \left[\Psi \right] \left[T(s) \right] + \left[R_d \right] \quad (92)$$

$$\left[H_v(s) \right] = \left[\Psi \right] \left[\Lambda \right] \left[T(s) \right] + \left[R_v \right] \quad (93)$$

$$\left[H_a(s) \right] = \left[\Psi \right] \left[\Lambda \right]^2 \left[T(s) \right] + \left[R_a \right] \quad (94)$$

where:

- $\left[R_d \right]_{N_0 \times N_1}$ = residual effects for the displacement
- $\left[R_v \right]_{N_0 \times N_1}$ = residual effects for the velocity
- $\left[R_a \right]_{N_0 \times N_1}$ = residual effects for the acceleration

3.2.3 Second order model

In this section the more general second order model, that was developed at the Catholic University of Leuven, will be discussed. In the next section, the special case of this model, namely the first order model, will be discussed.

Equations (92), (93) and (94) can be combined into one equation:

$$\begin{bmatrix} H_d(s) \\ H_e(s) \\ H_a(s) \end{bmatrix}_{3N_0 \times N_1} = \begin{bmatrix} \Psi \\ \Psi \\ \Psi \end{bmatrix} \begin{bmatrix} \Lambda \\ \Lambda \\ \Lambda \end{bmatrix}^2_{3N_0 \times 2N} [T(s)] + \begin{bmatrix} R_d \\ R_e \\ R_a \end{bmatrix}_{3N_0 \times N_1} \quad (95)$$

Substituting Eqs. (87) and (90) into Eq. (95) gives:

$$\begin{bmatrix} H_d(s) \\ s [H_d(s) - \Psi] [L] \\ s^2 [H_d(s) - s \Psi] [L] - \Psi [\Lambda] [L] \end{bmatrix} = \begin{bmatrix} \Psi \\ \Psi \\ \Psi \end{bmatrix} \begin{bmatrix} \Lambda \\ \Lambda \\ \Lambda \end{bmatrix}^2 [T(s)] + \begin{bmatrix} R_d \\ R_e \\ R_a \end{bmatrix} \quad (96)$$

Concentrating on the $\begin{bmatrix} \Psi \\ \Psi \\ \Psi \end{bmatrix} \begin{bmatrix} \Lambda \\ \Lambda \\ \Lambda \end{bmatrix}^2$ matrix, this matrix has $3N_0$ rows while $2N$ columns. This means that this matrix represents $2N$ vectors in a $3N_0$ dimensional space. When $3N_0 \geq 2N$ these vectors define a subspace of a $3N_0$ dimensional space. This means that at least $3N_0 - 2N$ linear independent vectors can be found that are orthogonal to these $2N$ vectors. If the $2N$ vectors are not independent, more than $3N_0 - 2N$ orthogonal vectors can be found. If the independent orthogonal vectors are represented by a matrix $[Q]$, the previous property can be expressed in next equation:

$$[Q]_{(3N_0-2N) \times 3N_0}^T [\Theta] = [0]_{(3N_0-2N) \times 2N} \quad (97)$$

where:

$$[\Theta]_{3N_0 \times 2N} = \begin{bmatrix} \Psi \\ \Psi \\ \Psi \end{bmatrix} \begin{bmatrix} \Lambda \\ \Lambda \\ \Lambda \end{bmatrix}^2$$

Under the restriction $3N_0 - 2N \geq N_0$ or $N_0 \geq N$, at least N_0 orthogonal vectors can be found and Eq. (97) becomes:

$$[Q]_{N_0 \times 3N_0}^T [\Theta] = [0]_{N_0 \times 2N} \quad (98)$$

The reason for the restriction $N_0 \geq N$ is that a matrix polynomial is desired that will have poles equal to the system poles. This restriction for a matrix polynomial is necessary so that the coefficients will be square matrices.

The fact that the different vectors in the $[\Theta]$ matrix appear in complex conjugates, makes the orthogonal vectors monophase vectors. A monophase vector is a vector that can be scaled such that all elements of the vector are real or imaginary. An example of a monophase vector is the normal mode vector. The monophase property can be proven as follows:

Assume that the complex vectors of the $[\Theta]$ are written as

$$[\Theta] = [\Theta_r] + j [\Theta_i]$$

and their complex conjugate vectors as:

$$[\Theta]^* = [\Theta_r] - j [\Theta_i]$$

where $j = \sqrt{-1}$

Writing the orthogonal vector as the sum of real plus imaginary part:

$$[Q] = [Q_r] + j [Q_i]$$

In order for the $[Q]$ matrix to be orthogonal to the complex conjugate vectors, it has to satisfy next two sets of equations:

$$([Q_r] + j [Q_i]) \times ([Q_r] + j [Q_i]) = [0]$$

and

$$([Q_r] + j [Q_i]) \times ([Q_r] - j [Q_i]) = [0]$$

These two sets of equations can be reduced to the following constraints:

$$\begin{cases} [Q_r] [Q_r] = [0] \\ [Q_r] [Q_i] = [0] \\ [Q_i] [Q_r] = [0] \\ [Q_i] [Q_i] = [0] \end{cases}$$

In order to satisfy these restrictions, the orthogonal vectors have to be monophasic. Practically, this means that these orthogonal vectors might be complex, but a scaling factor can always be found such that this scaling factor makes them real vectors. Mathematically, this comes down to:

$$[Q] = \alpha [\Xi]$$

where:

- $[\Xi]$ is a real matrix
- α is a complex scaling factor

The matrix $[Q]$ can be divided into submatrices:

$$[Q] = \begin{bmatrix} [B_0] \\ [B_1] \\ [B_2] \end{bmatrix}$$

Without affecting the generality of Eq. (98) and taking advantage of the monophasic property, the matrix $[Q]$ can be normalized with respect to the submatrix $[B_2]$ and the $[Q]$ matrix becomes:

$$[Q] = \begin{bmatrix} [A_0] \\ [A_1] \\ [I] \end{bmatrix} \quad (99)$$

Substituting Eq. (99) into Eq. (98) gives:

$$\begin{bmatrix} [A_0]^T & [A_1]^T & [I] \end{bmatrix}_{N_0 \times 3N_0} \begin{bmatrix} \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} \\ \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix}^2 \end{bmatrix}_{3N_0 \times 2N} = \begin{bmatrix} 0 \end{bmatrix}_{N_0 \times 2N} \quad (100)$$

Equation (100) represents a matrix polynomial of order 2. Since the matrix coefficients have a dimension of N_0 this matrix polynomial has $2N_0$ poles. The $2N$ system poles represented by the matrix $[\Lambda]$ are a subset of these $2N_0$ poles. Since N_0 can be a lot larger than $2N$, this can cause a numerical problem. This numerical problem can be minimized and this will be discussed later in this text.

Bringing the residue matrix in Eq. (96) to the left hand side, the next equation is obtained:

$$\begin{bmatrix} H_d(s) - [R_d] \\ s [H_d(s) - [\Psi] [L] - [R_e]] \\ s^2 [H_d(s) - s [\Psi] [L] - [\Psi] [\Lambda] [L] - [R_a]] \end{bmatrix}_{3N_0 \times N_i} = \begin{bmatrix} \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} \\ \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix}^2 \end{bmatrix}_{3N_0 \times 2N} [T(s)]_{2N \times N_i} \quad (101)$$

Postmultiplying Eq. (100) with the $[T]$ matrix and substituting Eq. (101) into the obtained equation gives:

$$\begin{bmatrix} [A_0]^T & [A_1]^T & [I] \end{bmatrix} \begin{bmatrix} H_d(s) - [R_d] \\ s [H_d(s) - [\Psi] [L] - [R_e]] \\ s^2 [H_d(s) - s [\Psi] [L] - [\Psi] [\Lambda] [L] - [R_a]] \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad (102)$$

This equation means that the measured FRF's can be described by a linear frequency domain model with real matrix coefficients:

$$\begin{aligned} [A_0]^T [H_d(s)] + s [A_1]^T [H_d(s)] + s^2 [H_d(s)] = \\ \begin{bmatrix} [A_1]^T [\Psi] + s [\Psi] + [\Psi] [\Lambda] [L] + [A_0]^T [R_d] + [A_1]^T [R_e] + [R_a] \end{bmatrix} \\ = s [R_1] + [R_0] \end{aligned} \quad (103)$$

where:

- $[R_1] = [\Psi] [L]$ and is equal to the residue matrix.
- $[R_0] = [A_1]^T [\Psi] + [\Psi] [\Lambda] [L] + [A_0]^T [R_d] + [A_1]^T [R_e] + [R_a]$

This equation holds for each value of s ($s = j\omega$ for frequency response function) of the frequency band where Eq. (82) is valid. It defines a linear model for the measured FRF's between N_0 response locations and N_i inputs or references. The unknowns of this model are the matrix coefficients $[A_0]$, $[A_1]$, $[R_1]$ and $[R_0]$.

To summarize, the algorithm proceeds as follows: first the unknown matrices $[A_0]$, $[A_1]$, $[R_1]$ and $[R_0]$ are calculated by solving Eq. (103) in a least squares sense, then the solution of the generalized eigenvalue problem, defined by Eq. (100), directly yields the complex system poles and the corresponding mode shapes for the structure.

As already mentioned, Eq. (100) is a matrix polynomial that has $2N_0$ poles. In the case where $N_0 > N$, the solution of the eigenvalue problem of dimension $2N_0$ will generate a lot of

computational modes, since only $2N$ physical modes (including the complex conjugate system poles) contribute to the measurement data. Moreover, the solution process for $[A_0]$, $[A_1]$, $[R_1]$ and $[R_0]$ may then become ill conditioned. To solve this problem, the data can first be condensed using principle component analysis. [66,67]

3.2.4 First order model

This section explains the first order model, which is a special case of the second order model which was discussed in previous section. The first order model was developed and implemented at the University of Cincinnati.

Instead of combining Eqs. (92), (93) and (94) to develop a second order model, Eqs. (92) and (93) can be combined into one equation to develop a first order model:

$$\begin{bmatrix} H_d(s) \\ H_e(s) \end{bmatrix} = \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} T(s) + \begin{bmatrix} R_d \\ R_e \end{bmatrix} \quad (104)$$

Substituting Eq. (87) into Eq. (104) and bringing the residual matrix to left hand side:

$$\begin{bmatrix} H_d(s) - R_d \\ s [H_d(s) - \Psi] [L] - R_e \end{bmatrix}_{2N_0 \times N_1} = \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix}_{2N_0 \times 2N} T(s)_{2N \times N_1} \quad (105)$$

Through the same reasoning as in the previous section, it can be shown that if $N_0 \geq 2N$ there are at least N_0 vectors that are orthogonal with respect to the $2N$ vectors in $\begin{bmatrix} \Psi \\ \Lambda \end{bmatrix}$. This property can be expressed in Eq. (106) where the $[B]$ matrix containing the orthogonal vectors is already expressed in terms of submatrices:

$$\begin{bmatrix} B_0 \\ B_1 \end{bmatrix}^T \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} = [0] \quad (106)$$

Since the $2N$ vectors in $\begin{bmatrix} \Psi \\ \Lambda \end{bmatrix}$ appear as pairs of complex conjugate vectors, the orthogonal vectors are monophasic vectors. The matrix $[B]$ can be normalized with respect to $[B_1]$ and Eq. (106) becomes:

$$\begin{bmatrix} A_0 \\ I \end{bmatrix}^T \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} = [0] \quad (107)$$

Equation (107) is a first order matrix polynomial where the matrix coefficients are real coefficients with dimension N_0 . Therefore, the matrix polynomial has N_0 poles. Notice that the matrix polynomial for the second order model was of order 2 with $2N_0$ poles, which could lead to numerical instability. Therefore, by choosing a first order model, this problem is minimized.

Postmultiplying Eq. (107) with the $[T]$ matrix and substituting Eq. (105), the obtained equation becomes:

$$\begin{bmatrix} [A_0]^T & [I] \end{bmatrix}_{N_0 \times 2N_0} \begin{bmatrix} [H_d(s)] - [R_d] \\ s [H_d(s)] - [\Psi] [L] - [R_*] \end{bmatrix}_{2N_0 \times N_1} = [0]_{N_0 \times N_1} \quad (108)$$

or

$$\begin{aligned} [A_0]^T [H_d(s)] &= -s [H_d(s)] + [\Psi] [L] + [R_*] + [A_0]^T [R_d] \\ &= -s [H_d(s)] + [B_1] \end{aligned} \quad (109)$$

where: $[B_1] = [\Psi] [L] + [R_*] + [A_0]^T [R_d]$

This means that the measurement data $[H]$ can be described by a model which is linear in the real unknowns $[A_0]$ and $[B_1]$.

It is possible to reduce the dimensions of the matrices by taking advantage of some system properties. It is known that elements of $[\Psi]$, $[A]$ and $[L]$ occur in complex conjugate pairs. This property can be used advantageously to reduce the dimensions of the different matrices used in the first order model.

If the complex conjugate terms are ignored in the matrices $[\Psi]$, $[A]$, $[T]$ and $[L]$, the dimensions reduce from $2N$ to N and the reduced matrices can be represented by:

$$[\Psi']_{N_0 \times N} \quad [A']_{N \times N} \quad [T'(s)]_{N \times N_1} \quad [L']_{N \times N_1}$$

Using this notation Eq. (105) becomes:

$$\begin{bmatrix} [H_d(s)] - [R_d'] \\ [H_d(s)] - [\Psi'] [L'] - [R_*'] \end{bmatrix}_{2N_0 \times N_1} = \begin{bmatrix} [\Psi'] \\ [\Psi'] [A'] \end{bmatrix}_{2N_0 \times N} [T'(s)]_{N \times N_1} \quad (110)$$

where the matrices $[R_d']$ and $[R_*']$ no longer only compensate for the residuals effects of the out-of-band modes, but also compensate for the error that is introduced by neglecting the complex conjugate term in the different matrices.

The matrix $[\begin{bmatrix} \Psi' \\ \Psi' [A'] \end{bmatrix}]^T$ has only N vectors in a $2N_0$ dimensional space. Therefore, when $N_0 \geq N$, there exists N_0 vectors that are orthogonal to these N vectors. This is expressed in Eq. (111):

$$\begin{bmatrix} [B_0'] \\ [B_1'] \end{bmatrix}^T \begin{bmatrix} [\Psi'] \\ [\Psi'] [A'] \end{bmatrix} = [0] \quad (111)$$

The orthogonal vectors can be normalized. In this case, Eq. (111) becomes:

$$\begin{bmatrix} [A_0'] \\ [I] \end{bmatrix}^T \begin{bmatrix} [\Psi'] \\ [\Psi'] [A'] \end{bmatrix} = [0] \quad (112)$$

The disadvantage of this reduction in the matrix size, when the complex conjugate terms are ignored, is that the matrix $[A_0']$ is no longer a real matrix but a complex matrix. Postmultiplying Eq. (112) by the $[T']$ matrix and substituting Eq. (110) yields:

$$\begin{bmatrix} [A_0']^T & [I] \end{bmatrix}_{N_0 \times 2N_0} \begin{bmatrix} [H_d(s)] - [R_d'] \\ s [H_d(s)] - [\Psi'] [L'] - [R_v'] \end{bmatrix}_{2N_0 \times N_1} = [0]_{N_0 \times N_1} \quad (113)$$

or

$$\begin{aligned} [A_0']^T [H_d(s)] &= -s [H_d(s)] + [\Psi'] [L'] + [A_0']^T [R_d'] + [R_v'] \\ &= -s [H_d(s)] + [B_1'] \end{aligned} \quad (114)$$

where:

$$[B_1'] = [\Psi'] [L'] + [A_0']^T [R_d'] + [R_v']$$

This means that the measurement data $[H]$ can be described by a model which is linear in the complex unknowns $[A_0']$ and $[B_1']$.

3.2.5 Principle Component Reduction

In order to reduce the size of the matrices involved in the Polyreference Frequency Domain Methods, the measured data can be transformed to their principle components by means of a linear transformation^[40,66,67]. This process minimizes not only the computer requirements (storage memory and execution time) but also provides a good way for estimating the number of modes contributing to the response of the structure.

The measurement data $[H]$ and the principle components, represented by $[H^e]$, are interrelated by following pair of linear transformations:

$$[H^e(s)]_{N_e \times N_1} = [C]_{N_e \times N_0} [H(s)]_{N_0 \times N_1} \quad (115)$$

or

$$[H(s)]_{N_0 \times N_1} = [C]^T_{N_0 \times N_e} [H^e(s)]_{N_e \times N_1} \quad (116)$$

where N_e is the number of principle components and is in general much smaller than the number of measurements N_0 .

The principle component data $[H^e]$ will satisfy the same linear model as the actual measurement data $[H]$. Indeed, combining Eq. (82) and (115) results in:

$$\begin{aligned} [H^e] &= [C] [\Psi] e^{[A]t} [L] \\ &= [\Psi^e] e^{[A]t} [L] \end{aligned} \quad (117)$$

Therefore, $[H^e]$ will also satisfy the second order linear model with real coefficients as stated in Eq. (103):

$$\begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \begin{bmatrix} H^e(s) \\ H^e(s) \end{bmatrix} + s \begin{bmatrix} A_1 \\ H^e(s) \end{bmatrix} + s^2 \begin{bmatrix} H^e(s) \\ H^e(s) \end{bmatrix} = s \begin{bmatrix} R_1^e \\ R_0^e \end{bmatrix} \quad (118)$$

where:

- $[R_1^e] = [\Psi^e] [L]$
- $[R_0^e] = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix} [\Psi^e] + [\Psi^e] [A] \begin{bmatrix} L \\ L \end{bmatrix} + [A_0] [R_d'] + [A_1] [R_s'] + [R_s]$

The same algorithm can be applied to the condensed principle component data to calculate the modal parameters. The results for these data will be a set of eigenvalues $[A]$ and eigenvectors $[\Psi]$. Note that from Eq. (82) and (100), $[A]$ remains the same for the principle component data. The mode shapes $[\Psi]$ can be calculated from the mode shapes $[\Psi^e]$ of the principle component data by the following transformation:

$$[\Psi]_{N_0 \times 2N} = [C]_{N_0 \times N_0}^T [\Psi^e]_{N_0 \times 2N} \quad (119)$$

3.2.6 The Modal Participation Matrix

The calculation of the modal participation factors $[L]$ of Eq. (82) can be performed on the actual data as well as on the principle component data. The system of equations to be solved is:

$$\begin{bmatrix} H(s) \end{bmatrix}_{N_0 \times N_1} = \begin{bmatrix} \Psi \end{bmatrix}_{N_0 \times 2N} \begin{bmatrix} s [I] - [A] \end{bmatrix}_{2N \times 2N}^{-1} \begin{bmatrix} L \end{bmatrix}_{2N \times N_1} \quad (120)$$

where the mode shapes $[\Psi]$ and the poles $[A]$ are already known. To keep the modal participation factors global, all output stations (or all principle components) should be included in the calculation.

However, the matrices $[R_0^e]$ and $[R_1^e]$, for the case when a principle component reduction is performed, contain sufficient information to solve for the modal participation factors, since the matrices $[A]$ and $[\Psi]$ are known in this phase of the parameter estimation. From Eq. (118) the matrix $[L]$ can be expressed as a function of the known variables:

$$\begin{bmatrix} L \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \Psi^e \\ A_1 \end{bmatrix} \begin{bmatrix} \Psi^e \\ \Psi^e \end{bmatrix} + \begin{bmatrix} \Psi^e \\ \Psi^e \end{bmatrix} [A] \end{bmatrix}^+ \begin{bmatrix} R_1^e \\ R_0^e \end{bmatrix} \quad (121)$$

To investigate the accuracy of the parameter estimation process, Eq. (84) can now be evaluated in order to curve-fit the measurement data for some or all output and/or input locations. The errors between the actually measured and the synthesized frequency response functions will be minimum in a global linear least squares sense.

3.3 Multiple-Reference Ibrahim Time Domain

3.3.1 Introduction

The Ibrahim Time Domain technique was originally developed for the analysis of single-input/multiple-output free decay or impulse response time domain data [17,28]. This technique can be expanded into the multiple-input/multiple-output algorithm by introducing the relation of mode shapes and modal participation factors.

Principal Component Analysis, Minimal Realization Approach and the generalized inverse technique have been found to be very useful during the development of the Polyreference Frequency Domain [42,43] and the Eigensystem Realization Approach[34]. Principal Component Analysis is used to evaluate the rank of an auto correlation matrix and to aid in the selection of the number of modes. Minimal Realization Algorithm is used to reduce the size of the system matrix of an eigenvalue problem to the approximate rank of the matrix. The generalized inverse technique is used to solve the equation $[B]\{x\}=\{b\}$ with or without the normal equations method. Since these mathematical approaches are also used in the development of multiple-reference Ibrahim Time Domain, this method is similar to the Polyreference Frequency Domain and the Eigensystem Realization Algorithm. It can be said that Multiple-Reference Ibrahim Time Domain is a time domain version of Polyreference Frequency Domain or another approach of Eigensystem Realization Algorithm.

3.3.2 Theory

The mathematical derivation starts from the time domain equations:

$$[h(t)] = [\Psi] [e^{\lambda_i t}] [L] \quad (122)$$

where:

- $[\Psi] = N_o$ by $2N$ mode shape matrix
- $[L] = 2N$ by N_i modal participation factor matrix
- N = number of modes
- N_o = number of outputs
- N_i = number of inputs

Expanding these matrices with respect to sampling times:

$$[H_{\Delta}(t)] = [\bar{\Psi}] [e^{\lambda_i \Delta t}] [\bar{L}] \quad (123)$$

where:

$$\bullet [H_{\Delta}(t)] = \begin{bmatrix} [h(t)] & [h(t+\Delta t)] & \cdots & [h(t+(k-1)\Delta t)] \\ [h(t+\Delta t)] & [h(t+2\Delta t)] & \cdots & [h(t+k\Delta t)] \\ \vdots & \vdots & \ddots & \vdots \\ [h(t+(i-1)\Delta t)] & [h(t+i\Delta t)] & \cdots & [h(t+(i+k-2)\Delta t)] \end{bmatrix}$$

$$\bullet [\bar{\Psi}] = \begin{bmatrix} [\Psi] \\ [\Psi] e^{\lambda_r \Delta t} \\ \vdots \\ [\Psi] e^{\lambda_r (i-1) \Delta t} \end{bmatrix}$$

$$\bullet [\bar{L}] = \left[[L] \mid e^{\lambda_r \Delta t} [L] \mid \dots \mid e^{\lambda_r (k-1) \Delta t} [L] \right]$$

- i = number of time shifts in column direction
- k = number of time shifts in row direction

The matrix $[H_{ik}(t)]$ is called Hankel's matrix. Note that the matrices $[H_{ik}(t)]$ and $[\bar{\Psi}]$ are normally rectangular matrices, therefore, only their generalized inverse exists.

Using one sampling time shift on Eq. (123):

$$[H_{ik}(t + \Delta t)] = [\bar{\Psi}] e^{\lambda_r \Delta t} e^{\lambda_r t} [\bar{L}] \quad (124)$$

Premultiply Eq. (123) by $[\bar{\Psi}]^+$, the generalized inverse of the $[\bar{\Psi}]$ matrix.

$$[\bar{\Psi}]^+ [H_{ik}(t)] = e^{\lambda_r t} [\bar{L}] \quad (125)$$

The right hand side of this equation is a common term in Eq. (124). Substituting Eq. (125) into Eq. (124) yields

$$[H_{ik}(t + \Delta t)] = [\bar{\Psi}] e^{\lambda_r \Delta t} [\bar{\Psi}]^+ [H_{ik}(t)] \quad (126)$$

Denoting $[\bar{\Psi}] e^{\lambda_r \Delta t} [\bar{\Psi}]^+$ as matrix $[B]$:

$$[H_{ik}(t + \Delta t)] = [B] [H_{ik}(t)] \quad (127)$$

This is a matrix homogeneous difference equation which also can be derived via Prony's algorithm. The matrix $[B]$, is the matrix of coefficient matrices of the matrix polynomial equation.

To obtain the eigenvalue problem, postmultiply Eq. (126) by the generalized inverse of $[[\bar{\Psi}]^+ [H_{ik}(t)]]$, since $[[\bar{\Psi}]^+ [H_{ik}(t)]]$ is in general not a square matrix.

$$[H_{ik}(t + \Delta t)] [[\bar{\Psi}]^+ [H_{ik}(t)]]^+ = [\bar{\Psi}] e^{\lambda_r \Delta t} \quad (128)$$

Using the property:

$$[[\bar{\Psi}]^+ [H_{ik}(t)]]^+ = [H_{ik}(t)]^+ [\bar{\Psi}]$$

Equation (128) can be simplified to :

$$[B] [\bar{\Psi}] = [\bar{\Psi}] e^{\lambda_r \Delta t} = [\bar{\Psi}] e^{s_r \Delta t} \quad (129)$$

where:

$$\bullet [B] = [H_{ik}(t + \Delta t)] [H_{ik}(t)]^+$$

Equation (129) is an eigenvalue problem; the size of the system matrix $[B]$ is $N_o i \times N_o i$. The eigenvalues that are found are of the form $z_r = e^{\lambda_r \Delta t}$.

Mode shapes are calculated as eigenvectors of Eq. (129). The eigenvalues are translated to damped natural frequencies and damping factors as follows ($\lambda_r = \sigma_r + i\omega_r$):

$$\sigma_r = \text{Re}(\ln z_r) / \Delta t \quad \omega_r = \text{Im}(\ln z_r) / \Delta t \quad (130)$$

3.3.2.1 Minimal Realization Approach

The minimal realization approach is a technique that is used to reduce the size of the system matrix $[B]$ of an eigenvalue problem to the approximate rank of the matrix.

The matrix $[H_{ik}(t)]$ is decomposed as follows:

$$[H_{ik}(t)]_{N_o i \times N_i k} = [P]_{N_o i \times N_o i} [D]_{N_o i \times N_i k} [Q]^T_{N_i k \times N_i k} \quad (131)$$

Denoting the rank of the matrix $[H_{ik}(t)]$ as p , the matrix $[D]$ is written as follows:

$$[D] = \begin{bmatrix} d_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \quad (132)$$

These diagonal terms are called singular values. The number of significant singular values and, therefore, the rank p of the matrix, may be evaluated using the definition of condition number and/or the ratio of successive singular values. This procedure is explained as follows:

Condition numbers are defined as follows:

$$CN = \frac{d_1}{d_k} \quad (133)$$

Since the first singular value is the largest numerical value, the condition number is always larger than zero. If the condition number is evaluated for different submatrices of the $[D]$ matrix by taking successive singular values, the rank p of the matrix is defined when the condition number first becomes very large.

The rank p can also be determined from the ratio of successive singular values calculated as follows:

$$\text{ratio} = \frac{d_k}{d_{k+1}} \quad (134)$$

If the ratio of successive singular values is evaluated, this ratio will be approximately equal to one when the successive singular values are significant. This will also be true when the successive values are trivial. At the transition between these two subsets of the singular values, the ratio of successive singular values will peak indicating the rank p ($p = k$). Then, the dimensionality of Eq.(131) is denoted as follows:

$$[H_{ik}(t)]_{N_o i \times N_i k} = [P]_{N_o i \times p} [D]_{p \times p} [Q]^T_{p \times N_i k} \quad (135)$$

This process is called the Principal Component Analysis. Theoretically, this rank number p is equal

to twice of the number of modes.

The minimal realization process begins from this point. Premultiplying Eqs.(123) and (124) by the transfer matrix $[P]^T$:

$$[P]^T [H_{ik}(t)] = [P]^T [\bar{\Psi}]^T e^{\lambda_i t} [\bar{L}] \quad (136)$$

$$[P]^T [H_{ik}(t+\Delta t)] = [P]^T [\bar{\Psi}]^T e^{\lambda_i \Delta t} e^{\lambda_i t} [\bar{L}] \quad (137)$$

Note that the matrix product $[P]^T [\bar{\Psi}]$ represents the reduced mode shapes and can be represented by $[\bar{\Psi}]$. Premultiplying Eq.(136) by the generalized inverse of $[\bar{\Psi}]$

$$[\bar{\Psi}]^+ [P]^T [H_{ik}(t)] = e^{\lambda_i t} [\bar{L}] \quad (138)$$

The right hand side of this equation is a common term in Eq.(137). Substituting Eq.(138) into Eq.(137) yields:

$$[P]^T [H_{ik}(t+\Delta t)] = [P]^T [\bar{\Psi}]^T e^{\lambda_i \Delta t} [\bar{\Psi}]^+ [P]^T [H_{ik}(t)] \quad (139)$$

Simplifying:

$$[H'_{ik}(t+\Delta t)] = [\bar{\Psi}]^T e^{\lambda_i \Delta t} [\bar{\Psi}]^+ [H'_{ik}(t)] \quad (140)$$

where:

- $[H'_{ik}(t+\Delta t)] = [P]^T [H_{ik}(t+\Delta t)]$
- $[H'_{ik}(t)] = [P]^T [H_{ik}(t)]$

The normal equation of Eq.(140) is:

$$[H'_{ik}(t+\Delta t)] [H'_{ik}(t)]^T = [\bar{\Psi}]^T e^{\lambda_i \Delta t} [\bar{\Psi}]^+ [H'_{ik}(t)] [H'_{ik}(t)]^T \quad (141)$$

To obtain the system matrix of an eigenvalue problem, postmultiply Eq.(141) by the generalized inverse, $[[H'_{ik}(t)] [H'_{ik}(t)]^T]^+$

$$[H'_{ik}(t+\Delta t)] [H'_{ik}(t)]^T [[H'_{ik}(t)] [H'_{ik}(t)]^T]^+ = [\bar{\Psi}]^T e^{\lambda_i \Delta t} [\bar{\Psi}]^+ \quad (142)$$

Postmultiplying both sides of Eq. (142) by $[\bar{\Psi}]$ and simplifying gives:

$$[B] [\bar{\Psi}] = [\bar{\Psi}]^T e^{\lambda_i \Delta t} [\bar{\Psi}] = [\bar{\Psi}]^T e^{\lambda_i \Delta t} \quad (143)$$

where:

- $[B] = [H'_{ik}(t+\Delta t)] [H'_{ik}(t)]^T [[H'_{ik}(t)] [H'_{ik}(t)]^T]^+$

This is an eigenvalue problem; the size of the system matrix $[B]$ is now $p \times p$, which is smaller than $N_o i \times N_o i$.

This is the Minimal Realization Approach. The eigenvectors are translated into the mode shapes as follows:

$$[\bar{\Psi}] = [P] [\bar{\Psi}] \quad (144)$$

Transfer matrix $[P]$ is obtained from the correlation matrix by solving the following eigenvalue

problem.

$$[H_{ik}(t)][H_{ik}(t)]^T [P] = [P]^T D^2 \quad (145)$$

This algorithm is referred to as the Multiple-Reference Ibrahim Time Domain.

3.3.3 Modified Multiple-Reference Ibrahim Time Domain

In practice, the matrix size of $[H_{ik}(t)][H_{ik}(t)]^T$ ($N_{oi} \times N_{oi}$) is too large to be installed in many mini-computers. Only a few measurement data can be included. This difficulty is overcome by some matrix manipulations.

Transposing Eq.(123) and Eq.(124)

$$[H_{ik}(t)]^T = [\bar{L}]^T [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (146)$$

$$[H_{ik}(t+\Delta t)]^T = [\bar{L}]^T [e^{\lambda_i \Delta t}] [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (147)$$

Decomposing the matrix $[H_{ik}(t)]^T$ as follows:

$$[H_{ik}(t)] = [P]^T D [Q]^T \quad (148)$$

or

$$[H_{ik}(t)]^T = [Q]^T D [P]^T \quad (149)$$

Using this matrix $[Q]$ as the transformation matrix, Eq.(146) and Eq.(147) can be modified as follows:

$$[Q]^T [H_{ik}(t)]^T = [Q]^T [\bar{L}]^T [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (150)$$

or,

$$[H'_{ik}(t)]^T = [\bar{L}']^T [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (151)$$

where:

- $[H'_{ik}(t)] = [Q]^T [H_{ik}(t)]^T$
- $[\bar{L}']^T = [Q]^T [\bar{L}]^T$ = reduced modal participation matrix

Likewise :

$$[Q]^T [H_{ik}(t+\Delta t)]^T = [Q]^T [\bar{L}]^T [e^{\lambda_i \Delta t}] [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (152)$$

or,

$$[H'_{ik}(t+\Delta t)]^T = [\bar{L}']^T [e^{\lambda_i \Delta t}] [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (153)$$

where:

- $[H'_{ik}(t+\Delta t)] = [Q]^T [H_{ik}(t+\Delta t)]^T$

Premultiplying Eq.(151) by $[[\bar{L}']^T]^+$:

$$[[\bar{L}^T]^T]^T [H_{ik}(t)] = [e^{\lambda_i t}] [\bar{\Psi}]^T \quad (154)$$

The right hand side of Eq.(154) is a common term in Eq.(152). Substituting Eq.(154) into (153) yields

$$[H_{ik}(t+\Delta t)]^T = [\bar{L}^T]^T [e^{\lambda_i \Delta t}] [[\bar{L}^T]^T]^T [H_{ik}(t)] \quad (155)$$

The normal equation of Eq.(155) is

$$[H_{ik}(t+\Delta t)] [H_{ik}(t)]^T = [\bar{L}^T]^T [e^{\lambda_i \Delta t}] [[\bar{L}^T]^T]^T [H_{ik}(t)] [H_{ik}(t)]^T \quad (156)$$

Taking a generalized inverse of $[[H_{ik}(t)] [H_{ik}(t)]^T]$ to obtain the system matrix of an eigenvalue problem:

$$[H_{ik}(t+\Delta t)] [H_{ik}(t)]^T \left[[H_{ik}(t)] [H_{ik}(t)]^T \right]^+ = [\bar{L}^T]^T [e^{\lambda_i \Delta t}] [[\bar{L}^T]^T]^T \quad (157)$$

Postmultiplying Eq. (157) by $[\bar{L}^T]^T$ and simplifying

$$[B] [\bar{L}^T]^T = [\bar{L}^T]^T [e^{\lambda_i \Delta t}]$$

where:

$$[B] = [H_{ik}(t+\Delta t)] [H_{ik}(t)]^T \left[[H_{ik}(t)] [H_{ik}(t)]^T \right]^+$$

This is the eigenvalue problem to be solved; the size of this system matrix $[B]$ is $p \times p$. In this algorithm, modal participation factors are calculated as eigenvectors. They are transformed as follows:

$$[\bar{L}]^T = [Q] [\bar{L}^T]^T \quad (158)$$

The transfer matrix $[Q]$ is calculated from the correlation matrix by the following eigenvalue problem:

$$[H_{ik}(t)]^T [H_{ik}(t)] [Q] = [Q] [D^2] \quad (159)$$

The size of the matrix $[[H_{ik}(t)]^T [H_{ik}(t)]]$ is $N_i k \times N_i k$. This size is usually smaller than $N_o i \times N_o i$ in Eq.(145), and this algorithm is, therefore, easy to implement on mini-computers. This algorithm is referred to as the Modified Multiple-Reference Ibrahim Time Domain. The modal vectors can be calculated by using the partial fraction method or by setting up an overdetermined system using Eq. (123). This overdetermined system, that has as unknowns only the modal vectors, can be solved in a least squares sense. These two methods have been explained in detail in the Polyreference Time Domain Technique.

3.3.4 Conclusions

Two approaches of Multiple-Reference Ibrahim Time Domain modal parameter estimation techniques, named Multiple-Reference Ibrahim Time Domain and Modified Multiple-Reference Ibrahim Time Domain, were developed, and attained sufficient accuracy for modal parameter estimation. As an additional tool to predict the convergence of the calculated modal parameters, the Modal Confidence Factor (MCF) was also included.

There is a limitation on the developments of the Multiple-Reference Ibrahim Time Domain. This technique involves the formulation of a very large system matrix for an eigenvalue problem in

order to include all measurement data. Since several data time shifts must be taken in the system matrix in order to increase the rank of the system matrix, this matrix is sometimes larger than 100×100 . This limitation is avoided in the Modified Multiple-Reference Ibrahim Time Domain algorithm. The size of the system matrix is always fixed to an arbitrary dimension; usually this size is limited to 40×40 which yields 20 modes.

The condition number and rank chart is also developed for the selection of the number of modes and the effective rank in order to better estimate the modal parameters. The rank chart usually indicates the number of modes in the measurement data within a certain frequency range, and the rank number at this mode indication normally satisfies the required accuracy for modal parameters. Where more precise results are required, the selection of a higher rank will yield better results for modal parameter estimates and convergence of MCFs.

The Multiple-Reference Ibrahim Time Domain algorithm calculates resonant frequencies, damping factors and mode shapes during the first stage of modal parameter estimation. The residues are obtained with respect to a specific reference point using the normal equations method at the second stage. Since this process is done in the time domain, residual effects are not considered in this algorithm.

The Modified Multiple-Reference Ibrahim Time Domain obtains modal participation factors during the first stage estimation the same as the Polyreference Time Domain technique. These modal participation factors are identical to those of Polyreference Time Domain; therefore, the second stage residue calculation is performed by a general polyreference residue calculation algorithm. This algorithm includes time domain and frequency domain normal equation methods, Section 3.1, and also has capability to obtain real normal modes. These options, therefore, are also available with the Modified Multiple-Reference Ibrahim Time Domain technique.

Time domain modal parameter estimation algorithms usually are more numerically robust solutions than frequency domain algorithms for lightly damped systems. The presented modal parameter estimation techniques in this section, Multiple-Reference Ibrahim Time Domain and Modified Multiple-Reference Ibrahim Time Domain, both use time domain data. For this reason, these methods are numerically very robust and consequently the Modified Multiple-Reference Ibrahim Time Domain technique is comparable to the Polyreference Time Domain technique. The Modified Multiple-Reference Ibrahim Time Domain technique is able to detect repeated roots, since it is a multiple-reference algorithm, and has the advantage to produce less computational poles in comparison with the Polyreference Time Domain technique.

3.4 Multiple-Reference Orthogonal Polynomial

3.4.1 Introduction

The recent trend of experimental modal analysis is to apply a multiple excitation force and to measure multiple response functions [51]. This approach results in a more uniform distribution of the excitation energy applied to the structure, and hence, more uniform response level; thus, higher quality and more consistent data can be obtained. Another practical advantage is that the data acquisition time can be reduced if the appropriate equipment is available.

In the last few years, several "multiple-reference" parameter estimation methods have been developed to take advantage of the consistency of the frequency response measurements relative to different reference locations. Such methods include Polyreference Time and Frequency Domain Method[18,42], Eigensystem Realization Algorithm[34], and Direct Parameter Estimation[68]. With these methods, great improvement in identifying experimental modal models have been achieved. Moreover, the multiple reference approach can be used to identify highly coupled and repeated modes, which is difficult or even impossible to attack for single reference approach.

Most of the early multiple reference work was based upon time domain algorithms. These algorithms were characterized by numerical stability, ease of use, and were particularly good at analyzing wide frequency bands. However, there are several important limitations:

- Averaging is often performed in the frequency domain and Fourier transformed into the time domain for analysis. For this case, time domain "leakage" can be a problem, particularly for small frequency bands.
- These time domain methods are based upon sampled data, with the result that all of the computed frequencies fall into the Nyquist band (from 0 to 1/2 of the sampling frequency). This can make it more difficult to sort out computational modes. This is becoming increasingly more important for the application of active control of large structures (Space Station, etc) where modal/servo systems exist. These systems have both low and high damped poles making it particularly difficult to separate the computational poles from the real poles.
- Sine testing is becoming more popular with the advent of systems capable of making many simultaneous measurements. In sine testing, the frequency interval may not be equal making it difficult to transform into the time domain.
- It is difficult to weight the data in a given frequency band with time domain methods.

As a result, there has been a great deal of recent interest in developing multiple reference frequency domain algorithms as witnessed by the recent development of the Frequency Domain Polyreference and Eigensystem Realization Algorithm (ERA). The rational fraction polynomial is one of the frequency domain algorithms, which can be extended to multiple references formulation with the advantage of increasing frequency resolution by the use of multiple reference data.

The general formulation for the Frequency Response Function (FRF) is in the form of rational fraction polynomials. In the frequency domain parameter estimation, some numerical difficulty prevents one from using this simple formulation directly. One of the difficulties is that the formulated equations are usually ill-conditioned, resulting from the dynamic range of the polynomial terms. Thus, one of the natural extensions is to use orthogonal polynomials instead of ordinary power polynomials. As the property of orthogonality is exercised, the normal equation used in the solution process is easily decoupled, and the size of the system matrix is greatly reduced.

In this section, a rational fraction formulation in orthogonal polynomials, based on the earlier

derivation by Richardson and Formenti^[37], and Hou^[69], is extended to a matrix formulation, ie. A matrix Auto-Regressive Moving Average (ARMA) model in the Laplace domain^[40]. This method can include multiple reference FRF calculations and thus is able to detect highly coupled and repeated modes. The matrix polynomial coefficients of the AR part is first estimated in least square sense, and then the system poles can be solved from the characteristic equation in companion matrix form. The residue matrices can be calculated from the matrix polynomial coefficients of the MA part, which is estimated in the same least square formulation as AR part. Therefore, the global estimation of both residues as well as poles is achieved.

A singular value decomposition method has been used to develop a complex mode indicator function (CMIF)^[70], which can be used to help determine the number of poles before the analysis. The CMIF is formed by performing a singular value decomposition of all of the FRF's at each spectral line. The singular values are plotted as functions of frequency. These singular values can also be used to generate appropriate weighting functions in order to enhance some certain frequency band and/or to enhance weak modes.

3.4.2 Theory

For a N degree-of-freedom linear system with viscous damping, one can formulate an Auto-Regressive Moving Average (ARMA) model of order (m,n) as

$$\sum_{k=0}^m [a_k] s^k [H(s)] = \sum_{k=0}^n [b_k] s^k \quad (160)$$

or in orthogonal polynomial basis

$$\sum_{k=0}^m [\alpha_k] [p_k] [H(s)] = \sum_{k=0}^n [\beta_k] [p_k] \quad (161)$$

where:

- N degree-of-freedom of the system or the number of modes.
- N_i number of excitation location.
- N_o number of response point.
- m order of matrix polynomial chosen in the Auto-Regressive part. $mN_i \geq N$.
- n order of matrix polynomial chosen in the Moving-Average part. $n \geq m+2$.
- $[H(s)]$ transfer function matrix of size N_i by N_o .
- $[a_k]$ matrix polynomial coefficient of size N_i by N_i .
- $[b_k]$ matrix polynomial coefficient of size N_i by N_o .
- $[\alpha_k]$ matrix polynomial coefficient for orthogonal polynomials of size N_i by N_i .
- $[\beta_k]$ matrix polynomial coefficient for orthogonal polynomials of size N_i by N_o .
- $[p_k]$ orthogonal polynomial of order k , which is a scalar function.

Since $[\alpha_k]$ and $[\beta_k]$ are both unknown parameters in Eq.(161), one can always choose one of the matrix coefficients to be the identity matrix. By choosing $[\alpha_m] = [I]$, Eq.(161) can be arranged and rewritten in matrix form for all frequency channels.

$$\begin{bmatrix} [H]^H [P_0]^H & \dots & [H]^H [P_{m-1}]^H & -[P_0']^H & \dots & -[P_n']^H \\ \vdots & & \vdots & & & \\ \vdots & & \vdots & & & \\ \vdots & & \vdots & & & \\ \vdots & & \vdots & & & \end{bmatrix} \begin{bmatrix} [\alpha_0]^T \\ [\alpha_1]^T \\ \vdots \\ [\alpha_{m-1}]^T \\ [\beta_0]^T \\ [\beta_1]^T \\ \vdots \\ [\beta_n]^T \end{bmatrix} = \begin{bmatrix} -[H]^H [P_m]^H \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \quad (162)$$

(for $i=1,2,\dots, l$ frequency channel)

where:

- $[P_k]^H$ product of p_k^* and identity matrix $[I]$ of size N_i by N_i .
- $[P_k']^H$ product of p_k^* and identity matrix $[I]$ of size N_o by N_o .

In order to solve for the matrix coefficients, one can formulate the normal equation and further decouple it by taking the advantage of the orthogonal property.

$$\begin{bmatrix} [Y] & [X]^T \\ [X] & [I] \end{bmatrix} \begin{bmatrix} [\alpha] \\ [\beta] \end{bmatrix} = \begin{bmatrix} [Y_n] \\ [X_n] \end{bmatrix} \quad (163)$$

or

$$([X]^T [X] - [Y]) [\alpha] = [X]^T [X_n] - [Y_n] \quad (164)$$

$$[\beta] = [X_n] - [X] [\alpha] \quad (165)$$

where:

$$\bullet [X] = \text{Re} \begin{bmatrix} -\sum_{i=1}^l [P_0'] [H]^H [P_0]^H & \dots & -\sum_{i=1}^l [P_0'] [H]^H [P_{m-1}]^H \\ \vdots & & \vdots \\ -\sum_{i=1}^l [P_n'] [H]^H [P_0]^H & \dots & -\sum_{i=1}^l [P_n'] [H]^H [P_{m-1}]^H \end{bmatrix}$$

$$\bullet [X_n] = \text{Re} \begin{bmatrix} \sum_{i=1}^l [P_0']^H [H]^H [P_m]^H \\ \vdots \\ \sum_{i=1}^l [P_n'] [H]^H [P_m]^H \end{bmatrix}$$

$$\bullet [Y] = \text{Re} \begin{bmatrix} \sum_{i=1}^l [P_0] [H] [H]^H [P_0]^H & \dots & \sum_{i=1}^l [P_0] [H] [H]^H [P_{m-1}]^H \\ \vdots & & \vdots \\ \sum_{i=1}^l [P_{m-1}] [H] [H]^H [P_0]^H & \dots & \sum_{i=1}^l [P_{m-1}] [H] [H]^H [P_{m-1}]^H \end{bmatrix}$$

$$\bullet [Y_n] = \text{Re} \begin{bmatrix} - \sum_{i=1}^l [P_0][H][H]^H[P_m]^H \\ \vdots \\ - \sum_{i=1}^l [P_{m-1}][H][H]^H[P_m]^H \end{bmatrix}$$

$$\bullet [\alpha] = \begin{bmatrix} [\alpha_0]^T \\ \vdots \\ [\alpha_{m-1}]^T \end{bmatrix} \quad [\beta] = \begin{bmatrix} [\beta_0]^T \\ \vdots \\ [\beta_n]^T \end{bmatrix}$$

Note here that the size of the system matrix in Eq.(164) is mN_i by mN_i , which is significantly smaller than the size of the previous system matrix in Eq.(163).

The orthogonal polynomial with any weighting function $q(s)$ is defined and normalized as:

$$\sum_{i=1}^l p_j(s_i) q(s_i) p_k(s_i)^* = \delta_{jk} \quad (166)$$

A step by step procedure for generating the orthogonal polynomials is developed by Forsythe^[73]. A simplified Forsythe orthogonal polynomials, namely Real Half-Function Orthogonal Polynomials^[37], for complex variable function can be derived as:

$$p_k(\omega_i) = (j)^k S_k(\omega_i) \quad (167)$$

Normalized function

$$S_{-1}(\omega_i) = 0$$

$$S_0(\omega_i) = \frac{1}{d_0}$$

$$d_0 = \left(2 \sum_{i=1}^l q_i \right)^{\frac{1}{2}}$$

$$S_1(\omega_i) = \frac{R_1(\omega_i)}{d_1}$$

$$d_1 = \left(2 \sum_{i=1}^l R_1(\omega_i)^2 q_i \right)^{\frac{1}{2}}$$

$$S_2(\omega_i) = \frac{R_2(\omega_i)}{d_2}$$

Nonnormalized function

$$R_{-1}(\omega_i) = 0$$

$$R_0(\omega_i) = 1$$

$$R_1(\omega_i) = \omega_i S_0(\omega_i)$$

$$\nu_0 = 0$$

$$R_2(\omega_i) = \omega_i S_1(\omega_i) - \nu_1 S_0(\omega_i)$$

$$d_2 = \left(2 \sum_{i=1}^l R_2(\omega_i)^2 q_i \right)^{\frac{1}{2}}$$

$$\vdots$$

$$S_k(\omega_i) = \frac{R_k(\omega_i)}{d_k}$$

$$d_k = \left(2 \sum_{i=1}^l R_k(\omega_i)^2 q_i \right)^{\frac{1}{2}}$$

$$\nu_1 = 2 \sum_{i=1}^l \omega_i S_1(\omega_i) S_0(\omega_i) q_i$$

$$\vdots$$

$$R_k(\omega_i) = \omega_i S_{k-1}(\omega_i) - \nu_{k-1} S_{k-2}(\omega_i)$$

$$\nu_{k-1} = 2 \sum_{i=1}^l \omega_i S_{k-1}(\omega_i) S_{k-2}(\omega_i) q_i$$

A transformation matrix $[\Gamma]$ can also be calculated in the generating procedure. Thus the power polynomial coefficients can easily be calculated from orthogonal polynomial coefficients $[\alpha]$ and $[\beta]$.

$$\begin{Bmatrix} b_0 \\ \vdots \\ b_n \end{Bmatrix}_{ik} = [\Gamma] \begin{Bmatrix} \beta_0 \\ \vdots \\ \beta_n \end{Bmatrix}_{ik} \quad \text{and} \quad \begin{Bmatrix} a_0 \\ \vdots \\ a_m \end{Bmatrix}_{ik} = [\Gamma] \begin{Bmatrix} \alpha_0 \\ \vdots \\ \alpha_m \end{Bmatrix}_{ik} \quad (168)$$

where, the subscript ik indicate only one of the entry ik of the matrix polynomial coefficients is considered.

$$[\Gamma] = \begin{bmatrix} \frac{1}{d_0} & 0 & (j)^2 \left(-\frac{\gamma_{00}\nu_1}{d_2} \right) & 0 & (j)^4 \left(-\frac{\gamma_{02}\nu_3}{d_4} \right) & \dots \\ 0 & \frac{1}{d_0 d_1} & 0 & (j)^2 \left(\frac{\gamma_{02} - \gamma_{11}\nu_2}{d_3} \right) & 0 & \dots \\ 0 & 0 & \frac{1}{d_0 d_1 d_2} & 0 & (j)^2 \left(-\frac{\gamma_{13} - \gamma_{22}\nu_3}{d_4} \right) & \dots \\ 0 & 0 & 0 & \frac{1}{d_0 d_1 d_2 d_3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

Once the $[a_k]$'s are found, one can solve for the poles, natural frequencies and dampings characteristics, by setting the determinant of the characteristic equation equal to zero.

$$\det \left(\sum_{k=0}^m [a_k] s^k \right) = 0 \quad (169)$$

Instead of solving the matrix characteristic equation, one can reformulate Eq.(169) in the companion matrix form and solve the eigenvalue problem for the poles.

$$\begin{bmatrix} -[a_{n-1}] & -[a_{n-2}] & \cdots & -[a_0] \\ [I] & 0 & \cdots & 0 \\ 0 & [I] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [I] & 0 \end{bmatrix} \begin{bmatrix} s^{n-1}\{v\} \\ s^{n-2}\{v\} \\ s^{n-3}\{v\} \\ \vdots \\ s^0\{v\} \end{bmatrix} = s \begin{bmatrix} s^{n-1}\{v\} \\ s^{n-2}\{v\} \\ s^{n-3}\{v\} \\ \vdots \\ s^0\{v\} \end{bmatrix} \quad (170)$$

Note that in this formulation $\{v\}$ is one of the columns of the modal participation matrix, which is proportional to the modal coefficients at the reference degrees of freedom for one mode.

Assuming that no exact repeated roots exist, which is always the case, then the residue matrix for mode r can be defined as

$$[A_r] \equiv \lim_{s \rightarrow \lambda_r} (s - \lambda_r) [H(s)] \quad (171)$$

by applying Eq.(160)

$$[A_r] = \lim_{s \rightarrow \lambda_r} (s - \lambda_r) [a(s)]^{-1} [b(s)] \quad (172)$$

where:

$$\bullet [a(s)] = \sum_{k=0}^m [a_k] s^k$$

$$\bullet [b(s)] = \sum_{k=0}^n [b_k] s^k$$

Since $[a(s)]$ is singular as $s \rightarrow \lambda_r$, one can formulate a Singular Value Decomposition to determine the generalize inverse of $[a(s)]$

$$[a(s)] = [U(s)] [\Sigma(s)] [V(s)]^H \quad (173)$$

or

$$[a(s)] = \sum_{i=1}^{N_1} \sigma_i(s) \{u_i(s)\} \{v_i(s)\}^H \quad (174)$$

where:

$$\bullet [\Sigma(s)] = \begin{bmatrix} \sigma_1(s) & & & \\ & \sigma_2(s) & & \\ & & \ddots & \\ & & & \sigma_{N_1}(s) \end{bmatrix} \quad \text{singular value matrix, which is a diagonal matrix.}$$

$$\bullet [U(s)] = [\{u_1(s)\} \{u_2(s)\} \cdots \{u_{N_1}(s)\}] \quad \text{left singular matrix, which is a unitary matrix.}$$

$$\bullet [U(s)]^H [U(s)] = [I]$$

$$\bullet [V(s)] = [\{v_1(s)\} \{v_2(s)\} \cdots \{v_{N_1}(s)\}] \quad \text{right singular matrix, which is a unitary matrix.}$$

$$\bullet [V(s)]^H [V(s)] = [I]$$

The generalized inverse is obtained as

$$[a(s)]^{-1} = [V(s)] [\Sigma(s)]^{-1} [U(s)]^H \quad (175)$$

or

$$[a(s)]^{-1} = \sum_{i=1}^{N_1} \sigma_i(s)^{-1} \{v_i(s)\} \{u_i(s)\}^H \quad (176)$$

Applying Eq. (176) to the indeterminate term in Eq.(172):

$$\lim_{s \rightarrow \lambda_r} (s - \lambda_r) [a(s)]^{-1} = \lim_{s \rightarrow \lambda_r} (s - \lambda_r) \sum_{i=1}^{N_1} \sigma_i(s)^{-1} \{v_i(s)\} \{u_i(s)\}^H \quad (177)$$

If Eq.(174) is evaluated at $s = \lambda_r$, at least one of the singular values, $\sigma_i(\lambda_r)$, must be zero, which indicates that the matrix $[a(\lambda_r)]$ is not of full rank. This is consistent with the inability to determine the eigenvector in an absolute sense. For the case of repeated roots at $s = \lambda_r$, more than one of the singular values will be zero. For the case of no repeated roots, only one of the singular values, $\sigma_i(\lambda_r)$, will be zero. Thus, Eq.(177) reduces to:

$$\lim_{s \rightarrow \lambda_r} (s - \lambda_r) [a(s)]^{-1} = \lim_{s \rightarrow \lambda_r} \frac{(s - \lambda_r)}{\sigma_i(s)} \{v_i(\lambda_r)\} \{u_i(\lambda_r)\}^H \quad (178)$$

For the identity that $\sigma_i(\lambda_r) = 0$, the reciprocal of the indeterminate term in Eq.(178) can be expressed as:

$$\lim_{s \rightarrow \lambda_r} \frac{\sigma_i(s)}{s - \lambda_r} = \lim_{s \rightarrow \lambda_r} \frac{\sigma_i(s) - \sigma_i(\lambda_r)}{s - \lambda_r} = \left. \frac{d\sigma_i(s)}{ds} \right|_{s=\lambda_r} \quad (179)$$

Therefore, Eq.(178) becomes:

$$\lim_{s \rightarrow \lambda_r} (s - \lambda_r) [a(s)]^{-1} = \left(\left. \frac{d\sigma_i(s)}{ds} \right|_{s=\lambda_r} \right)^{-1} \{v_i(\lambda_r)\} \{u_i(\lambda_r)\}^H \quad (180)$$

In order to determine $\left. \frac{d\sigma_i(s)}{ds} \right|_{s=\lambda_r}$, one can take the derivative of Eq.(174) as

$$\left. \frac{d[a(s)]}{ds} \right|_{s=\lambda_r} = \sum_{i=1}^{N_1} \left(\frac{d\sigma_i(s)}{ds} \{u_i(s)\} \{v_i(s)\}^H + \sigma_i(s) \frac{d\{u_i(s)\}}{ds} \{v_i(s)\}^H + \sigma_i(s) \{u_i(s)\} \frac{d\{v_i(s)\}^H}{ds} \right) \Big|_{s=\lambda_r} \quad (181)$$

Then, by pre-multiplying Eq.(181) by $\{u_i(\lambda_r)\}^H$, post-multiplying by $\{v_i(\lambda_r)\}$, and using the orthogonality property of a unitary matrix, only one term remains on the right hand side.

$$\{u_i(\lambda_r)\}^H \left. \frac{d[a(s)]}{ds} \right|_{s=\lambda_r} \{v_i(\lambda_r)\} = \left. \frac{d\sigma_i(s)}{ds} \right|_{s=\lambda_r} \quad (182)$$

Substituting this result into Eq.(180),

$$\lim_{s \rightarrow \lambda_r} (s - \lambda_r) [a(s)]^{-1} = \frac{\{v_i(\lambda_r)\} \{u_i(\lambda_r)\}^H}{\{u_i(\lambda_r)\}^H \frac{d[a(s)]}{ds} \big|_{s=\lambda_r} \{v_i(\lambda_r)\}} \quad (183)$$

Therefore, the residue matrix in Eq.(172) can be calculated as,

$$[A_r] = \frac{\{v_i(\lambda_r)\} \{u_i(\lambda_r)\}^H}{\{u_i(\lambda_r)\}^H \frac{d[a(s)]}{ds} \big|_{s=\lambda_r} \{v_i(\lambda_r)\}} [b(\lambda_r)] \quad (184)$$

3.4.3 Conclusions

The Multiple-Reference Orthogonal Polynomial has demonstrated some attractive characteristics for modal parameter estimation. Some of these are:

- It can use any frequency spacing data, and thus is suitable for step sine measurement data.
- Works well in narrow frequency bands
- Global parameter estimation for all FRF measurement.
- It includes multiple reference measurements calculation and, therefore, it is good for system with highly coupled and repeated modes
- Easy to increase the order of the polynomials for residual calculation and error compensation.
- Weighting as a function of frequency can be used
- Additional weighting function can be used to detect or enhance weak modes.
- By direct use of FRF data, time domain "leakage" is avoided.
- Fewer computational modes introduced, thus it is suitable for combined modal/servo systems which can include both low and high damping modes.
- CMIF can be used to indicate close or repeated eigenvalues before the parameter estimation procedure, so that the proper order of the polynomials can be chosen.

3.5 Multi-Mac

3.5.1 Introduction

Multi-Mac is a spatial domain method of determining modal parameters based on multiple-reference frequency response functions. Multi-Mac is an extension of the concept of Modal Assurance Criterion (MAC). MAC is a calculation that is used to gain confidence in the estimates of modal vectors either by verifying that the estimates of different modal vectors are unique, that normal modes have been estimated, or that estimates from different rows or columns of the residue matrix for the same mode are identical. Multi-Mac uses principal component analysis to determine the number of independent vectors that make up the residue vectors calculated from a number of reference locations. The number of significant eigenvalues is used to determine the number of independent vectors. Using the eigenvectors from the principal component analysis, a set of orthogonal mode shapes can be calculated.

3.5.2 Theory

For linear systems with normal modes, the residue of any input/output combination is made up of three parts, the modal vector at the input location, the modal vector at the output location, and the scaling constant for that mode^[50,54,76-78].

$$A_{pqr} = Q_r \psi_{pr} \psi_{qr} \quad (185)$$

Where:

- Q_r = Scaling constant for mode r
- ψ_{pr} = Modal vector for location p of mode r
- ψ_{qr} = Modal vector for location q of mode r
- A_{pqr} = Residue

Equation (185) can be expanded to form the complete residue matrix for any mode r .

$$[A_r] = Q_r \begin{bmatrix} \psi_{1r}\psi_{1r} & \psi_{1r}\psi_{2r} & \psi_{1r}\psi_{3r} & \dots \\ \psi_{2r}\psi_{1r} & \psi_{2r}\psi_{2r} & \psi_{2r}\psi_{3r} & \dots \\ \psi_{mr}\psi_{1r} & \psi_{mr}\psi_{2r} & \psi_{mr}\psi_{3r} & \dots \end{bmatrix} \quad (186)$$

From Equations (186) it can be seen that there must be a linear relationship between any row or any column of the residue matrix for a particular mode. For example, if the structure was excited at points 1 and 2, then in column one every residue would have ψ_{1r} in common. Likewise, every residue in row two would have ψ_{2r} in common. The modal vectors at the output locations, ψ_{1r} through ψ_{mr} , must therefore be related since they define the same mode. If this linear relationship does not exist, either measurement errors have contaminated the data, the modes are closely coupled or, there is a repeated root at that pole location so that the residue is a linear combination based on input location.

Also, from the basic theory of modal analysis, the modal vectors of two different modes must be orthogonal with each other and a weighting matrix such as the mass matrix.

If estimates of the residues for the same mode but for different rows or columns is used, a principal component analysis can be used to determine the number of independent vectors that make up those residues. If one significant eigenvalue is found from the principal component analysis, then one mode is presented. If more than one significant eigenvalue is calculated, then there are more than one set of independent vectors that make up the residue matrix for that mode.

Equation (186) can be used to set up the principal component analysis. The size of the matrices is determined by the number of modal vector estimates and the number of measurement locations. The residue matrix can have more than one estimate of the modal vector for the same input/output locations. These estimates could be from different curve fitting algorithms. The matrix $[U]$ is the matrix of eigenvectors for each eigenvalue.

$$[A_r][A_r]^H = [W][U][\lambda][U]^T[W]^T \quad (187)$$

Where:

- $[A]$ = Residue matrix
- $[W]$ = Diagonal weighting matrix
- $[U]$ = Eigenvector matrix
- $[\lambda]$ = Diagonal eigenvalue matrix

Equation (187) will yield as many eigenvalues as the number of rows in the residue matrix. Some of these eigenvalues will be zero or of insignificant value. The number of significant eigenvalues will indicate the number of independent vectors that make up the residues at that frequency. Associated with each eigenvalue will be the eigenvectors for that eigenvalue which will number as many as the number of rows in the residue matrix. The weighting matrix that was used in Eq. (187) for this work was the identity matrix. A better weighting matrix would be to use the inverse of the square root of the mass matrix or an estimate of the inverse of the square root of the mass matrix from a finite element analysis. The matrix could also be an error matrix or a matrix that would allow different types of data in the residue matrix. For example, if both acceleration and displacement data was contained in the residue matrix, the multiplication of acceleration and displacement would not be dimensionally correct. But, the weighting matrix could be used to make the multiplication dimensionally correct^[70].

Any estimate of the residue may be used in this procedure. Since, at resonance, the quadrature part of the frequency response function is proportional to the residue, the simplest method is to use the peak of the imaginary part of the frequency response estimate as an estimate of the residue^[16]. If leakage is present in the data, it is advantageous to use spectral lines that are adjacent to the highest peak in the imaginary part. These spectral lines will be less contaminated by leakage. Therefore, the multi-mac calculation can be used before any sophisticated parameter estimation is attempted to define the number of independent vectors at each pole location or can be used after all parameter estimation has been completed to gain confidence in the estimate modal vectors.

Once the eigenvalues and eigenvectors have been calculated by the principal component analysis, the residues can be transformed to a new coordinate space which are mathematically guaranteed to be orthogonal with each other and the assumed weighting matrix. Therefore, if more than one eigenvalue of significant value is found, the orthogonal mode shapes that make up the estimate of the residue at that frequency can be computed. For example, if there are repeated roots, the residues that make up that repeated root will be calculated. If the modes are heavily coupled, unique modes will be estimated. But, the transformed modes will have the contamination of other modes removed and will be orthogonal with each other.

Equation (187) defines the transformation from the original residues to the new space. This transformation would be used for each eigenvalue of significant value. Therefore, if more than one

eigenvalue of significant value is found, the transformation will yield orthogonal estimates of the residues. If more than one estimate of the residue is used for each row or column, these transformed estimates are summed together to reduce the variance. The matrix multiplication will therefore yield one residue for each measurement location. The variance of the modal vectors will be reduced in a least squares sense by $[1/n]^{1/2}$.

$$A_{r,m,1} = \sum [U]^T [A_r] \quad (188)$$

3.5.3 Experimental Procedure

In practice, a number of reference locations are used to estimate the frequency response functions. These can be from either a multiple input test or an impact test with a number of reference accelerometers. In this way, a number of rows equal to the number of reference accelerometers or a number of columns equal to the number of inputs of the residue matrix will be estimated. To get good estimates of the natural frequencies of the system, a summation of the power spectrum of the quadrature response is calculated^[16]. This summation, which can use all frequency response estimates or a subset of the frequency response estimates, enhances the global modes of the system and give less weight to any local modes.

From these starting values, the quadrature part of the frequency response estimates are used. In practice, the peak location and one or two spectral lines on either side of the peak is used. In this way, the errors caused by leakage can be reduced. If there is no leakage or if a large number of references is used, it is not necessary to use more than the peak value. These three or five frequencies are used as an estimate of the residue for that mode for a given reference location.

A principal component analysis, which is the number of spectral lines times the number of references in size, is then done on these estimates of the residue. This will yield a number of eigenvalues equal to the number of spectral lines times the number of references. If only one eigenvalue of significant value is calculated, then there is one independent vector that makes up the residue vectors at those frequencies. If more than one significant eigenvalue is found then there are that many independent vectors that make up the residue vectors at that frequency.

The residue vectors can then be transformed by the eigenvectors found from the principal component analysis. These new mode shapes are mathematically guaranteed to be orthogonal with each other. These transformed residues could be weighted by the mass matrix to yield mode shapes of the system.

Using these transformed estimates of the residue vectors, an enhanced frequency response function can be computed which can be fit for estimates of the frequency and damping of that mode^[50].

4. CASE STUDIES

4.1 Circular Plate

Utilizing impact testing, data was taken on a circular plate structure. Seven accelerometers were permanently mounted on the structure, while the force was moved to 36 different locations. Therefore, a data set was obtained with seven references. Figure 5 shows the circular plate modal model with the seven reference points indicated.

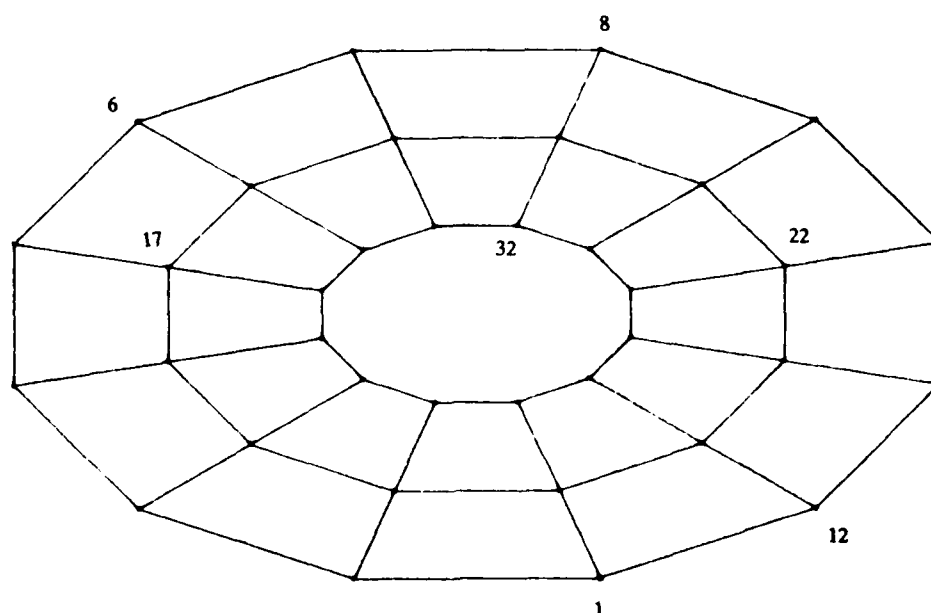


Figure 5. Circular Plate Model

The following algorithms were used to analyze the data for frequency and damping characteristics:

- Polyreference Time Domain (PTD)
- Polyreference Frequency Domain (PFD)
- Orthogonal Polynomial (OP)
- Multiple-Reference Ibrahim Time Domain (MRITD)
- Modified Multiple-Reference Ibrahim Time Domain (MMRITD)

Due to memory limitations in the implementation of the algorithms, the data was analyzed with only six references. The references used for the analysis were arbitrarily chosen. The data was obtained using baseband analysis from 0 Hz. to 2560 Hz., with a time domain blocksize of 1024. Therefore, the frequency resolution, Δf , was 5 Hz. A typical driving point frequency response function is shown in Figure 6.

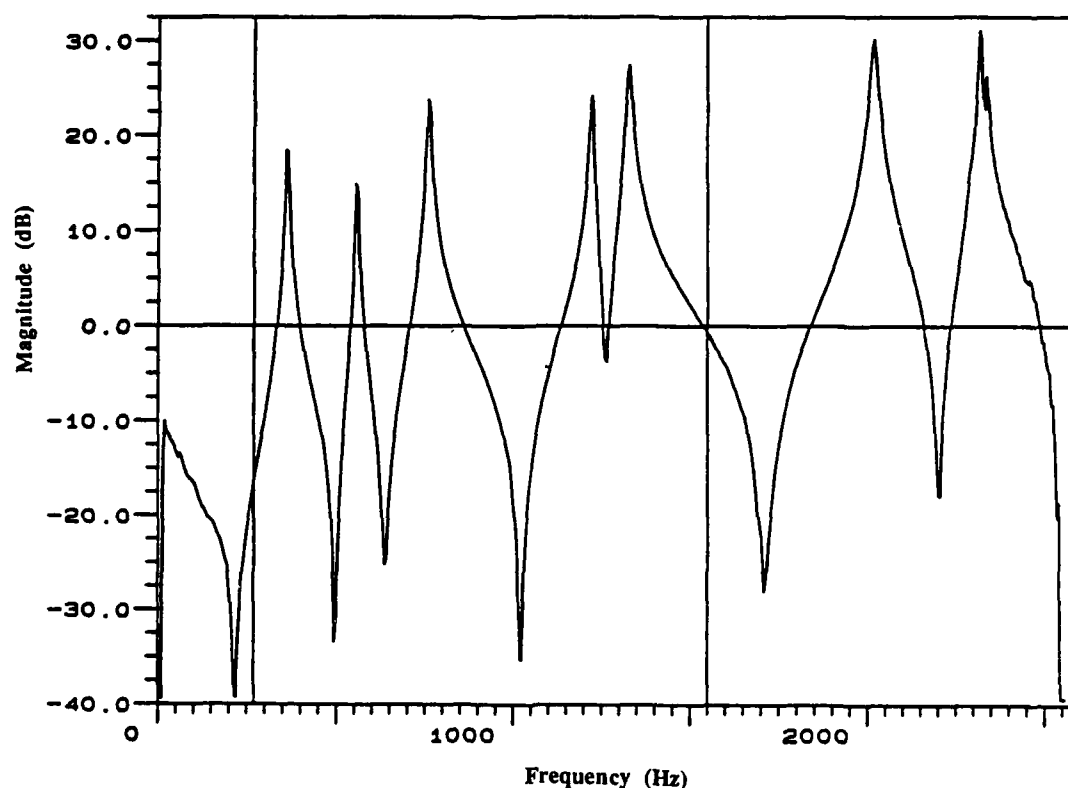


Figure 6. Circular Plate Driving Point Frequency Response Function

The data was processed in several analyses. First, the data was analyzed from spectral line 50 to spectral line 178 or, from 250 Hz. to 890 Hz. The second analysis band was from spectral line 200 to spectral line 328 or, from 1000 Hz. to 1640 Hz. The frequency and damping results of these analyses are presented in Table 1.

TABLE 1. Circular Plate Frequency and Damping Analysis

Mode		Frequency/Damping Estimation Method				
		PTD	PFD	OP	MRITD	MMRITD
1	Freq. (Hz)	362.276	362.368	362.474	362.208	362.254
	Damp. (%)	0.8665	0.8296	0.8876	0.8580	0.868
2	Freq. (Hz)	363.612	363.562	363.603	363.629	363.567
	Damp. (%)	0.9368	0.8850	0.9554	0.9013	0.946
3	Freq. (Hz)	557.041	557.124	557.078	557.085	557.045
	Damp. (%)	0.5129	0.5683	0.5389	0.5154	0.523
4	Freq. (Hz)	761.143	761.309	761.185	761.228	761.171
	Damp. (%)	0.6654	0.6287	0.6667	0.6653	0.663
5	Freq. (Hz)	764.138	764.166	764.146	764.340	764.190
	Damp. (%)	0.3371	0.3259	0.3338	0.3335	0.339
6	Freq. (Hz)	1223.021	1223.119	1223.006	1223.033	1223.006
	Damp. (%)	0.3337	0.3445	0.3357	0.3372	0.334
7	Freq. (Hz)	1224.126	1223.814	1224.145	1224.034	1224.092
	Damp. (%)	0.3262	0.3195	0.3211	0.3314	0.332
8	Freq. (Hz)	1328.063	1328.213	1328.030	1328.070	1328.069
	Damp. (%)	0.5031	0.5207	0.4999	0.4936	0.496
9	Freq. (Hz)	1328.760	1328.687	1328.754	1328.772	1328.781
	Damp. (%)	0.4129	0.4079	0.4084	0.4032	0.408

Notes: PTD Polyreference Time Domain
PFD Polyreference Frequency Domain
OP Orthogonal Polynomial
MRITD Multiple-Reference Ibrahim Time Domain
MMRITD Modified Multiple-Reference Ibrahim Time Domain

4.2 Body-In-White

Frequency response functions were obtained from a "body-in-white" automobile structure using dual-shaker random excitation with triaxial responses. The data set contains 432 frequency response functions; from the 72 measurement locations, measured in the three global directions, and the two references. Figure 7 shows the "body-in-white" modal model with the two reference points indicated.

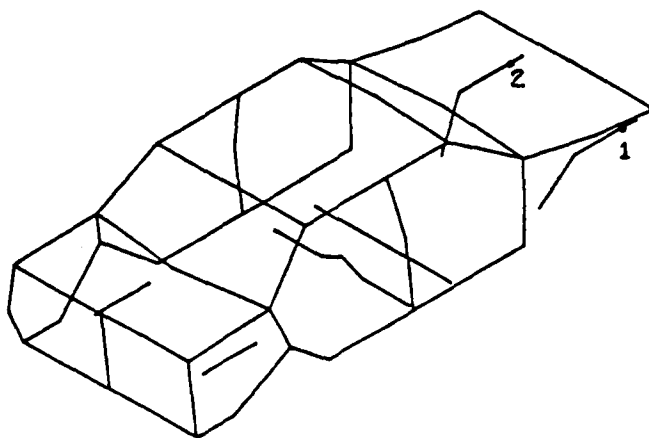


Figure 7. Body-In-White Model

The frequency response functions were obtained using baseband analysis from 0 Hz. to 60 Hz., with a time domain blocksize of 1024. Therefore, the frequency resolution, Δf , was 0.117 Hz. A typical driving point frequency response function is shown in Fig. 8.

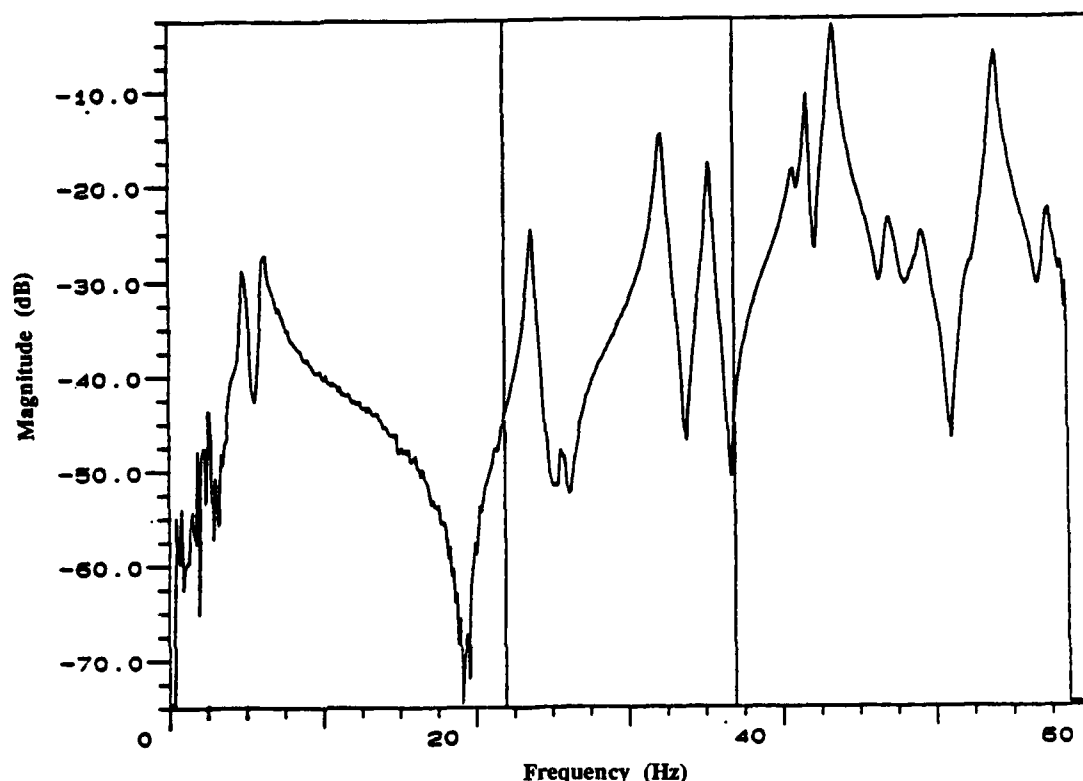


Figure 8. Body-In-White Driving Point Frequency Response Function

The following algorithms were used to analyze the data for frequency and damping characteristics:

- Polyreference Time Domain (PTD)
- Eigensystem Realization Algorithm (ERA)
- Modified Multiple-Reference Ibrahim Time Domain (MMRITD)

The frequency and damping evaluation presented in Table 2 was performed as a class project at the University of Cincinnati^[65,79].

4.3 Multi-Mac Experimental Examples

To demonstrate the value of the Multi-Mac calculation, a number of test cases were evaluated.

First, a H-frame was tested. Figure 9 shows the H-frame. Twelve reference accelerometers were used, four tri-axial accelerometers at the four ends as shown in Figure 9. The H-frame was impacted at 29 points in the vertical direction and 26 points in the horizontal direction.

Figure 10 shows a typical driving point frequency response function estimate in the range of 430 Hertz to 630 Hertz. There appears to be 3 or 4 natural frequencies in this range. Figure 11 is a summation of the power spectrum of the quadrature responses using points 1Z, 13Z, 17Z, 29Z, 1Y, and 13Y as the references. This function shows that, using these references, there are at least 7 global modes in this range.

TABLE 2. Body-In-White Frequency and Damping Analysis

Mode		Frequency/Damping Estimation Method		
		PTD	ERA	MMRITD
1	Freq. (Hz)	23.704	23.740	23.693
	Damp. (%)	0.6848	0.9602	0.637
2	Freq. (Hz)	32.174	32.173	32.175
	Damp. (%)	0.6265	0.6161	0.625
3	Freq. (Hz)	35.293	35.294	35.293
	Damp. (%)	0.4460	0.4653	0.445
4	Freq. (Hz)	40.636	40.658	40.663
	Damp. (%)	0.5425	0.4779	0.501
5	Freq. (Hz)	41.691	41.713	41.719
	Damp. (%)	0.4263	0.3306	0.404
6	Freq. (Hz)	42.760	42.767	42.750
	Damp. (%)	0.3728	0.3762	0.374
7	Freq. (Hz)	43.383	43.382	43.385
	Damp. (%)	0.4357	0.4264	0.439
8	Freq. (Hz)	46.748	46.784	46.728
	Damp. (%)	0.7213	0.9369	0.656
9	Freq. (Hz)	48.819	48.955	48.943
	Damp. (%)	0.8949	0.8937	0.955
10	Freq. (Hz)	53.926	53.938	n/a
	Damp. (%)	0.3771	0.3487	

Notes: PTD Polyreference Time Domain
ERA Eigensystem Realization Algorithm
MMRITD Modified Multiple-Reference Ibrahim Time Domain
n/a not analyzed

To demonstrate the value of Multi-Mac, a principal component analysis was performed on each mode. Tables 3 through 9 are the first 20 eigenvalues from the principal component analysis. These eigenvalues are from five spectral lines and six references locations using the quadrature part of the frequency response estimate. The tables show the most significant eigenvalues as a function of the rate of change in the eigenvalues.

Figures 12 through 25 are the transformed mode shapes using the eigenvectors found from the principal component analysis. If more than one significant eigenvalue was found, then that many modes were calculated. These modes are orthogonal with each other due to the transformation. Although none of the modes are repeated roots, the Multi-Mac calculation shows the high degree of modal coupling for many of the modes in this range. As an example, the estimate of the residue vector at 576 Hertz is contaminated by both the mode at 567 Hertz and the mode at 581 Hertz. This can be seen by noting that the one mode computed for 576 Hertz is the same as the 567 Hertz and the modes at 581 are also similar to the 576 Hertz estimate. But, as shown by Table 10, the estimates are orthogonal to each other. This is true for any of the frequencies that have more than one significant eigenvalue.

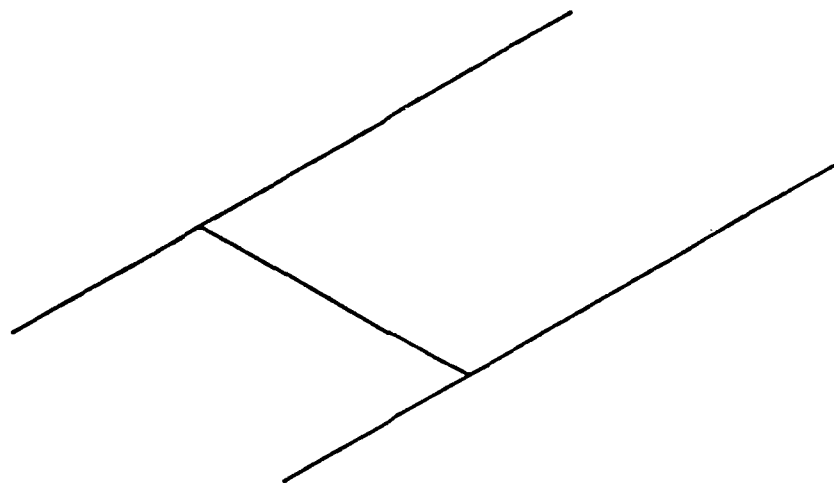


Figure 9. H-frame Test Object

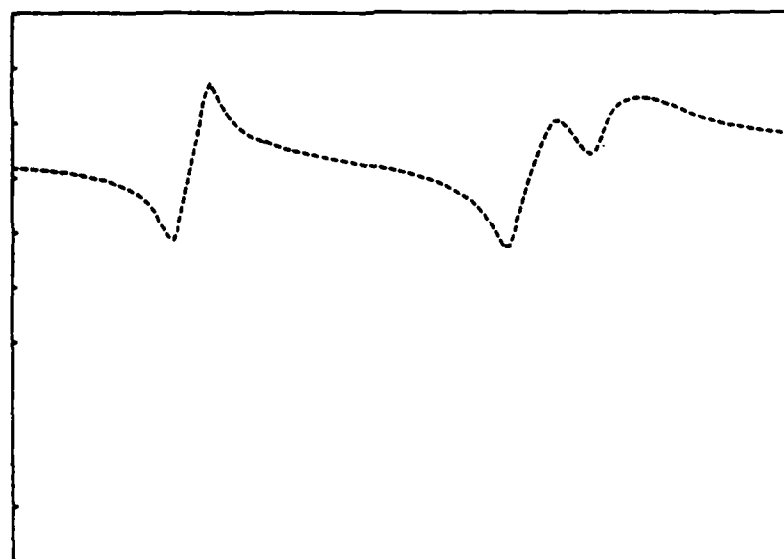


Figure 10. H-frame Driving Point Frequency Response Estimate

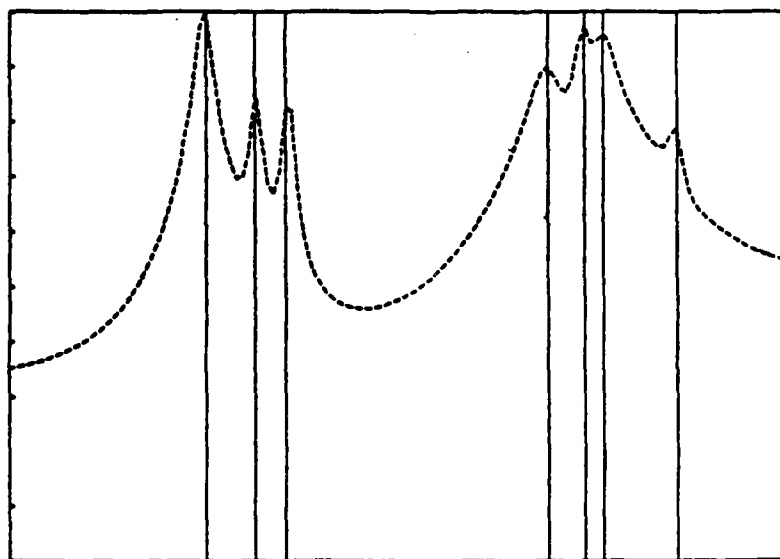


Figure 11. H-frame Summation of the Power Spectrum of the Quadrature Responses

TABLE 3. H-frame Eigenvalues for 479 Hertz

```

*****
***  CRITERIA TO JUDGE RANK  ***
*****
NO.=  1  E.VAL=  .127152E+03@*****@
NO.=  2  E.VAL=  .257414E+00@*      @
NO.=  3  E.VAL=  .823126E-01@*      @
NO.=  4  E.VAL=  .314892E-02@*      @
NO.=  5  E.VAL=  .909428E-02@*      @
NO.=  6  E.VAL=  .391440E-02@*      @
NO.=  7  E.VAL=  .544588E-03@*      @
NO.=  8  E.VAL=  .486090E-03@*      @
NO.=  9  E.VAL=  .318055E-03@*      @
NO.= 10  E.VAL=  .138132E-03@*      @
NO.= 11  E.VAL=  .930068E-04@*      @
NO.= 12  E.VAL=  .522677E-04@*      @
NO.= 13  E.VAL=  .409133E-04@*      @
NO.= 14  E.VAL=  .146818E-04@*      @
NO.= 15  E.VAL=  .102797E-04@*      @
NO.= 16  E.VAL=  .869041E-05@*      @
NO.= 17  E.VAL=  .786539E-05@*      @
NO.= 18  E.VAL=  .689718E-05@*      @
NO.= 19  E.VAL=  .483071E-05@*      @
NO.= 20  E.VAL=  .347911E-05@*      @
SELECTED RANK OF DD MATRIX= 1

```

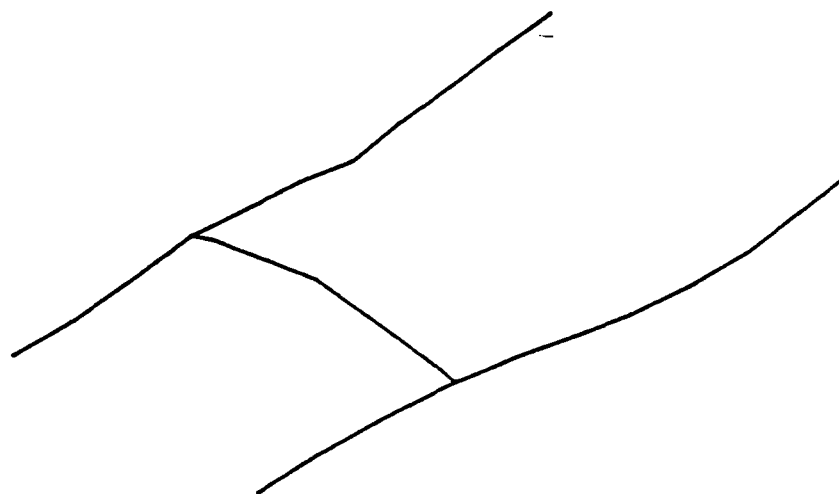


Figure 12. H-frame Mode Shape at 479 Hertz

TABLE 4. H-frame Eigenvalues for 492 Hertz

```

*****
***  CRITERIA TO JUDGE RANK  ***
*****
NO.= 1  E.VAL= .220459E+02@*****@
NO.= 2  E.VAL= .647909E+00@*****@
NO.= 3  E.VAL= .282519E-01@**@
NO.= 4  E.VAL= .961900E-02@*@
NO.= 5  E.VAL= .528314E-02@***@
NO.= 6  E.VAL= .126037E-02@*@
NO.= 7  E.VAL= .843033E-03@*@
NO.= 8  E.VAL= .344217E-03@*@
NO.= 9  E.VAL= .263810E-03@*@
NO.= 10 E.VAL= .162976E-03@*@
NO.= 11 E.VAL= .920806E-04@*@
NO.= 12 E.VAL= .729814E-04@*@
NO.= 13 E.VAL= .427178E-04@*@
NO.= 14 E.VAL= .314986E-04@*@
NO.= 15 E.VAL= .208977E-04@*@
NO.= 16 E.VAL= .831395E-05@*@
NO.= 17 E.VAL= .512812E-05@*@
NO.= 18 E.VAL= .376602E-05@*@
NO.= 19 E.VAL= .180069E-05@*@
NO.= 20 E.VAL= .140457E-05@*@
SELECTED RANK OF DD MATRIX= 2

```

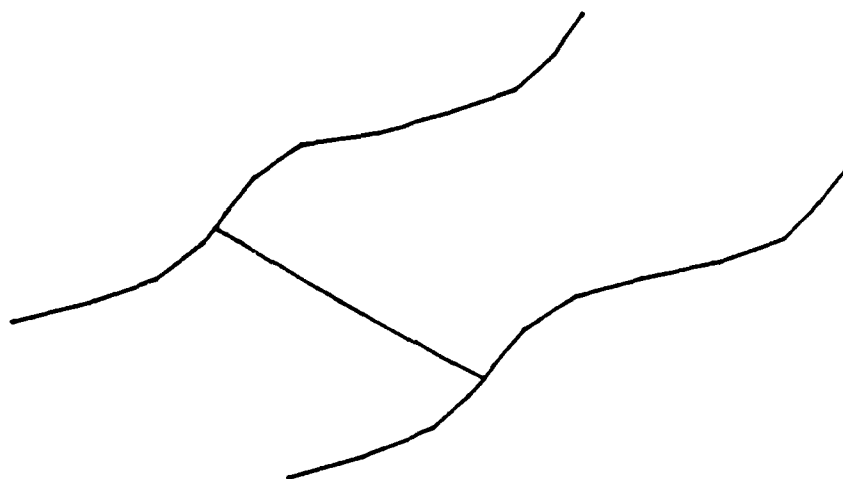


Figure 13. H-frame First Mode Shape at 492 Hertz

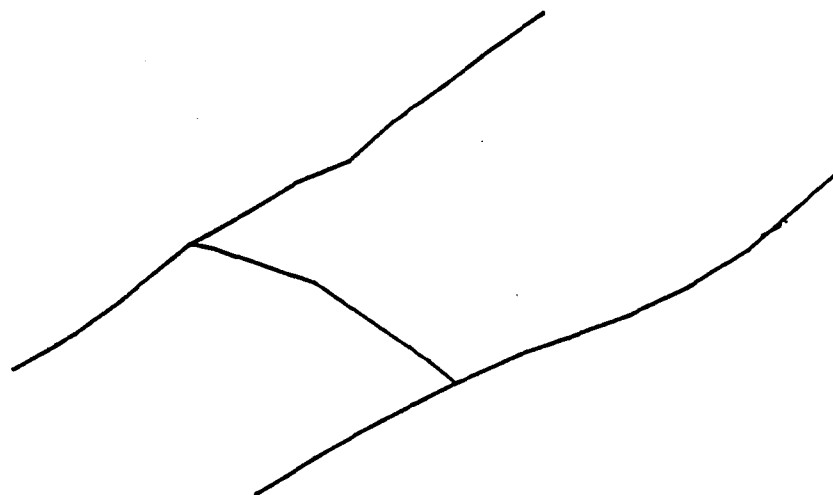


Figure 14. H-frame Second Mode Shape at 492 Hertz

TABLE 5. H-frame Eigenvalues for 500 Hertz

```

*****
***  CRITERIA TO JUDGE RANK  ***
*****
NO.=  1  E.VAL=  .194833E+02@*****@
NO.=  2  E.VAL=  .211252E+00@*      @
NO.=  3  E.VAL=  .995534E-01@*      @
NO.=  4  E.VAL=  .158864E-01@*      @
NO.=  5  E.VAL=  .743090E-02@***    @
NO.=  6  E.VAL=  .518315E-03@*      @
NO.=  7  E.VAL=  .400569E-03@*      @
NO.=  8  E.VAL=  .341622E-03@*      @
NO.=  9  E.VAL=  .156928E-03@*      @
NO.= 10  E.VAL=  .116135E-03@*      @
NO.= 11  E.VAL=  .685376E-04@*      @
NO.= 12  E.VAL=  .428652E-04@*      @
NO.= 13  E.VAL=  .256207E-04@*      @
NO.= 14  E.VAL=  .100953E-04@*      @
NO.= 15  E.VAL=  .558249E-05@*      @
NO.= 16  E.VAL=  .469059E-05@*      @
NO.= 17  E.VAL=  .313182E-05@*      @
NO.= 18  E.VAL=  .173096E-05@*      @
NO.= 19  E.VAL=  .123087E-05@*      @
NO.= 20  E.VAL=  .883329E-06@*      @
SELECTED RANK OF DD MATRIX=  1

```

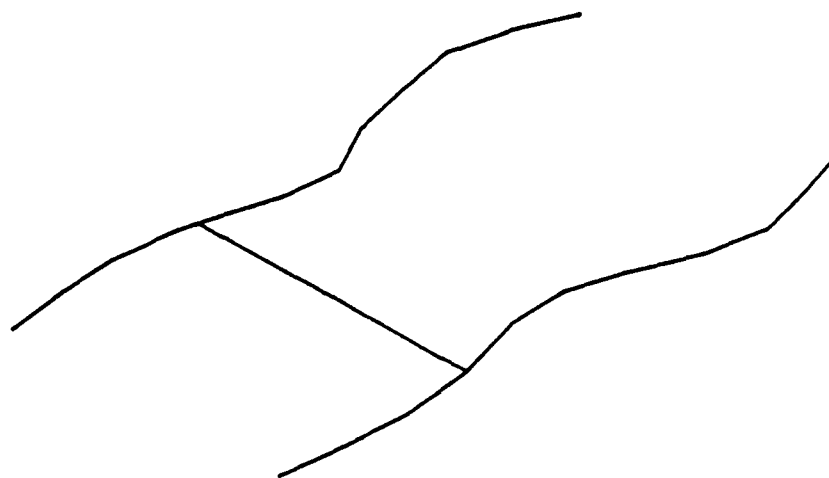



Figure 15. H-frame Mode Shape at 500 Hertz

TABLE 6. H-frame Eigenvalues for 567 Hertz

```

*****
***  CRITERIA TO JUDGE RANK  ***
*****
NO.=  1  E.VAL= .835621E+02@*****@
NO.=  2  E.VAL= .712774E+00@*      @
NO.=  3  E.VAL= .295527E+00@*      @
NO.=  4  E.VAL= .151509E+00@*      @
NO.=  5  E.VAL= .336923E-01@*      @
NO.=  6  E.VAL= .132355E-01@*      @
NO.=  7  E.VAL= .332226E-02@*      @
NO.=  8  E.VAL= .237506E-02@*      @
NO.=  9  E.VAL= .110479E-02@*      @
NO.= 10  E.VAL= .488129E-03@*      @
NO.= 11  E.VAL= .400007E-03@*      @
NO.= 12  E.VAL= .157472E-03@*      @
NO.= 13  E.VAL= .994866E-04@*      @
NO.= 14  E.VAL= .550801E-04@*      @
NO.= 15  E.VAL= .318778E-04@*      @
NO.= 16  E.VAL= .111433E-04@*      @
NO.= 17  E.VAL= .965316E-05@*      @
NO.= 18  E.VAL= .759809E-05@*      @
NO.= 19  E.VAL= .520880E-05@*      @
NO.= 20  E.VAL= .394904E-05@*      @
SELECTED RANK OF DD MATRIX= 1

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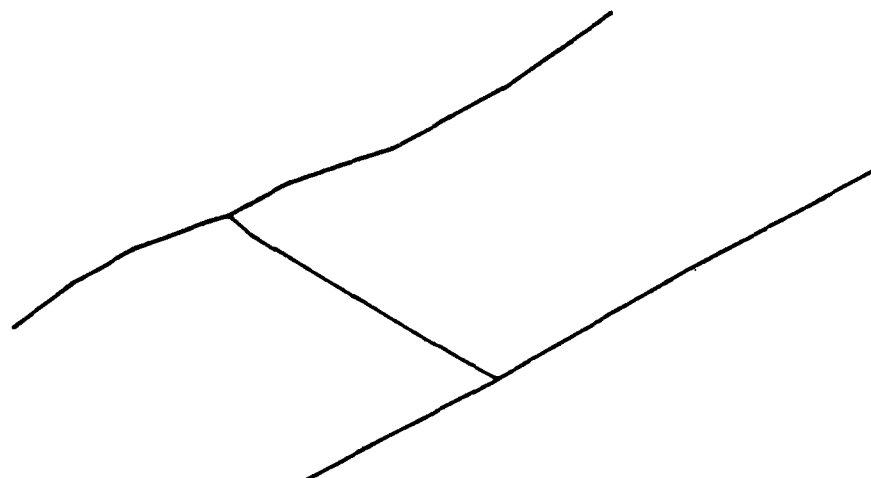


Figure 16. H-frame Mode Shape at 567 Hertz

TABLE 7. H-frame Eigenvalues for 576 Hertz

```

*****
***   CRITERIA TO JUDGE RANK   ***
*****
NO.=  1  E.VAL=  .785431E+02@*
NO.=  2  E.VAL=  .393105E+02@***
NO.=  3  E.VAL=  .595365E+01@*****
NO.=  4  E.VAL=  .119642E+00@*
NO.=  5  E.VAL=  .764095E-01@**
NO.=  6  E.VAL=  .191642E-01@*
NO.=  7  E.VAL=  .176129E-01@*
NO.=  8  E.VAL=  .117179E-01@*
NO.=  9  E.VAL=  .426094E-02@*
NO.= 10  E.VAL=  .338080E-02@*
NO.= 11  E.VAL=  .121294E-02@*
NO.= 12  E.VAL=  .726578E-03@*
NO.= 13  E.VAL=  .501546E-03@*
NO.= 14  E.VAL=  .258313E-03@*
NO.= 15  E.VAL=  .110423E-03@*
NO.= 16  E.VAL=  .797348E-04@*
NO.= 17  E.VAL=  .600097E-04@*
NO.= 18  E.VAL=  .260145E-04@*
NO.= 19  E.VAL=  .211416E-04@*
NO.= 20  E.VAL=  .166732E-04@*
SELECTED RANK OF DD MATRIX=  3

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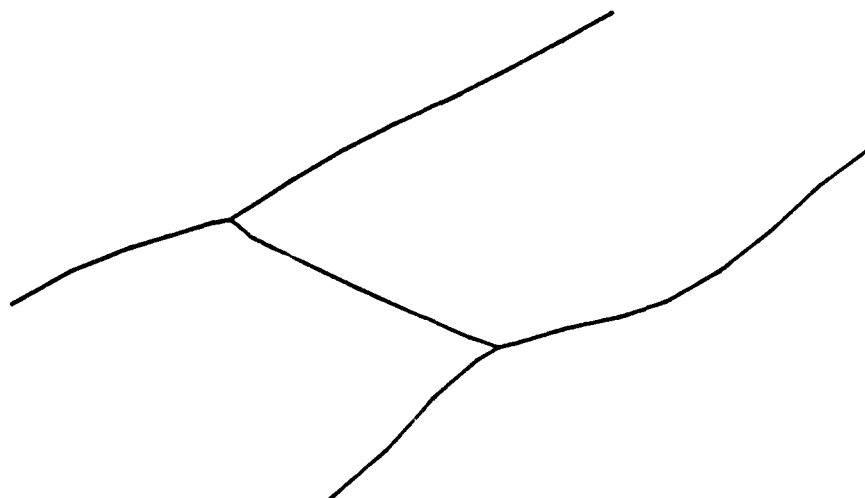


Figure 17. H-frame First Mode Shape at 576 Hertz

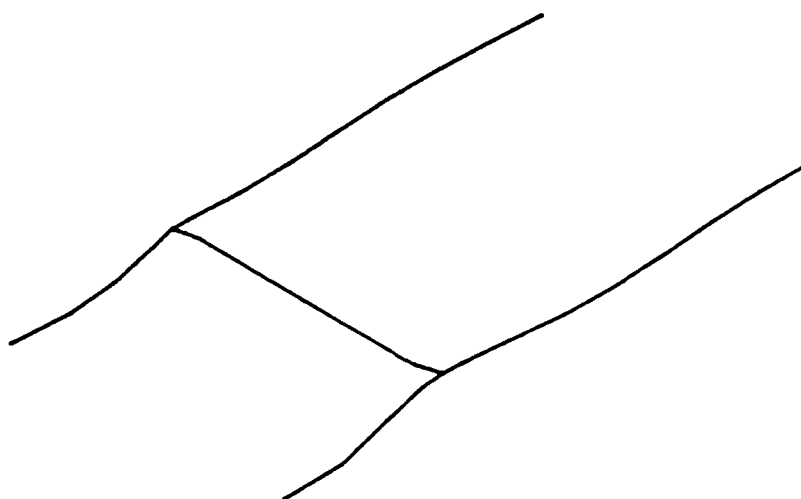


Figure 18. H-frame Second Mode Shape at 576 Hertz

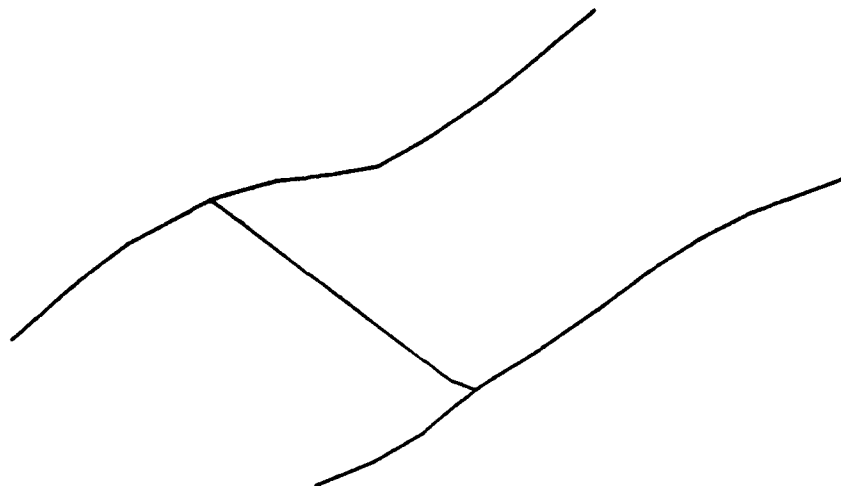


Figure 19. H-frame Third Mode Shape at 576 Hertz

TABLE 8. H-frame Eigenvalues for 581 Hertz

```

*****
***   CRITERIA TO JUDGE RANK   ***
*****
NO.=  1  E.VAL=  .119269E+03@*****@
NO.=  2  E.VAL=  .105354E+02@*
NO.=  3  E.VAL=  .765341E+01@*****@
NO.=  4  E.VAL=  .948675E-01@*
NO.=  5  E.VAL=  .730457E-01@*
NO.=  6  E.VAL=  .443483E-01@*
NO.=  7  E.VAL=  .874408E-02@*
NO.=  8  E.VAL=  .512513E-02@*
NO.=  9  E.VAL=  .387253E-02@*
NO.= 10  E.VAL=  .133609E-02@*
NO.= 11  E.VAL=  .736777E-03@*
NO.= 12  E.VAL=  .376061E-03@*
NO.= 13  E.VAL=  .213631E-03@*
NO.= 14  E.VAL=  .144119E-03@*
NO.= 15  E.VAL=  .122016E-03@*
NO.= 16  E.VAL=  .879375E-04@*
NO.= 17  E.VAL=  .343842E-04@*
NO.= 18  E.VAL=  .232610E-04@*
NO.= 19  E.VAL=  .120455E-04@*
NO.= 20  E.VAL=  .903840E-05@*
SELECTED RANK OF DD MATRIX=  3

```

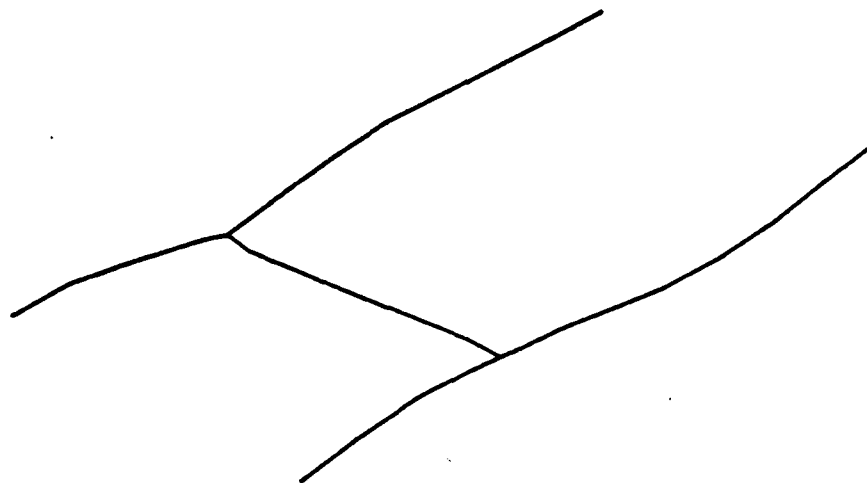


Figure 20. H-frame First Mode Shape at 581 Hertz

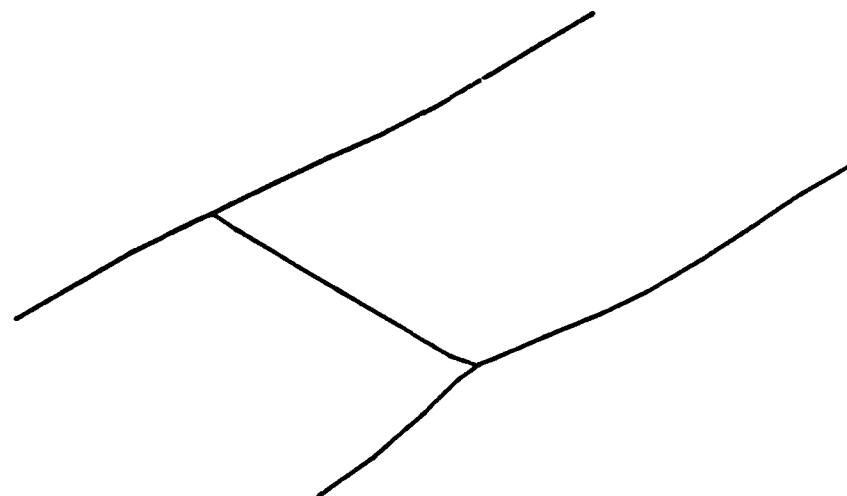


Figure 21. H-frame Second Mode Shape at 581 Hertz

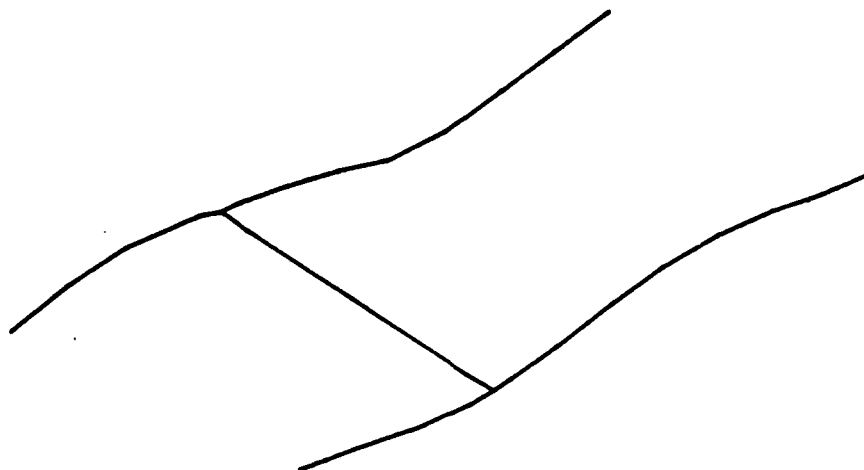


Figure 22. H-frame Third Mode Shape at 581 Hertz

TABLE 9. H-frame Eigenvalues for 600 Hertz

```

*****
***  CRITERIA TO JUDGE RANK  ***
*****
NO.= 1  E.VAL= .944627E+01@***** @
NO.= 2  E.VAL= .218838E+01@* @
NO.= 3  E.VAL= .128650E+01@***** @
NO.= 4  E.VAL= .509607E-01@*** @
NO.= 5  E.VAL= .171052E-01@*** @
NO.= 6  E.VAL= .628757E-02@** @
NO.= 7  E.VAL= .297248E-02@*** @
NO.= 8  E.VAL= .984154E-03@** @
NO.= 9  E.VAL= .491233E-03@* @
NO.= 10 E.VAL= .326437E-03@* @
NO.= 11 E.VAL= .292134E-03@* @
NO.= 12 E.VAL= .168954E-03@* @
NO.= 13 E.VAL= .908950E-04@** @
NO.= 14 E.VAL= .456719E-04@* @
NO.= 15 E.VAL= .313949E-04@* @
NO.= 16 E.VAL= .169647E-04@* @
NO.= 17 E.VAL= .116909E-04@* @
NO.= 18 E.VAL= .871389E-05@*** @
NO.= 19 E.VAL= .344043E-05@* @
NO.= 20 E.VAL= .244828E-05@* @
SELECTED RANK OF DD MATRIX= 3

```

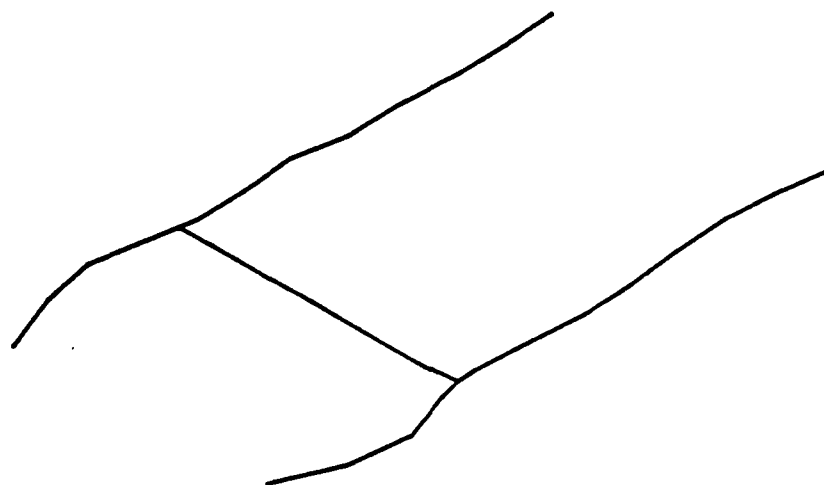


Figure 23. H-frame First Mode Shape at 600 Hertz

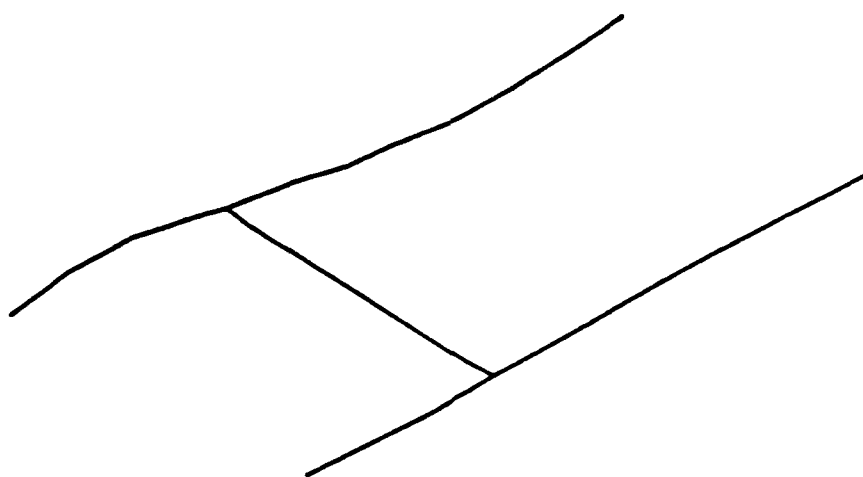


Figure 24. H-frame Second Mode Shape at 600 Hertz

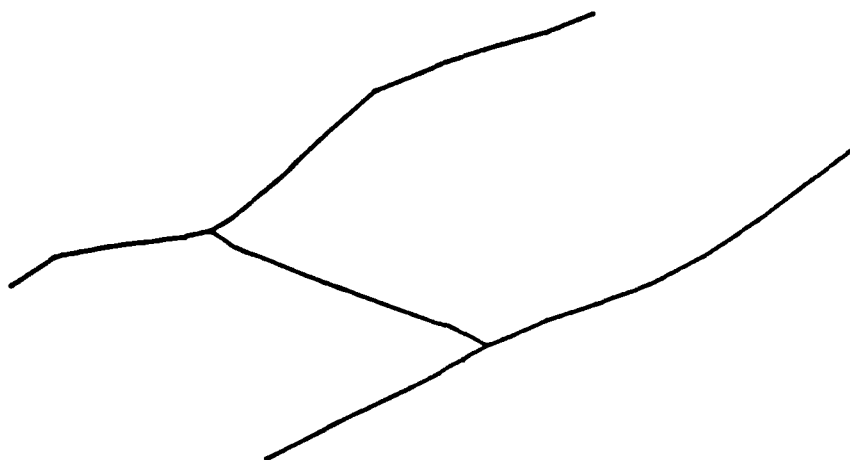


Figure 25. H-frame Third Mode Shape at 600 Hertz

TABLE 10. H-frame Modal Assurance Criterion for Transformed Modes

REFERENCE MODE	ANALYSIS MODE	M.A.C.	M.S.F. (REAL)	M.S.F. (IMAG)
1	1	1.0000000	1.0000000	0.0000000
1	2	.0000000	.0000000	.0000000
1	3	.0000000	.0000001	.0000000
2	2	1.0000000	1.0000000	0.0000000
2	3	.0000000	-.0000002	0.0000000
3	3	1.0000000	1.0000000	0.0000000

M.S.F Modal Scaling Factor (complex)
M.A.C. Modal Assurance Criterion

Next, the Multi-Mac calculation was tried on a circular plate that is expected to have repeated roots. Again, 6 reference accelerometers were attached to the structure and 5 channels around the peak in the quadrature part were used in the calculation. Figure 26 is a typical frequency response estimate and Figure 27 is the summation of the power spectrum of the quadrature response. Both functions indicate that there are two mode in the frequency range. Tables 11 and 12 are the eigenvalues from the principal component analysis. Both analysis show two significant eigenvalues indicating two independent modal vectors. Figures 28 and 29 are the transformed mode shapes using the eigenvectors from the principal component analysis. Both show that there exists repeated roots at both frequencies. Tables 13 and 14 show that these transformed mode shapes are orthogonal with each other.

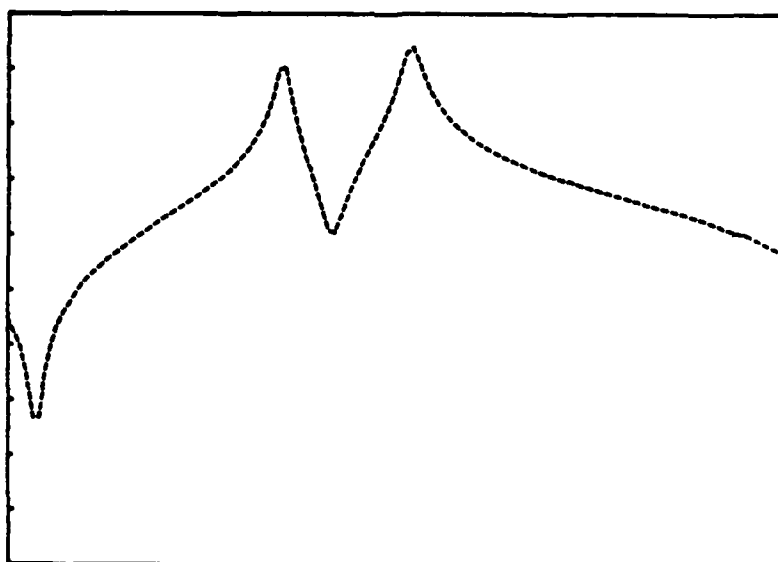


Figure 26. Circular Plate Driving Point Frequency Response Estimate

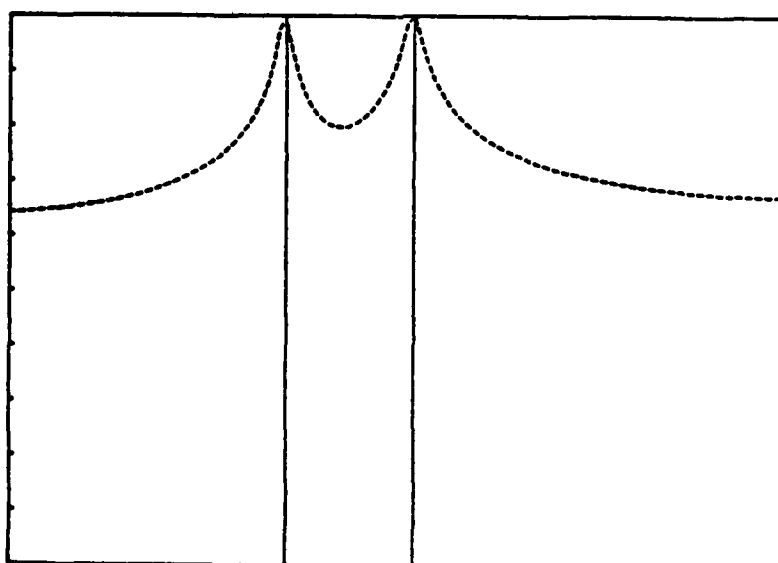


Figure 27. Circular Plate Summation of the Power Spectrum of the Quadrature Responses

TABLE 11. Circular Plate Eigenvalues for 1225 Hertz

*** CRITERIA TO JUDGE RANK ***			

NO.= 1	E.VAL=	.175307E+05@*	@
NO.= 2	E.VAL=	.600946E+04@*****	@
NO.= 3	E.VAL=	.816603E+01@*	@
NO.= 4	E.VAL=	.241708E+01@*	@
NO.= 5	E.VAL=	.129267E+01@*	@
NO.= 6	E.VAL=	.331926E+00@*	@
NO.= 7	E.VAL=	.207313E+00@*	@
NO.= 8	E.VAL=	.163597E+00@*	@
NO.= 9	E.VAL=	.133060E+00@*	@
NO.= 10	E.VAL=	.895497E-01@*	@
NO.= 11	E.VAL=	.351897E-01@*	@
NO.= 12	E.VAL=	.281372E-01@*	@
NO.= 13	E.VAL=	.244295E-01@*	@
NO.= 14	E.VAL=	.191932E-01@*	@
NO.= 15	E.VAL=	.100598E-01@*	@
NO.= 16	E.VAL=	.729477E-02@*	@
NO.= 17	E.VAL=	.548224E-02@*	@
NO.= 18	E.VAL=	.321106E-02@*	@
NO.= 19	E.VAL=	.140953E-02@*	@
NO.= 20	E.VAL=	.109937E-02@*	@
SELECTED RANK OF DD MATRIX= 2			

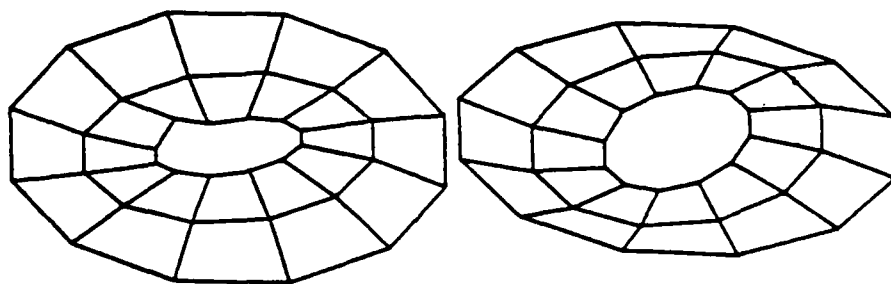


Figure 28. Circular Plate Mode Shapes at 1225 Hertz

TABLE 12. Circular Plate Eigenvalues for 1330 Hertz

 *** CRITERIA TO JUDGE RANK ***

NO.= 1	E.VAL=	.220557E+05@*	@
NO.= 2	E.VAL=	.123343E+05@*****	@
NO.= 3	E.VAL=	.453680E+01@*	@
NO.= 4	E.VAL=	.370407E+01@*	@
NO.= 5	E.VAL=	.882260E+00@*	@
NO.= 6	E.VAL=	.255783E+00@*	@
NO.= 7	E.VAL=	.127655E+00@*	@
NO.= 8	E.VAL=	.650219E-01@*	@
NO.= 9	E.VAL=	.469805E-01@*	@
NO.= 10	E.VAL=	.386407E-01@*	@
NO.= 11	E.VAL=	.225943E-01@*	@
NO.= 12	E.VAL=	.209756E-01@*	@
NO.= 13	E.VAL=	.936288E-02@*	@
NO.= 14	E.VAL=	.488921E-02@*	@
NO.= 15	E.VAL=	.391385E-02@*	@
NO.= 16	E.VAL=	.283658E-02@*	@
NO.= 17	E.VAL=	.189168E-02@*	@
NO.= 18	E.VAL=	.150162E-02@*	@
NO.= 19	E.VAL=	.123286E-02@*	@
NO.= 20	E.VAL=	.593683E-03@*	@
SELECTED RANK OF DD MATRIX=			2

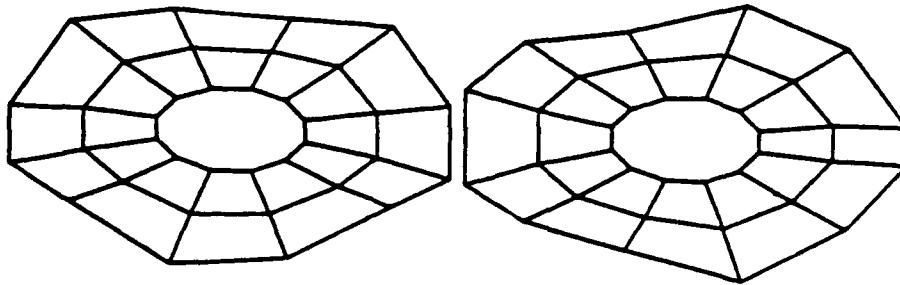


Figure 29. Circular Plate Mode Shapes at 1330 Hertz

TABLE 13. Circular Plate Modal Assurance Criterion for Transformed Modes 1225 Hertz Mode

REFERENCE MODE	ANALYSIS MODE	M.A.C.	M.S.F. (REAL)	M.S.F. (IMAG)
1	1	1.0000000	1.0000000	1.0000000
1	2	.0000000	-.0000002	0.0000000
2	2	1.0000000	1.0000000	0.0000000

M.S.F Modal Scaling Factor (complex)
M.A.C. Modal Assurance Criterion

TABLE 14. Circular Plate Modal Assurance Criterion for Transformed Modes 1330 Hertz Mode

REFERENCE MODE	ANALYSIS MODE	M.A.C.	M.S.F. (REAL)	M.S.F. (IMAG)
1	1	1.0000000	1.0000000	1.0000000
2	2	1.0000000	1.0000000	0.0000000

M.S.F Modal Scaling Factor (complex)
M.A.C. Modal Assurance Criterion

5. CONCLUSIONS

One of the conclusions reached in a previous Air Force Contract (F33615-77-C3059) was that the area of modal parameter estimation was, to in the future, advance rapidly due to technology transfer from other fields involved in parameter estimation. This certainly has occurred as indicated by the drastic increase in the number of parameter estimation algorithms which have been described in the literature in the last five years. This effort has been international in scope, with many of the newer techniques being variations of each other.

These methods range from single reference single degree-of-freedom (SDOF) methods to sophisticated multi-reference, multi-response, multiple degree-of-freedom (MDOF) methods. The algorithm of choice depends upon a number of conditions:

- **Modal Application**

- **Trouble Shooting**--For many of the problem solving, or trouble shooting applications, the simpler SDOF, or single reference MDOF methods are used, since simple test procedures and a quick look are desirable.
- **Model Verification**--There has been increased emphasis on finite element verification. These applications require a higher level test and parameter identification procedures.
- **Model Generation, or Correction**--There is also increased emphasis on; the generation of modal models based upon experimental data, and/or the correction of existing models. These applications require the highest level of test and parameter identification procedures.

- **Equipment Considerations**

The parameter identification methods reviewed in this report depend heavily upon the testing methods (single input, or multiple input) and testing equipment. These new algorithms place a severe requirement upon the testing methods to obtain consistent data bases, particularly for the more advanced multi-input multi-output methods.

- **Wideband vs Narrowband**

Wideband verses narrowband refers to the frequency bandwidth of the frequency response measurements. In general, for very broad frequency range measurements, time domain algorithms work well, while frequency domain algorithms seem to perform well for the narrow, or zoom bands. Recently, there has been increased emphasis in sine testing. Sine testing, not in the classical sense, but in terms of multi-input multi-output test and parameter estimation methods. This emphasis will provide the impetus to refine the frequency domain algorithms to efficiently use the increased spatial information that multi-input multi- output sine testing yields.

- **Modal Density**

The choice of the parameter estimation method depends heavily upon the modal density. For cases with low modal density, single input SDOF or MDOF methods work well. For the high modal density cases the multi-input methods, especially ones which use spatial information, are the methods of choice. It should be noted that the advanced methods require consistent data and place additional constraints on the testing methods.

A summary of the characteristics of the modal parameter identification methods is shown in Table 15. All of the methods which were discussed in detail in this report are briefly summarized in this table.

It should be again noted that all of the methods covered in this report can be described in terms of a characteristic space, where a particular parameter identification algorithm uses as input, measured values in this characteristic space, to deconvolve the systems characteristics. The more advanced methods use information from all three axes of the characteristic space simultaneously. From the measurement standpoint, it is increasingly more important that the measured data be consistent.

TABLE 15. Summary of Modal Parameter Estimation Methods

Modal Parameter Estimation Characteristics							
	Time, Frequency, or Spatial Domain	Single or Multiple Degrees-of-Freedom	Global Modal Frequencies and Damping Factors	Repeated Modal Frequencies and Damping Factors	Global Modal Vectors	Global Modal Participation Factors	Residuals
Quadrature Amplitude	Frequency	SDOF	No	No	No	No	No
Kennedy-Pancu Circle Fit	Frequency	SDOF	No	No	No	No	Yes
SDOF Polynomial	Frequency	SDOF	Yes/No	No	No	No	No
Non-Linear Frequency Domain	Frequency	MDOF	No	No	No	No	Yes
Complex Exponential	Time	MDOF	No	No	No	No	No
Least Squares Complex Exponential (LSCE)	Time	MDOF	Yes	No	No	No	No
Ibrahim Time Domain (ITD)	Time	MDOF	Yes	No	Yes	No	No
Multi-reference Ibrahim Time Domain (MITD)	Time	MDOF	Yes	Yes	Yes/No	No/Yes	No
Eigenystem Realization Algorithm (ERA)	Time	MDOF	Yes	Yes	Yes	Yes	No
Orthogonal Polynomial	Frequency	MDOF	Yes	No	No	No	Yes
Multi-reference Orthogonal Polynomial	Frequency	MDOF	Yes	Yes	Yes	Yes	Yes
Polyreference Time Domain	Time	MDOF	Yes	Yes	No	Yes	No
Polyreference Frequency Domain	Frequency	MDOF	Yes	Yes	Yes	Yes	Yes
Time Domain Direct Parameter Identification	Time	MDOF	Yes	Yes	Yes	Yes	No
Frequency Domain Direct Parameter Identification	Frequency	MDOF	Yes	Yes	Yes	Yes	Yes
Multi-MAC	Spatial	SDOF	No	Yes	Yes	No	No
Multi-MAC / CMIF / Enhanced FRF	Spatial	MDOF	Yes	Yes	Yes	Yes	No

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NOMENCLATURE

Matrix Notation

$\{.. \}$	braces enclose column vector expressions
$\{.. \}^T$	row vector expressions
$[..]$	brackets enclose matrix expressions
$[..]^H$	complex conjugate transpose, or Hermitian transpose, of a matrix
$[..]^T$	transpose of a matrix
$[..]^{-1}$	inverse of a matrix
$[..]^+$	generalized inverse (pseudoinverse)
$[..]_{q \times p}$	size of a matrix: q rows, p columns
$[..]$	diagonal matrix

Operator Notation

A^*	complex conjugate
F	Fourier transform
F^{-1}	inverse Fourier transform
H	Hilbert transform
H^{-1}	inverse Hilbert transform
\ln	natural logarithm
L	Laplace transform
L^{-1}	inverse Laplace transform
$\text{Re} + j\text{Im}$	complex number: real part "Re", imaginary part "Im"
\dot{x}	first derivative with respect to time of dependent variable x
\ddot{x}	second derivative with respect to time of dependent variable x
\bar{y}	mean value of y
\hat{y}	estimated value of y
$\sum_{i=1}^n A_i B_i$	summation of $A_i B_i$ from $i = 1$ to n
$\frac{\partial}{\partial t}$	partial derivative with respect to independent variable "t"
$\det[..]$	determinant of a matrix
$\ .. \ _2$	Euclidian norm

Roman Alphabet

A_{pqr}	residue for response location p, reference location q, of mode r
C	damping
COH	ordinary coherence function†
COH_{ik}	ordinary coherence function between any signal i and any signal k†
COH^n	conditioned partial coherence†
e	base e (2.71828...)
F	input force

F_q	spectrum of q^{th} reference†
GFF	auto power spectrum of reference†
GFF_{qq}	auto power spectrum of reference q †
GFF_{ik}	cross power spectrum of reference i and reference k †
$[GFFX]$	reference power spectrum matrix augmented with the response/reference cross power spectrum vector for use in Gauss elimination
GXF	cross power spectrum of response/reference†
GXX	auto power spectrum of response†
GXX_{pp}	auto power spectrum of response p †
$h(t)$	impulse response function†
$h_{pq}(t)$	impulse response function for response location p , reference location q †
$H(s)$	transfer function†
$H(\omega)$	frequency response function, when no ambiguity exist, H is used instead of $H(\omega)$ †
$H_{pq}(\omega)$	frequency response function for response location p , reference location q , when no ambiguity exist, H_{pq} is used instead of $H_{pq}(\omega)$ †
$H_1(\omega)$	frequency response function estimate with noise assumed on the response, when no ambiguity exist, H_1 is used instead of $H_1(\omega)$ †
$H_2(\omega)$	frequency response function estimate with noise assumed on the reference, when no ambiguity exist, H_2 is used instead of $H_2(\omega)$ †
$H_S(\omega)$	scaled frequency response function estimate, when no ambiguity exist, H_S is used instead of $H_S(\omega)$ †
$H_v(\omega)$	frequency response function estimate with noise assumed on both reference and response, when no ambiguity exist, H_v is used instead of $H_v(\omega)$ †
$[I]$	identity matrix
j	$\sqrt{-1}$
K	stiffness
L	modal participation factor
M	mass
M_r	modal mass for mode r
$MCOH$	multiple coherence function†
N	number of modes
N_i	number of references (inputs)
N_o	number of responses (outputs)
p	output, or response point (subscript)
q	input, or reference point (subscript)
r	mode number (subscript)
R_I	residual inertia
R_F	residual flexibility
s	Laplace domain variable
t	independent variable of time (sec)
t_k	discrete value of time (sec)
	$t_k = k \Delta t$
T	sample period
x	displacement in physical coordinates
X	response
X_p	spectrum of p^{th} response†
z	Z domain variable

Greek Alphabet

$\delta(t)$	Dirac impulse function
Δf	discrete interval of frequency (Hertz or cycles/sec)

Δt	discrete interval of sample time (sec)
ϵ	small number
η	noise on the output
λ_r	r^{th} complex eigenvalue, or system pole $\lambda_r = \sigma_r + j\omega_r$
$[\Lambda]$	diagonal matrix of poles in Laplace domain
ν	noise on the input
ω	variable of frequency (rad/sec)
ω_r	imaginary part of the system pole, or damped natural frequency, for mode r (rad/sec) $\omega_r = \Omega_r \sqrt{1 - \zeta_r^2}$
Ω_r	undamped natural frequency (rad/sec) $\Omega_r = \sqrt{\sigma_r^2 + \omega_r^2}$
ϕ_{pr}	scaled p^{th} response of normal modal vector for mode r
$\{\phi\}_r$	scaled normal modal vector for mode r
$[\Phi]$	scaled normal modal vector matrix
$\{\psi\}$	scaled eigenvector
ψ_{pr}	scaled p^{th} response of a complex modal vector for mode r
$\{\psi\}_r$	scaled complex modal vector for mode r
$[\Psi]$	scaled complex modal vector matrix
σ	variable of damping (rad/sec)
σ_r	real part of the system pole, or damping factor, for mode r
ζ	damping ratio
ζ_r	damping ratio for mode r
\dagger	vector implied by definition of function

$$\frac{\partial E}{\partial a_1} = \sum_{r=1}^N 2 \left[y_r - (a_1 x_r + a_0) \right] [-x_r] = 0 \quad (\text{a3})$$

$$\frac{\partial E}{\partial a_0} = \sum_{r=1}^N 2 \left[y_r - (a_1 x_r + a_0) \right] [-1] = 0 \quad (\text{a4})$$

Equations (a3) and (a4) can be solved for the unknown parameters a_1 and a_0 .

$$a_1 = \frac{N \sum_{r=1}^N x_r y_r - \left(\sum_{r=1}^N x_r \right) \left(\sum_{r=1}^N y_r \right)}{N \sum_{r=1}^N x_r^2 - \left(\sum_{r=1}^N x_r \right)^2} \quad (\text{a5})$$

$$a_0 = \frac{\left(\sum_{r=1}^N y_r \right) \left(\sum_{r=1}^N x_r^2 \right) - \left(\sum_{r=1}^N x_r \right) \left(\sum_{r=1}^N x_r y_r \right)}{N \sum_{r=1}^N x_r^2 - \left(\sum_{r=1}^N x_r \right)^2} \quad (\text{a6})$$

The a_1 and a_0 values computed from Eqs. (a5) and (a6) represent the characteristics of a straight line which would "best" describe the x and y sets of values.

Similarly, one could compute a_1 and a_0 parameters with the criterion that the sum of the square of errors in the x (e_x) would be minimum. Another approach could be the minimization of the sum of the square of errors in the x and y (e_s).

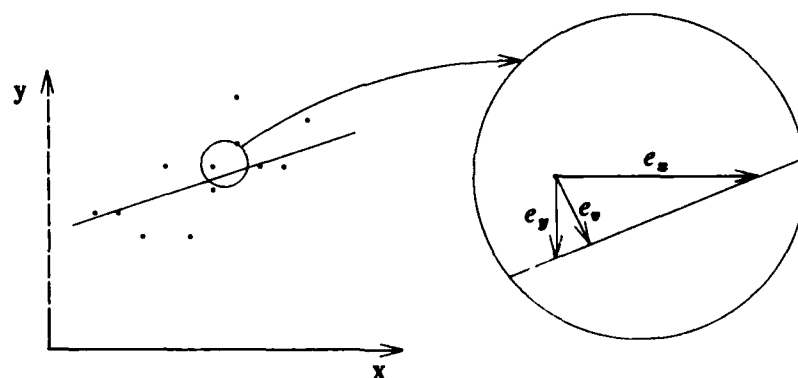


Figure A-2. Errors in Least Squares Estimation

The least squares problem can be formulated in matrix notation as:

$$\{Y\} = [X]\{A\} \quad (\text{a7})$$

$$\text{where, } \{Y\} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix} \quad [X] = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \quad \{A\} = \begin{Bmatrix} a_1 \\ a_0 \end{Bmatrix}$$

The equations presented by matrix Eq. (a7) are in general a set of inconsistent and overdetermined equations. Inconsistent, since it is not usually possible to find $\{A\}$ that would satisfy all the individual equations of Eq. (a7), and overdetermined, since the number of equations is larger than the number of unknowns. The least square solution of Eq. (a7) is:

$$[X]^T\{Y\} = [X]^T[X]\{A\} \quad (a8)$$

and solving for unknown vector $\{A\}$

$$\{A\} = ([X]^T[X])^{-1} [X]^T\{Y\} \quad (a9)$$

provided that $([X]^T[X])^{-1}$ exists. In the case that the inverse of $([X]^T[X])$ does not exist, one could use numerical techniques to solve for vector $\{A\}$.

In a more general case where matrices A , X , and Y are complex valued, the transpose notation, T , must be replaced with hermitian notation, H , in Eqs. (a8) and (a9). Where the hermitian operator, H , is the complex conjugate transpose. Hence, the unknown vector $\{A\}$ is given by:

$$\{A\} = ([X]^H[X])^{-1} [X]^H\{Y\} \quad (a10)$$

Individual equations in matrix Eq. (a7) could be multiplied by a weighting factor to give that equation more or less weight in the computation. The weighting factors could be presented in form of a diagonal ($N \times N$) matrix, W . The diagonal element in row i represents the weighting factor corresponding to equation i , and the off diagonal terms are all zero. Matrix W is pre-multiplied to both sides of Eq. (a7):

$$[W]\{Y\} = [W][X]\{A\} \quad (a11)$$

where,

$$[W] = \begin{bmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_N \end{bmatrix}$$

Solving for vector $\{A\}$ in Eq. (a11):

$$\{A\} = ([X]^H[W]^H[W][X])^{-1} [X]^H[W]^H[W]\{Y\} \quad (a12)$$

In general the relationship of Eq. (a1) could be in the form of:

$$y = a_N x^N + a_{N-1} x^{N-1} + \dots + a_1 x + a_0 \quad (a13)$$

A set of equations similar to the formulation above could be written and solved to obtain the least squares estimation of unknown parameters $a_N, a_{N-1}, \dots, a_1, a_0$.

The least squares method stated above could easily be extended to problems involving more than one independent variable. For example, z could be expressed in terms of x and y :

$$z = a_2 y + a_1 x + a_0 \quad (a14)$$

The corresponding normal equations for the least squares problem of Eq. (a14) are:

$$\frac{\partial E}{\partial a_2} = \sum_{r=1}^N 2 \left[z_r - (a_2 y_r + a_1 x_r + a_0) \right] [-y_r] = 0 \quad (a15)$$

$$\frac{\partial E}{\partial a_1} = \sum_{r=1}^N 2 \left[z_r - (a_2 y_r + a_1 x_r + a_0) \right] [-x_r] = 0 \quad (a16)$$

$$\frac{\partial E}{\partial a_0} = \sum_{r=1}^N 2 \left[z_r - (a_2 y_r + a_1 x_r + a_0) \right] [-1] = 0 \quad (a17)$$

Equations (a15) - (a17) can be solved for the unknown parameters a_2 , a_1 , and a_0 . The computed values a_2 , a_1 , and a_0 represent the characteristics of a plane which would "best" describe the x , y , and z sets of values.

The above theory and formulation could be expanded to least squares estimation of a surface and eventually to higher order dimensions.

A.2 Correlation Coefficient

The "goodness" of the least squares estimation process is measured by the coefficient of correlation parameter which is defined in terms of *total variation* and *explained variation*. The total variation of y is defined as $\sum_{r=1}^N (y_r - \bar{y})^2$, which is the sum of the squares of the deviations of y_r from the mean value, \bar{y} . Total variation consists of two parts: (1) the explained variation, $\sum_{r=1}^N (\hat{y}_r - \bar{y})^2$; and (2) the unexplained variation, $\sum_{r=1}^N (y_r - \hat{y}_r)^2$. The terms explained variation and unexplained variation are used to denote the fact that the deviations $\hat{y}_r - \bar{y}$ have a definite pattern, while, the deviations $y_r - \hat{y}_r$ are random and unpredictable.

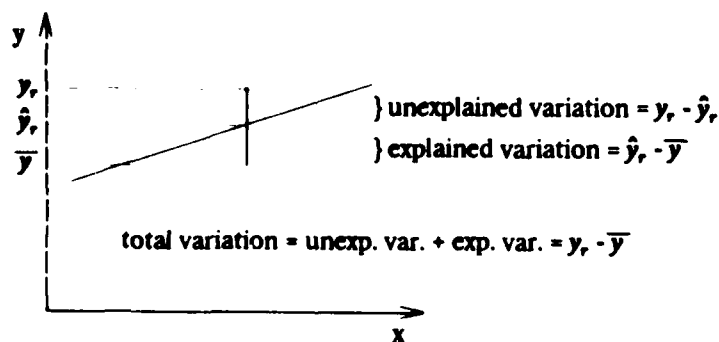


Figure A-3. Variations in Data

Therefore, the coefficient of correlation γ^2 , is defined as:

$$\gamma^2 = \frac{\sum_{r=1}^N (\hat{y}_r - \bar{y})^2}{\sum_{r=1}^N (y_r - \bar{y})^2} \quad (\text{a18})$$

The magnitude of γ^2 varies between 0 and 1. A value of 0 indicates no correlation between dependent and independent variable(s), while a value of 1 indicates perfect correlation.

It should be pointed out that the coefficient of correlation computed for a set of data and a assumed model, only indicates the relationship of data based on the assumed model. That is, the coefficient of correlation measures the degree to which the assumed model describes the relationship for a set of data.

A.3 Examples

A.3.1 Example 1

For the data given in Table A-1 and the assumed model equation:

$$y = a_1 x + a_0 \quad (\text{a19})$$

x	65	63	67	64	68	62	70	66	68	67	69	71
y	68	66	68	65	69	66	68	65	71	67	68	70

TABLE A-1. x and y Values for Least Squares Fit

- Find the least squares solution of a_1 and a_0 .
- Find the coefficient of correlation.

Solution:

- The following normal equations must be solved for a_1 and a_0

$$a_0 N + a_1 \sum_{r=1}^N x_r = \sum_{r=1}^N y_r \quad (\text{a20})$$

$$a_0 \sum_{r=1}^N x_r + a_1 \sum_{r=1}^N x_r^2 = \sum_{r=1}^N x_r y_r \quad (\text{a21})$$

substituting the appropriate terms in Eqs. (a20) and (a21):

$$\begin{aligned} 12 a_0 + 800 a_1 &= 811 \\ 800 a_0 + 53418 a_1 &= 54107 \end{aligned}$$

from which we find $a_1 = 0.476$ and $a_0 = 35.82$ or

$$y = 0.476x + 35.82 \quad (\text{a22})$$

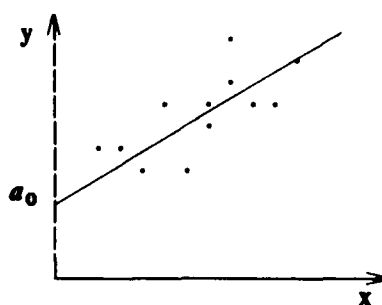


Figure A-4. Least Squares Fit of Data

$$\text{b) Explained variation} = \sum_{r=1}^N (\hat{y}_r - \bar{y})^2 = 19.22$$

$$\text{Total variation} = \sum_{r=1}^N (y_r - \bar{y})^2 = 38.92$$

$$\text{Coefficient of correlation} = r^2 = \frac{19.22}{38.92} = 0.7027$$

A.3.2 Example 2

Given the model for the sampled impulse response function between two points on a structure as,

$$h(t_k) = \sum_{r=1}^{2N} A_r e^{\lambda_r k \Delta t} \quad (\text{a23})$$

where,

$$k = 0, 1, 2, \dots, 2N$$

Δt = value of time subinterval

$$t_k = k \Delta t$$

and the known pole information λ_r , formulate the least squares solution of estimating the residues, A_r .

Solution:

For simplicity and conciseness let,

$$z_r = e^{\lambda_r \Delta t}$$

and therefore Eq. (a23) can be rewritten as,

$$h(t_k) = \sum_{r=1}^{2N} A_r z_r^k \quad (\text{a24})$$

expanding Eq. (a24) for time values of t_0 to t_{2N-1} will result in the following $2N$ equations

$$\begin{aligned} h(t_0) &= A_1 + A_2 + \cdots + A_{2N} \\ h(t_1) &= A_1 z_1 + A_2 z_2 + \cdots + A_{2N} z_{2N} \\ h(t_2) &= A_1 z_1^2 + A_2 z_2^2 + \cdots + A_{2N} z_{2N}^2 \\ &\vdots \\ h(t_{2N-1}) &= A_1 z_1^{2N-1} + A_2 z_2^{2N-1} + \cdots + A_{2N} z_{2N}^{2N-1} \end{aligned} \quad (\text{a25})$$

presenting the $2N$ equations of Eq. (a25) in matrix form gives,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_{2N} \\ z_1^2 & z_2^2 & \cdots & z_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{2N-1} & z_2^{2N-1} & \cdots & z_{2N}^{2N-1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2N} \end{Bmatrix} = \begin{Bmatrix} h(t_0) \\ h(t_1) \\ h(t_2) \\ \vdots \\ h(t_{2N-1}) \end{Bmatrix} \quad (\text{a26})$$

or,

$$[z] \{A\} = \{h\} .$$

Solving Eq. (a26) with more than $2N$ rows for vector $\{A\}$ will result in the least squares solution of the residues.

A.4 References

- [1] Ben Nobel, James W. Daniel, *Applied Linear Algebra*, Prentice-Hall, Inc., 1977.
- [2] Gilbert Strang, *Introduction to Applied Mathematics*, Wellesley-Cambridge press, 1986.
- [3] Murray R. Spiegel, *Schaum's Outline Series "Statistics"*, McGraw-Hill Book, 1961.

APPENDIX B: SINGULAR VALUE DECOMPOSITION

B.1 Singular Value Decomposition

The singular value decomposition will decompose a matrix into the simplest possible form, that being diagonal. This decomposition will always be possible regardless of the rank, or dimension, of the matrix^[1].

Consider the right and left eigenvectors of the $m \times n$ matrix $[A]$, which is of rank k .

$$[A] \{v\} = \{u\} \sigma \quad (b1)$$

$$[A]^H \{u\} = \{v\} \sigma \quad (b2)$$

where:

- $\{u\} = m \times 1$ left singular vector
- $\{v\} = n \times 1$ right singular vector
- $\sigma =$ scalar singular value

By substituting Eq. (b1) into Eq. (b2) for $\{u\}$, the right singular vectors can be determined from:

$$[A]^H [A] \{v\} = \{v\} \sigma^2 \quad (b3)$$

$$| [A]^H [A] - \sigma^2 [I] | = 0$$

where:

- $i = 1 \rightarrow k \quad \sigma_i^2 > 0$
- $i = k+1 \rightarrow n \quad \sigma_i^2 = 0$
- $[V] = [\{v\}_1 \{v\}_2 \{v\}_3 \cdots \{v\}_n]$ right singular unitary matrix.

By substituting Eq. (b2) into Eq. (b1) for $\{v\}$, the left singular vectors can be determined from:

$$[A] [A]^H \{u\} = \{u\} \sigma^2 \quad (b4)$$

$$| [A] [A]^H - \sigma^2 [I] | = 0$$

where:

- $i = 1 \rightarrow k \quad \sigma_i^2 > 0$
- $i = k+1 \rightarrow m \quad \sigma_i^2 = 0$
- $[U] = [\{u\}_1 \{u\}_2 \{u\}_3 \cdots \{u\}_m]$ left singular unitary matrix.

If the eigenvector matrices $[U]$ and $[V]$ are unitary ($[U]^H = [U]^{-1}$ and $[V]^H = [V]^{-1}$), that is, both columns and rows form an orthonormal set, the linear transformation will preserve both angles and lengths. Interpreted geometrically, linear transformations defined by unitary matrices behave like simple rotations in space. The matrix $[A]$, can now be decomposed into diagonal form by appending the eigenvector matrices to Eq. (b1) and premultiplying by $[U]^H$, forming the matrix product:

$$[S] = [U]^H [A] [V] = \begin{bmatrix} \{u\}_1^H \\ \vdots \\ \{u\}_k^H \\ \vdots \\ \{u\}_m^H \end{bmatrix} \begin{bmatrix} \{u\}_1 \sigma_1 & \cdots & \{u\}_k \sigma_k & \{0\}_{k+1} & \cdots & \{0\}_n \end{bmatrix} . \quad (b5)$$

Using the unitary matrix property, $[U]^H [U] = [I]$, the right hand side can be further simplified as:

$$[S] = [U]^H [A] [V] = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_k \\ -[0] & - & - & - & [0] \end{bmatrix} \begin{bmatrix} [0] \\ [0] \end{bmatrix} = \begin{bmatrix} [\Sigma] & [0] \\ [0] & [0] \end{bmatrix} . \quad (b6)$$

Thus, the original matrix, $[A]$, is decomposed into the matrix $[S]$, with the singular values on the diagonal, by the unitary matrices $[U]$ and $[V]$.

Then the singular value decomposition of $[A]$ is defined by:

$$[A] = [U][S][V]^H \quad (b7)$$

Noting the partitioning of the matrix $[S]$ of Eq. (b6), the unitary transformation matrices $[U]$ and $[V]$ can be partitioned in the same way to yield:

$$[A] = [U][S][V]^H = \begin{bmatrix} [U]_1 & [U]_2 \end{bmatrix} \begin{bmatrix} [\Sigma] & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [V]_1^H \\ [V]_2^H \end{bmatrix} \quad (b8)$$

Thus, the singular value decomposition of $[A]$ can be further reduced to:

$$[A] = [U]_1 [\Sigma] [V]_1^H \quad (b9)$$

where:

- $[U]_1$ = left singular submatrix of size $m \times k$
- $[\Sigma]$ = diagonal singular value matrix of size $k \times k$
- $[V]_1^H$ = right singular submatrix of size $k \times n$.

The (Moore-Penrose) generalized inverse (pseudoinverse) ^[1-3], $[A]^+$, of $[A]$ is:

$$[A]^+ = [V] \begin{bmatrix} [\Sigma]^{-1} & [0] \\ [0] & [0] \end{bmatrix} [U]^H = [V]_1 [\Sigma]^{-1} [U]_1^H .$$

where:

- $[A]^+ = n \times m$ generalized inverse of $[A]$
- $[V]_1$ = right singular submatrix of size $n \times k$.
- $[\Sigma]^{-1}$ = inverse of diagonal singular value matrix of size $k \times k$
- $[U]_1^H$ = left singular submatrix of size $k \times m$.

B.2 References

- [1] Ben Nobel, James W. Daniel, *Applied Linear Algebra*, Prentice-Hall, Inc., 1977, pp. 323-337.
- [2] Gibert Strang, *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, 1986.
- [3] Gibert Strang, *Linear Algebra and Its Applications*, Academic Press, 1980.