

FATIOMAL ADVISORY COMMITTEE FOR AEROMAUTICS


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SOME FLEMENTARY PRINGIPLES OF SHELI STRESS AMALYSIS
WITH NOTES ON THE USE OF THE SHEAR GENTRR
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SUmbilry

The analysis of various types of shell under combined bending and torsion is discusser. The calculation and the use of the shear center are touched upon as incidental proinems. Twelve fully worted numerical examples are given in an appendix.

INTRODUGTION

The literature on siell analysis is quite scattered and some of it is not easily available. A definite need therefore existefor setting down in a reasonably comprehensive, out concise, manner tie principles and tho metiods used in the various phases of shell analysis. The present paper deals with the distribution of the stresses, chiefly the shearing stresses, over the cross sections of cantilever shells of constant cross section subjected to combined bending and torsion.

The subject matter is far from being new; as the apDended bibliography indicates, the main principlos vere well established in 1931. Continued discussion in the tochnical literature indicatos, however, that the knowledge of the subject is not so widely disseminated as it needs to bo. It is hopod tiat the mannor of presentation chosen for this papor, in particular tho collection of fully worled numorical oxamplos, will holp materially to achicve the ultimato goal, namely, to provido all practicing ongineors with a working knomledge of the subject.

N.A.O.A. Technical Note No. 691<br>THE ANALYSIS OF SHEI工S

Basic Assumptions and Theories

Typical cases of the problem to be treated are shown in figure l(a). As indicated by the figure, the shell will be assumed to have a constant cross section. It will also be assumed, in general, that the materlal effective in bencing is disposed symmetrically about the horizontal exis. The horizontal and the vertical axes will then be principal axes, and the load will be assumed to act vertio cally.

For purposes of strese analysis, the structures are idealized in the usual manner. A certain effective width of slcin is added to each actual longitudinal or flange to obtain the cross-sectional area effective in banding; this offective area is assumed to be concentrated at the con-. troid. The skin itself is assumed to carry only shear. Idealized sections aro ropresented as in figure $1(b)$.

Stresses caused by bending are obtained in first approximation by applying the engineering theory of bending to the idealized crpss sections. This theory is based on the assumptions that plane cross sections remain plane and that Hooke's law applies. The theory leads to the formuJas, for the normal stresses due to bending,

$$
\begin{equation*}
\sigma=\frac{M y}{I} \tag{1}
\end{equation*}
$$

and, for the shear stresses due to bending,

$$
\begin{equation*}
T=\frac{P Q}{b I} \tag{2a}
\end{equation*}
$$

The derivation of these formulas can be found in any textbook on strength of materials.

It should be noted that, in the computation of the static moments $Q$ as well as of the moments of-inertia $I$, consideration is given only to the material asgumed to be effective in bending.

In shell structures, it is often convonient to use, not the shear streas $T$, but tho shear force per inch length of shoet, which will be designated by

## $q=T t$

and which vill be calfed the "shear~force intonsity" or, briefly, "shear intensity." Formula (2a) then becomes

$$
\begin{equation*}
q=\frac{P Q}{I} \tag{2}
\end{equation*}
$$

for open sections $\boldsymbol{s}_{4}$ where $\quad b=t$.

(a)

(b)

Figure 1.

In the case of a closed cross section, an equivalent formula is obtained by considering the equilibrium of horizontal forces on a piece of the croses section as shown in figure 2

$$
\begin{equation*}
q_{m}-q_{r}=\frac{P Q}{I} \tag{2b}
\end{equation*}
$$

Where $Q$ is now the static moment about the neutral axis of the (effective) areas of the longitudinals lying between the skin panels $r$ and $m$ where the shear intensities are measured.


Figure 2.

Shear stresses caused by torsion in a tube (fig. 3) are obtained by the well-known formula

$$
\begin{equation*}
q=\tau t=\frac{T}{2 A} \tag{3}
\end{equation*}
$$

applicable to thin-wall tubes. The angle of trist per unit length of tube is given by

$$
\theta=\frac{T}{G J}
$$

Where the torsion constant $J$ is defined by

$$
J=\frac{4 A^{3}}{\oint \frac{a s}{t}}
$$

the symbol denoting an integration around the entire circumference of the tube.


In practical cases, the thickness is constant over large parts of the circumference; the calculation of the line integral therefore reduces to the addition of a fer..fractions of the type s/t. Substitute (3) into the formula for angle of twist, and there is obtained

$$
\theta=\frac{q}{2 G A} \delta \frac{d s}{t}
$$

In the most general case, $G$ and $q$ may be variable along the circumferenco. Variation of $G$ may be due to the use of differont matertals or to the formation of diagonal-tension fields. Variation of $q$ may be caused by attaching other torsion tubes to form multicollular tubes. In the goneral case, tho rormula for twist becomes

$$
\begin{align*}
& \theta=\frac{I}{2 A} \oint q \frac{d s}{G_{e} t} \\
& \theta=\frac{I}{2 A} \oint \frac{T}{G_{\theta}} d s \tag{4b}
\end{align*}
$$

$$
(4 a)
$$

or

Where $G_{e} i_{s}$ the effective shear stiffness It should be noted that longitudinals have no influence in the gimple torsion problem.

The derivation of these torsion formulas, which may be found in a number of standard textbooks, is based on the assumptions that the torques are applied as shear stresses distributed over the end faces according to the theory and that the cross sections are free to follow the tendency to warp that exists in most cases. In practical structures, it is usually not posaible to comply with those assumptions. The root section is usually built in more or less completely, and the resulting restraint on the warping causes normal stresses, or bending stresses, and a rodistribution of shearing stressos (referonce 1). These effects disappear quite rapidiy with increasing distance from the root and are usually negligible at a distance from the root equal to, or greater than, the midth of the box. At the root, however, they may be quite appreciable.

The commonly accepted theory of shells in combined bending and torsion uses the simple theories of bending and torsion. Corrections may therefore be neceasary near the root to account for the efrects of reatraint againgt warping just mentioned, which modify the simple theory of torsion. Corrections may also be necessary to account fior the effocts of shear deformation, which modify the simplo bending theory (reference 2).

An emphatic word of warning must be given relating to the use of the theories of bending and torsion. These theories give fairly reliable results, if they are ured With judgment. The theory that the entire cross section acts as a unit naturally cannot be expected to hold very woll if the joints are not perfect or if the changea in dimonsions and shape are too sudden. Nose covors attached With piano hinges and trailing edgos with their acuto anglos at the tips are the most usual examples of structural components that cannot be oxpected to be fully effoctive either in bending or in torsion.

## Sign Conventions

Extornal forces will be taken as positive when act= ing upward. External torques will be taken as positivo when clockwiae.

Boris notation mill be usod to dosignate cells and walls. The cells will be designated by letters from left to right, starting with "a."

Shear stresses and forces in the walls of a cell will be taken as positive when going clockrise around the cell. If a wall belongs to two cells, the sign will be established by assuming the wall to belong to the left-hand cell.

Iine integrations will be performed in a clockrise direction. It should be noted that, in a wall belenging to two cells, the sign of the shear must be reversed in. the right-hand cell when performing a lfie integration, because the arrow of the positive shear direction as es tablished will oppose the sense of positive direction of integration. In doubtful cases, and preferably in all cases, free-body diagrams should be drawn indicating the directions of all forces. The use of such diagrams vill materially reduce the chances of errors in sign and will do aray vith the necessity of adhering rigidiy to a set convention of signs, provided that care is taken in writing the equations of equilibrium.

Note that, in cases where the sign convention is adhered to rigidiy, the basic equation (3) must be written

$$
q=-\frac{T}{2 A}
$$

The Open Shell
Open shells (fig. 4) can be analyzed by applying formulas (1) and (2). After the shear forces in each part of the cross section have been found, tho rosultant of the intermal shear forces can bo found by ordinary statics. This resultant lies on a vertical line distant $\theta$ from tho open wall of tho shell. Tho point whoro this resultant intersects tho horizontal axis is celled tho "shear center." The external load $P$ must pess through the shoar conter if thero is to be no torsion. Tho torsional stiffnoss and strongth of an open section boing extromely small, it is nocessary to koop the external load. vory close to tho shear contor. Tho knowlodgo of the shear center is therefore important for an opon soctまor.
(a)

(b)

(c)

(d)

(e)

Figure 4.


Figure 5.

In curvod shoots with a constant shear intensity $q$, such as the webs of tho sections shown in figures 4(b) and $4(\mathrm{c})$, it is ofton convonient to replace the shear stresses acting on the curved cross section by a resultant force. Integrating horizontal components, vertical components, and moments of the elementary shear forces, there is obtained (fig. 5) the horizontal resultant

$$
H=0 \text {. }
$$

the vertical resultant

$$
\begin{equation*}
V=q h \tag{5}
\end{equation*}
$$

and the torque moment about any point

$$
\begin{equation*}
T=2 q A_{0} \tag{6}
\end{equation*}
$$

Which gives as the location of the resultant force $R=V$ the distance

$$
\begin{equation*}
\theta=\frac{2 A_{0}}{h} \tag{7}
\end{equation*}
$$

from the open face of the shell. In these formulas, $A_{0}$ is the area included betreen the contour of the sheet and the open face. It should de noted that the formulas do not apply to the entire sections shown in figures 4(d) and $4(e)$ subjected to bending loads, because the shear intensity would not be constant. The formulas would apply only to the part of the section included between the two longitudinals next to the neutral axis.

Numerical examples 1 to 5 illustrate the analysis of open sections.

The Two-Flange, Single-Cell Box (D-Section)
The two-flange, single-cell section (fig. 6) may be considered as a combination of a beam and a torsion tube. The beam can take bending moments only in a plane parallel to the plane of the two flanges, so that the load $P$ producing the bending must be paraliel to this plane. The torsion tube can take care of the torsion existing fif the load $P$ is not applied at the shear center of the shell.

Tho total shear force acting on any cross section may be resolved into two components (fig. $6(b)$ ): the shear force $S_{T /}$ acting in the plane web, and the shear force $S_{N}$ in the nose sheet acting at the shear center of the nose shoct; these forces aro knorin as to location and direction but unknown as to magnituae. There aro available two equations of static oquilibrium to find them:

$$
\begin{aligned}
& \Sigma V=P-S_{W}+S_{N}=0 \\
& \Sigma M=-P d+S_{N} e=0 . \\
& S_{N}=P \frac{d}{e}=P \frac{d h}{2 A_{a}} . \\
& S_{H}=P+S_{N}=P\left(I+\frac{d h}{2 A_{a}}\right)
\end{aligned}
$$

giving
and, finally

$$
\begin{align*}
& T_{N}=\frac{S_{N}}{h t_{N}}=P \frac{d}{2 A_{a} t_{N}}  \tag{8}\\
& \tau_{W}=\frac{S_{W}}{h t_{W}}=\frac{P}{t_{W}}\left(\frac{1}{B}+\frac{d}{2 A_{a}}\right) \tag{9}
\end{align*}
$$

The flange stresses are found by using equation (I), which simplifies to

$$
\sigma=\frac{M}{h A_{F}}
$$

$A_{F}$ being the effective area of the flange.
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(a)

(b)

Figure 6.


Figure 7.

If the angle of twist per unit length da is desired, it can be found by substituting (8) and (9) into formula (4b)

$$
\begin{equation*}
\theta=\frac{I}{2 A G}\left[\frac{P d p}{2 A_{a} t_{N}}+\frac{P}{t_{T}}\left(1+\frac{d h}{2 A_{a}}\right)\right] \tag{10}
\end{equation*}
$$

assuming $G_{e}=G$ for both webs and denoting by $p$ the perimeter, or developed length, of the nose contour.

From equation (9) it can be concluded that the shear stress in the plane web becomes zero if $P$ is located at

$$
a=-\frac{2 A}{h}
$$

Which is the location of the shear center of the nose web alone. Vice versa, if $P$ is located at the shear center of the plane web, i.e., at the plane wed or $\mathrm{a}=0$, the shear stress in the nose web becomes zero.

Although the section shown in figure 6 is the most common example of the two-flange single-cell aection, it is not the only one. Figure 7 shows another example, a two-spar box with three-point atbachment. In this case, the spar attachod with a single bolt cannot act as a boam, and the box might be termed a "rectangular D-section."

Analysis of the D-Section by the Shear-Center Metiod
If the distance $d$ is chosen so that the angle of trist $\theta$ becomes zero, the condition of bending without torsion is obtained, and the distance $d_{0}$ thus found locates the shear center, or the elastic conter, of the ontire section. Letting $\theta=0$ in equation (io),

$$
\begin{equation*}
a_{0}=-\frac{2 \dot{A} t_{N}}{h t_{N}+t_{N}} \tag{11}
\end{equation*}
$$

A laad $P$ applied at $d_{0}$ will cause only bending, the associated shear stresses being

$$
\begin{align*}
& \tau_{N B}=-\frac{p}{h t_{N}+p t_{W}}  \tag{12a}\\
& \tau_{W B}=\frac{P_{p}}{h\left(h t_{N}+p t_{W}\right)} \tag{I2b}
\end{align*}
$$

A load $P$ applied at a distance $d$ can be replaced by an equal load $P$ at $d_{0}$ and a torque

$$
T=P\left(\alpha-d_{0}\right)
$$

The shear stresses caused by the torque can be calculated by equation (3) and added to the stresses given by equations (12a) and (12b). Obviously this method of analysis using the shear center is much more laborious than the direct method of analysis.

Example 6 in the appendix illustrates the analysis of D-sections.

The Two-Cell Torsion Tube
Since a single-cell tube is sufficient to take torsion, a two-cell tube (fis. 8) is statically indeterminate.


In order to analyze it, imagine the two celins to be split open and unknown shears of the intensities $q_{a}=X$ and $q_{b}=Y$ applied to the celis. One oquation for these two unknowns $\ddagger$ furnished by the static equation stating that the sum of the torques must be zero; using equation (6),

$$
\begin{equation*}
X \times 2 A_{a}+Y \times 2 A_{b}+T=0 \tag{13}
\end{equation*}
$$

An additional equation is obtained from the condition that elastic continuity must be preserved, namely, that the angle of twist of cell a relative to coll b must bo zero, or that the twist of cell a must oqual that of cell b.

$$
\begin{equation*}
\theta_{a}=e_{b} \tag{114}
\end{equation*}
$$

Expressing- $\theta_{a}$ and $\theta_{b}$ as functions of $X$ and $Y$, $y$ using oquation (4b), as

$$
\theta_{a}=\frac{I}{2 A_{a}} \oint_{a} \cdot \frac{T}{G} d s \text { and } \theta_{b}=\frac{I}{2 A_{b}} \oint_{b} \frac{T}{G} d s
$$

Tho second rolation noedod to find $X$ and $Y$ ia obtained by oquating these two exprossions in conformance with (14).

If the angle of twist is desired, it is found by substituting the values for $X$ and $Y$ in the expression for $\theta_{a}$ or for $\theta_{b}$.

Bxample $\gamma$ illustrates the numerical procedure.

The Two-Flange Two-Gell Shell
The analysis of the two-flange two-cell shell in combined bending and torsion (fig. 9 (a)) is closely analogous to the emalysis of the two-cell torsion tube. Imagine the two cells to be split open (fig. $9(\mathrm{z})$ ) and the shearg of intensities $X$ and $Y$ appliod. Tho load $p$ located at d is replaced by a load $P$ located in the plane of tho flanges and a torque Pa. The shear intensity in the shoar web is then

$$
\begin{equation*}
q_{W}=X \rightarrow Y+\frac{P}{\bar{h}} . \tag{15}
\end{equation*}
$$

Wquations (13) and (14) are again used to find the shear intensities $X$ and $Y$, as in tho case of the torsion tube; the only difforonco between the two aases lios in tho appearance of tho term $P / h$ in the yeb shear intonsity.

It should be noted in figure $9(a)$ that there is only a single bolt attaching the auxiliary rear spar and that the flanges of the rear spar are dotted, indicating that they do not enter into the calculation.

Example 8 illustrates the analysis of a two-flange two-cell shell.

(a)


Analysis of the Two-Flange Two-Cell Shell by the Shear-Center Method

The location of the shear center is found as before from the condition that tho angle of twist must be zero.

Fquating $\epsilon_{a}$ and $\epsilon_{b}$ each to zero, two equations are obtained instead of the single equation (14). These tro equations together with (13) are sufficient to find the unknown location $d_{0}$ at which $P$ must be placed to produco bending without torsion as well as the shear stresses associated with this special case of bending.

After the shear center has been found and the solutions for bending only and for torque only have been completed, eny additional loading case to be investigated may be broken up into a combination of bending only and torsion only, as discussed fir the two-flange, two-iveb shell. The analysis of any given case then consists merely in multiplying the stresses from the two basic solutions by appropriate factors and adding them. This method requires Iess numerical work than setting up and solving equations (13) and (14) for each case. Consequentiy, the shearcenter method of analysis saves time if a sufficient number of cases are investigated, so that the total time saved on individual cases overbalances the time required for finding the shear conter and malsing the basic solutions.

It might be pointed out that the same adrantages can be had by using any arbitrary-Ioad case and the puretorsion case as basic cases. If the arbitrary case chosen as basic is for a load $F_{1}$ located at $d_{1}$, then a load $P_{2}$ located at $d_{2}$ can be replaced by a load $P_{a}$ at $d_{1}$ and a torque $P_{2}\left(d_{2}-d_{1}\right)$. The analysis of additional cares is therefore just as simple as if the shear-center method had boen used.

The shear-center method is illustrated by example 9.
The Three-Flange Single-CeIl Shell
The single-cell shell with three flanges is of interest as the practical example of a D-section capable of taking bending in any plane (fig. 10). This section can be easily analyzed for the general case of a section without symmetry. The location of the resiltant shear in aach web is known from formula (7), and tho equilibrant of the load $P$ can be resolved into three. forces alang these lines by statics. If the load is parallel to the plane of two flanges, the third flange is unstrcssed, and tho shear intensity $q$ is.constent for the two webs joining the third.flange. The analysis is then analogoug to that of a twonflange shell.
b.

d
Figure 10
, The MuItiflange Single-Cell Shell
The four-flange box (ifg. Il) may be considered from two points of view. If the upper cover is cut, the lower cover will also arop out of action. The structure is then the familiar two-spar wing. This structure is statically determinate (for vertical loads), torsicn being taken care. of by one spar bending down and the other one bending up: With the cover intact, the structure is statically indeterminate.

In the commonily accepted shell theory, the torsion taken by opposite bending of the spars is neglected. Torsion is assumed to be aisorbed entirely by the four walls acting together as a torsion tube. All flanges are assumed to act as a unit, namely, a single beam obeying the engineering theory of bending. The shell is then, in principle, analogous to the D-section, consisting of a combination of a torsion tube and a beam, and is statically detera minate. This conclusion remains valid if there are longitudinals attached to the cover sheets. Each longitudinal introduces one more unknown shear stress in the sheet and also one adaitional equation of equilibrium of forces along the $z$ axis.

The jusifification for using this theory in preference to the one first mentioned lies in the fact that, for allmetal stressed-skin wings, the torque taken by differential bending of the epars is very small compared with that taken by the torsion tupe except near the root. In the region of the root, corrections must be made to allow for this effect, as will be discussed later.

(a)


As example, the equation for the ample case of figure II will be developed.

$$
\begin{align*}
& \Sigma V: P+S_{1}-S_{R}=0  \tag{a}\\
& \Sigma M:-P d+S_{3} h+S_{2} V=0 \tag{b}
\end{align*}
$$

$I I$ on flange $I_{1} d F_{1}+\frac{S_{1}}{h} d z-\frac{S_{3}}{W} d z=0$
$\Sigma I$ on flange $\ddot{2}: d E_{a}+\frac{S_{3}}{W} d z-\frac{S_{a}}{h} d z=0$
Now

$$
\begin{equation*}
\frac{d}{d z}\left(F_{1}+F_{z}\right)=\frac{d}{d z}\left(\frac{M}{h}\right)=\frac{P}{h} \tag{d}
\end{equation*}
$$

Since the stresses in the tiro flanges are assumed to be equal

$$
\begin{equation*}
\frac{d F_{1}}{d z}=\frac{P}{h} \frac{A_{1}}{A} \text { and } \frac{d F_{a}}{d z}=\frac{P}{h} \frac{A_{a}}{A} \tag{e}
\end{equation*}
$$

Where

$$
A=A_{1}+A_{2}
$$

Substituting (e) into (c) and (a)

$$
\begin{equation*}
\frac{P A_{1}}{h A}+\frac{S_{1}}{h}-\frac{S_{3}}{W}=0 \tag{f}
\end{equation*}
$$

$$
\begin{equation*}
\frac{P A_{2}}{h A}+\frac{S_{3}}{w}-\frac{S_{2}}{h}=0 \tag{g}
\end{equation*}
$$

From (g) $\quad S_{3} h=S_{2} W-\frac{P A_{2} W}{A}$
Substituting into (b)

$$
\begin{align*}
-P d+S_{2} W & -\frac{P A_{a} W}{A}+S_{a}=0 \\
S_{a} & =\frac{P}{2}\left(\frac{d}{W}+\frac{A_{a}}{A}\right)  \tag{16}\\
S_{2} & =\frac{P}{2}\left(\frac{d}{W}+\frac{A_{a}}{A}-2\right)  \tag{17}\\
S_{3} & =\frac{P}{2 h}\left(d-W \frac{A_{a}}{A}\right) \tag{1.8}
\end{align*}
$$

In the more general case of the trapezoidal box (fig. la (a)), the equations become

$$
\begin{aligned}
& \dot{V}: P+S_{1}-S_{a}-S_{3} \frac{h_{1}-h_{a}}{W^{i}}=0 \\
& \Sigma M:-P d+S_{3} h_{1} d+S_{2} W=0 \\
& \Sigma I_{1}: d F_{1}+\frac{S_{1}}{h_{1}} d z-\frac{S_{3}}{W^{i}} d z=0 \\
& \Sigma I_{1 a}: d F_{a}+\frac{S_{3}}{W^{i}} d z-\frac{S_{3}}{h_{a}} d z=0
\end{aligned}
$$

In this case

$$
\frac{d F_{1}}{d Z}=\frac{h_{1} A_{1}}{2 I} P \quad \text { and } \quad \frac{d F_{2}}{d Z}=\frac{h_{8} A_{2}}{2 I} P
$$

so that

$$
S_{3}=S_{2} \frac{W:}{h_{a}}-\frac{h_{2} W^{\prime}}{2 I} A_{a} P
$$

and finally

$$
\begin{equation*}
s_{2}=P \frac{h_{2}}{h_{1}+h_{2}}\left[\frac{d}{w}+\frac{h_{1} h_{2}}{2 I}\right] \tag{19}
\end{equation*}
$$

Substituting $S_{s}$ into the proceding equation gives $S_{3}$, and the first equation then yiolds $S_{1}$.


Figure 12.
If a number of stringors with a total cross-sectional area $\Lambda_{3}$ are uniformiy distributod along the vidth of tho cover sheets (fig. $12(b)$ ), the formula for $S_{a}$ becomes

$$
\begin{equation*}
S_{2}=P \frac{h_{E}}{h_{1}+h_{2}}\left[\frac{d}{W}+\frac{h_{1} h_{2} h_{2}}{2 I}+\frac{h_{1}\left(h_{1}+2 h_{2}\right) A_{3}}{12 I}\right] \tag{19a}
\end{equation*}
$$

Since the analysis of a given case consistsmerely in substituting numerical values into equations (I6), (I7), and (18), no examples will be given hore. If examples aro desired, they may bo found in reference 3 , which covers in detail the analysis of the four-ilange box by the ehoarcenter method:

## The Multicellular Shell

Inasmuch as the multiflange single-cell shell is statically determinate, the multicell shell is statically indeterminate. The method of analysis is analogous to that used for the two -flange, multicell shell and will be illustrated .by the example of a four-flanges tro-cell shell (fig. 13 (a)).


Figure 13.

Imagine each cell cut open and shears of intensity $X$ and $Y$ applied. The transverse load $P$ causes shear loads $S_{1}$ and $S_{2}$ in the two spar webs, which are proportional to the moments of inertia of these spars

22

$$
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$$

$$
\begin{equation*}
S_{1}=P \frac{I_{1}}{I} \text { and } S_{a}=P \frac{I_{a}}{I} . \tag{20}
\end{equation*}
$$

with

$$
I_{1}=\frac{I}{2} A_{1} h_{1}^{2} \quad I_{2}=\frac{I}{2} A_{3} h_{2}^{2} \quad I=I_{1}+I_{2}
$$

The equations (4b) for angle of twist are written down, and the condition of continuity

$$
\theta_{a}=\theta_{b}
$$

furnishes one equatinn. The second equation is found, as before, from the static condition that the internal torque must balance the external torque.

- Talring moments about $A_{1}$

$$
\begin{equation*}
-P d+S_{a} T+2 A_{a} X+2 A_{0} Y=0 \tag{2I}
\end{equation*}
$$

The analysis is closely analogous to the analysis of the two-flange, two-cell soction previously discussod. The only difference is that the shear intensity in the rear spar is now $Y+\frac{S_{a}}{h_{2}}$ instead of $Y$.

Example lo illustratas the analyais of a four-flange, two-cell section by the direct method. Example 11 illustrates the method of finding the shear center and tho shoar stresses associatod with bonding.

The procodure for more complicatod casos (fig. 14) is merely a simple extonsion of the procedure discussod, so that no oxample will be required.


Figure 14.

As mentioned before, the simple theories of bending and of torsion used thus far may require corrections. These corrections are important only in the inboard region near the root, if they are important at all. Whenever they are to be made, it is very adrisable to separate bending from torsion at the outset. Such a procedure will make the calculations much clearer and will materially reduce the $\dot{\text { a }}$ anger of committing errors in aign.

The flange material in box beams is usually distributed ecross the cover in the form of individual stringers or corrugated sheet. The bending action of such beans differs from that assumed by the simple theory of bending, because the sheet deforms under the shear stresses imposed on it. The analysis of bending action under such circumstances is discussed in reference 2 and no discussion will be given here.

In tubes subjected to torsion, the cross sections usually have a tendency to warp out of their original planes. If this warping is prevented by attaching the tube to a rigid support, or oy conditions of symmetry in the midale of the span, then longitudinal (normal) stresses will arise, and the shear stresses will be redistributed. Reference 4 gives a method of calculating the effects for cross sections of arbitrary shape, but the method is of limited usefulness. Methods of analyzing rectangular tubes have been developed by a number of writers; the most important reports dealing with the methods are summarized in reference 1. This reforence also gives what appears to be the only published experimental data. Thoy do not agree very well with tho theory; fortunately, the effects of end rostraint are small in most practical cases, so that thoy need not be very accurately calculated.

For a rectangular tube symmetrical about both axes, such as shown in figure l5, the normal forces on the corner flanfes caused by complete restraint may be calculated by the formula

$$
\begin{equation*}
X= \pm 0.56 \frac{T}{A}\left(\frac{b}{t_{b}}-\frac{c}{t_{c}}\right) \cdot \sqrt{\frac{A_{P}}{\frac{b}{t_{b}}+\frac{c}{t_{c}}}} \tag{22}
\end{equation*}
$$

(reference 1 , equation ( $3 c$ ) with $G / E=0.4$ ). The sisn of the stresses is determined from the rule that the walle with the smaller ratio of width to thickness (in general, the vertical walls) act like independent spars, absorbing tho torque by bending in opposite diroctions.


Figure 15.

The effect of the end restraint on the shoar intensities is written most conveniontly in the form

$$
\begin{equation*}
\Delta q= \pm \frac{T}{2 A} \frac{\frac{b}{t_{b}}-\frac{c}{t_{c}}}{\frac{b}{t_{b}}+\frac{c}{t_{c}}} \tag{23}
\end{equation*}
$$

obtainod by using oquations (3c). (9), and (10) of reference $1 ; \Delta q$ is the correction to be applied to the shear intensity calculated on the assumption of no restraint, namely,

$$
q=\frac{T}{2 A}
$$

The negative sign in (23) is used for the walls with the larger ratio of width to thickness, in general the horizontal walls.

In actual cases, the box will seldom be symmetrical about both axes, as assumed in the derivation of formulas (22) and (23). The simplest procedure in such a case will be to use average values for $b, c, t_{b}$, and $t_{c}$. This procedure is somewhat unconservative, but formulas (22) and (2\%) are basically conservative because they assume infinitely clasely spaced bulkheads. Furthermore, except in such cases as wings continuous across the center line of the eirplane, the root section will not-be rigidly built in, because there will be play in the fittings and olastic yielding in the fittings and in the centor section.

The simple procedure outlincd here and used in oxamplo 12 méy, of courso, be ingufficiont in some casos; a moro detailed troatment, howevor, is boyond tho scopo of this papor.

Langloy Memorial Aeronautical Laboratory, Wational Advisory Committoo for Aoronautics, Langley Field, Va., January 20, 1939.

```
A PPENDIX
    Example I
```

Find the shear intensity and the shear center of the section shown in figure 16 for a vertical load $P$.

$$
\begin{aligned}
& I=2 A \bar{x}^{2}=2 \times \frac{I}{4} \times 4=2 \text { in. }^{4} \\
& Q=A \bar{x}=\frac{1}{4} \times 2=\frac{1}{2} \text { in. }^{3}
\end{aligned}
$$

By formula (2), the shear intensity is

$$
q=\frac{\underline{p}}{\mathrm{q}} \frac{\mathrm{l}}{2}=\frac{1}{4} \mathrm{PI} \mathrm{Ib} . / \mathrm{in} .
$$

In this simple case, the correct direction of the arrow can be found by inspection, and the sign convention is not noedod.

The force $H$ in the leg (ing. l6(b)) is

$$
H=q W=\frac{1}{4} P \times 3=\frac{3}{4} P
$$

$$
\text { H } \times 4=x \times P \quad x=3 \text { in. }
$$

or the location e of the shear center $\theta=x+w=6$ in. $=2 \pi$, which agrees with formula (7).

(a)

(b)

Figure 16.

## Example 2

Find the shear intensity and the shear center for the section shown in figure 17.

Bow's notation is used as indicated:

$$
\begin{gathered}
I=2 \times \frac{1}{4} \times 4+2 \times \frac{1}{4} \times 4=4 \mathrm{in} .^{4} \\
Q_{a c}=\frac{1}{4} \times 2=\frac{1}{2} \\
Q_{a b}=\frac{1}{4} \times 2+\frac{1}{4} \times 2=1
\end{gathered}
$$

The shear intensities are therefore

$$
q_{a c}=\frac{P \times \frac{1}{3}}{4}=\frac{1}{8} P \quad q_{a b}=\frac{P \times I}{4}=\frac{1}{4} P
$$

The horizontal force H is

$$
E=a_{a c} \times W=\frac{1}{8} P \times 3=\frac{3}{8} P
$$

Taking moments about the lower left corner of the channel as before,

$$
\begin{gathered}
P x=H h=\frac{3}{8} P \times 4 \quad x=\frac{3}{2} \operatorname{in} . \\
e=\frac{3}{2}+3=4 \frac{1}{2} \text { in. }=\frac{3}{2}
\end{gathered}
$$


(a)

(b)

Figure 77.

## Example 3

Find the shear intensities and the shear center of the section shown in figure 18 .

$$
\begin{aligned}
& I=6 \text { in. }{ }^{4} \\
& Q_{a d}=\frac{1}{2} \text { in. }^{3} \quad q_{a d}=P \times \frac{1}{12} \operatorname{lb} . / \operatorname{in} . \\
& Q_{a c}=I \text { in. }^{3} \quad q_{a c}=P \times \frac{1}{6} \mathrm{Ib} . / \text { in. } \\
& Q_{a b}=1 \frac{1}{2} \text { in. }{ }^{3} \quad q_{a b}=P \times \frac{1}{4} I b . / i n . \\
& H=q_{E d} \times 1.5+\dot{q}_{a c} \times 1.5=\frac{P}{12} \times 1.5+\frac{p}{6} \times 1.5=\frac{3}{8} P \\
& P_{X}=E h=\frac{3}{8} P \times 4=\frac{3}{2} P \\
& x=\frac{3}{2} \text { in. }
\end{aligned}
$$

as in the preceding case. The relation $e=\frac{3}{2}$ holds for all channels, if the effective material is uniformly distributed along the legs of the channel.

(a)


Figure 18.

## Example 4

Find the shear intensity and the shear center for the section shown in figure 19 .

$$
\begin{gathered}
I=2 A_{F} R^{2} \text { in. } \\
Q=A_{F} R \text { in. } \\
q=P \frac{A_{F} R}{2 A_{F} R}=\frac{D}{2 R} \mathrm{Ib} \cdot / \operatorname{In} .
\end{gathered}
$$

Taking moments about the center of the circle

$$
\begin{aligned}
& P_{e}=q \times \pi R \times R=\frac{P}{2 R} \times \pi R \times R \\
&=\frac{\pi P R}{2} \\
& e=\frac{\pi}{2} R \quad \text { in. }
\end{aligned}
$$



Figure $1 y$.

## Example 5

Find the shear center of the section shown in figure 20. In this case, all the sheet is effective in bending.

$$
I=\frac{1}{2} \pi R^{3} t \quad \operatorname{in} .^{4}
$$

$Q_{\theta}=\int_{0}^{\theta} R t d \theta \times R \cos \theta=R^{2} t \sin \theta \operatorname{in} .^{3}$
$q_{\theta}=P \times \frac{R^{2} t \sin \theta}{\frac{I}{2} \pi R^{3} t}=P \frac{2}{\pi} \frac{\sin \theta}{R} I n / i n$.
Taking moments about the center of the circle.

$$
\begin{gathered}
P_{\theta}=\int_{0}^{\pi} P \frac{2}{\pi} \frac{\sin \theta^{\prime}}{R} \operatorname{RiG} \times R \\
\theta=\frac{4}{\pi} R \text { in. }
\end{gathered}
$$



Figure 20.

Fxamplo 6
Given the D-soction shown in figuro 21.
(a) By diroct analysis, finit tho strossos in tho soction and the anglo of trist, assuming that no buckling occurs.
(b) Find the shear center of the section, and make the analysis by the shear-center method.
(c) Find the changes caused by the flat shoet doveloping a full diag-onal-tonsion field.
(a) Direct analy-
sis.- Find first the 10cation o of the shear center of the nose by formula (r)

$$
\theta_{N}=\frac{2 A_{a}}{h}=\frac{2 \frac{\pi}{2} R^{2}}{2 R}=\frac{\pi}{2} \cdot R=15.71 \text { in }
$$

Taking moments about the plane web

$$
\begin{gathered}
S_{N} \times 15.71=P Q=5,000 \times 5=25,000 \mathrm{in} .-1 \mathrm{~b} . \\
S_{N}=1,5911 \mathrm{~b} .
\end{gathered}
$$

which gives

$$
S_{W}=6,591 \mathrm{Ib} .
$$

The shear stresses are therefore.

$$
\begin{aligned}
& T_{N}=\frac{1591}{0.064 \times 20}=1,2441 \mathrm{~b} . / \mathrm{sq} \cdot \mathrm{in} . \\
& T_{W}=\frac{6591}{0.032 \times 20}=10,3001 \mathrm{n} . / \mathrm{sq} . \mathrm{in} .
\end{aligned}
$$

The angle of twist is obtained from the basic formula (4b)

$$
\begin{aligned}
\theta & =\frac{1}{2 A_{a}^{G}} \oint T d s \\
& =\frac{1}{2 \times \frac{\pi}{2} \times 100 G}[I, 244 \times \pi \times 10+10,300 \times 20] \\
& =\frac{1}{G} \times 780
\end{aligned}
$$

With $G=4 \times 10^{6}$, this value becomes

$$
\text { 今 }=195 \times 10^{-6} \text { radian per inch length. }
$$

(b) Shear-center analygis.- In this simple case, the location of the shear center can be found from formula (ll)

$$
d_{0}=-\frac{2 \times \frac{\pi}{2} \times 100 \times 0.064}{20 \times 0.064+\pi \times 10 \times 0.032}=-8.80 \mathrm{in} .
$$

In order to illustrate the procedure used in general cases, the solution will be carried through starting from fundamental principles.

The load $P$ is assumed to act at an unknown distance d. There are then three unknowns: TN, TM, and d. To find theso unknowns, there are available the static equations $\Sigma V=0$ and $\Sigma M=0$ and the elastic equation $\theta=0$.

$$
\begin{gathered}
\Sigma V=q_{N} h-q_{W} h+P=0 \\
\Sigma M(a b o u t \text { the } \pi e b)=q_{N} h \theta-P d_{0}=0 \\
\theta=\frac{1}{2 A_{a} G}\left[\frac{q_{N}}{t_{N}} \times P+\frac{q}{t_{N}} \times h\right]=0 \quad \text { (by equation (4a)) }
\end{gathered}
$$

Numericalzy

$$
\begin{gathered}
q_{N} \times 20-q_{\mathbb{N}} \times 20+5,000=0 \\
q_{N} \times 20 \times 15.71-5,000 d_{0}=0 \\
\theta G=\frac{1}{2 \times 157.1}\left[\frac{q_{N}}{0.064} \times \pi \times 10+\frac{q_{W}}{0.032} \times 20\right]=0
\end{gathered}
$$

These three equations are solved and yield

$$
\begin{gather*}
q_{N}=-140.11 \mathrm{~b} . / \mathrm{in} . ; \quad q_{W}=110.0 \mathrm{Ib} . / \mathrm{in} . \\
\alpha_{0}=-8.80 \mathrm{in} \tag{a}
\end{gather*}
$$

The shear intensities just obtained are those associated with bending caused by a load $P$ applied at the shear center. The actual load is applied at $d=5$, so that there is a torque

$$
T=-5,000(5+8.80)=-69,000 \text { in. }-1 \mathrm{~b} .
$$

giving a shear intensity

$$
\begin{equation*}
q=\frac{69000}{2 \times 157.1}=219.8 \tag{b}
\end{equation*}
$$

The total shear. intensities are then

$$
q_{N}=-140.1+219.8=79.7
$$

and

$$
\tau_{N}=\frac{q_{N}}{0.064}=1,245 \mathrm{Ib}, / \mathrm{sq} . \operatorname{in}
$$

$$
q_{V}=110.0+219.8=329.8
$$

$$
\tau_{W}=\frac{q_{W}}{0.032}=10,3001 \mathrm{~b} . / \mathrm{sq} . \mathrm{in}:
$$

(c) Changes caused by buckling in flat web. - When the flat wo is allowed to bucirlo into a full diafonal-tonsion field, the effective thickness becomes

$$
t_{\theta}=\frac{5}{8} t=\frac{5}{8} \times 0.032=0.020 \mathrm{in} .
$$

In order to evaluate the angle of tirist $\theta$, it is necessary to obtain an effective shear stress

$$
T_{e}=\frac{8}{5} T=\frac{8}{5} \times 10,300=16,500 \mathrm{Ib} \cdot / \mathrm{sq} \cdot \mathrm{in} .
$$

Substituting this value into the expression for $\theta$

$$
\begin{aligned}
\theta & =\frac{I}{100 \pi G}[1,244 \pi \times 10+16,500 \times 20] \\
& =\frac{1}{G} \times 1,174=293.5 \times 10^{-6} \text { radian per inch }
\end{aligned}
$$

The changed location $d_{0}$ of the elastic center is obtained by substituting $t_{e}$ instead of $t$ into formula (11)

$$
d_{0}=-\frac{2 \times \frac{\pi}{2} \times 100 \times 0.064}{20 \times 0.064+\pi \times 10 . \times 0.020}=-10.65 \mathrm{In.}
$$

## Example 7

Find the shear stresses and the angle of twist in the torsion tube shown in figure 22. Assume $G_{0}=G$ for all WaIls.


It rill be necessary to sot up expressions for tho angiles of twist $\theta$ in terms of the shear intensities $X$ and Y by using formula (Aa). As a preliminary stop, tho auxinmary parameters

$$
a=\int \frac{d g}{t}
$$

Will bo computed, so that formula (Aa) will bo used in the form

$$
\theta G=\frac{\lambda}{2 \AA} \oint \mathrm{aq}
$$

Bow's notation is used as indicated.

$$
\begin{aligned}
& a_{a c}=\frac{51}{0.020}=2,550 \\
& a_{b d}=\frac{44}{0.073}=602.5 \\
& a_{b e}=\frac{20}{0.036}=555 \\
& a_{b f}=\frac{44}{0.030}=1,467 \\
& a_{b a}=\frac{24}{0.051}=470
\end{aligned}
$$

(Note that $a_{b a}=a_{a b}$. )
The expresgions for the angles of twist are

$$
\begin{aligned}
\theta_{a} G & =\frac{1}{2 \times A_{a}}\left[X \times a_{a c}+(X-Y) a_{a b}\right] \\
& =\frac{1}{2 \times 192}[X \times 2,550+(X-Y) 470] \\
& =3.85 X-0.600 Y \ldots \\
\theta_{b} G & =\frac{I}{2 A_{b}}\left[Y a_{b d}+Y a_{b e}+Y a_{b e}+(Y-X) a_{b a}\right] \\
& =\frac{I}{2 \times 990}[Y \times 602.5+Y \times 555+Y \times 1,467+(Y-X) 470] \\
& =I .562 Y-0.2376 X
\end{aligned}
$$

$s$

Equating $\theta_{a}=\theta_{b}$, obtain

$$
\begin{equation*}
4.088 X=2.162 Y \tag{a}
\end{equation*}
$$

The equation of moment equilibrium (I3) is

$$
\begin{equation*}
X \times 2 \times 392+Y \times 2 \times 990-250,000=0 \tag{b}
\end{equation*}
$$

The solution of these two equations is

$$
X=55.3 \mathrm{lb} . / \text { in. } \quad Y=104.5 \text { 10. } / \mathrm{in} .
$$

The shear stresses are therefore

$$
\begin{aligned}
& T_{a c}=\frac{X}{t}=\frac{55.3}{0.020}=2,765 \mathrm{Ib} . / \mathrm{sq} . \mathrm{in} . \\
& \tau_{b d}=\frac{Y}{t}=\frac{104.5}{0.073}=1,431 \mathrm{Ib} . / \mathrm{sq} . \mathrm{in} . \\
& T_{b e}=\frac{Y}{t}=\frac{104.5}{0.036}=2,900 \mathrm{Ib} . / \mathrm{sq} . \mathrm{in} . \\
& \tau_{b f}=\frac{Y}{t}=\frac{104.5}{0.030}=3,48010 . / \mathrm{sq} \cdot \mathrm{in} . \\
& T_{a b}=\frac{X}{t}=\frac{Y}{t}=\frac{49.2}{0.051}=-9651 \mathrm{~b} . / \mathrm{sq.in} .
\end{aligned}
$$

The angle of twist is obtained by substituting into the expression for $e_{a}$

$$
\theta=\epsilon_{a}=\frac{1}{G}[3.85 X-0.600 Y]=\frac{1}{G} \times 150.3
$$

Pith $G=4 \times 10^{6}$ this value becomes

$$
\theta=37.6 \times 10^{-6} \text { radian per inch length, }
$$

Example 8
Find the shear stresses in the section shown in figpure 23. This section is identical with the one used in example 7, except that two flanges have been added to take care of beam action.

(a)


- Figure 23.

The parameters a can be taken from the preceding example.

The shear intensity caused by the load $P$ acting on the shear web is

$$
q_{W}=\frac{P}{h}=\frac{5000}{24}=208.21 \mathrm{~b} . / \mathrm{in} .
$$

The expressions for $\theta$ are now written exactly as in example 7 except for the addition of $\mathrm{g}_{\mathrm{W}}$.

$$
\begin{aligned}
& \text { N.A.C.A. Technical Mote No: 691. } \\
& \theta_{a} G=\frac{1}{2 \times 392}[\mathrm{X} \times 2,550+(X-Y+208.2) \times 470] \\
&=3.85 \mathrm{X}-0.600 \mathrm{Y}+124.96 . \\
& \theta_{\mathrm{b}} \mathrm{G}^{G}=\frac{1}{2 \times 990}[\mathrm{Y} \times 602.5+\mathrm{Y} \times 555+\mathrm{Y} \times 1467+(\mathrm{Y}-\mathrm{X}-208.2) \times 470] \\
&=1.562 \mathrm{Y}-0.2376 \mathrm{X}-49.5
\end{aligned}
$$

Equating $\theta_{a}{ }^{G}$ to. $\theta_{b} G$ gives

$$
\begin{equation*}
4.088 X-2.162 Y+174.46=0 \tag{a}
\end{equation*}
$$

The equation of moment equilibrium is taken around the shear web, to oliminate one term.

$$
\begin{equation*}
X \times 2 \times 392+Y \times 2 \times 990-5,000 \times 46=0 \tag{b}
\end{equation*}
$$

Solving equations (a) and (b), obtain

$$
X=15.5 \quad Y=110.1
$$

The shear stresses are therefore

$$
\begin{aligned}
& \mathrm{T}_{\mathrm{ac}}=\frac{15.5}{0.020}=775 \mathrm{Ib} . / \mathrm{sq} . \operatorname{In} . \\
& \tau_{b d}=\frac{110.1}{0.073}=1,5101 \mathrm{~b} . / \mathrm{sq} \cdot \mathrm{in} . \\
& T_{b e}=\frac{110.1}{0.036}=3,0601 \mathrm{~b} . / \mathrm{sq.in} . \\
& T_{b f}=\frac{110.1}{0.030}=3,670 \mathrm{Ib} . / \mathrm{sq} . \operatorname{in} . \\
& \tau_{a b}=\frac{15.5-110.1}{0.051}+208.2=2,2261 \mathrm{~b} . / \mathrm{sq.in} .
\end{aligned}
$$

## Example 9

For the section used in example 8 , find the shear center, and analyse the load case of example 8 by the shearcenter method.

Two equations are obtainea by equating to zero the ex~ pressions for $\theta_{a}$ and $\theta_{b}$, which are taken from example 8.

$$
\therefore
$$

$$
3.85 X-0.600 Y+124.96=0
$$

$$
-0.2376 X+1.562 Y-49.5=0
$$

The solution of these equations gives the shear intensities associated with torsion-free bending. .

$$
X=-28.21 b . / i n . \quad Y=27.421 b . / i n .
$$

The $\lambda i \operatorname{stance} d_{0}$ af the shear center from the shear web is obtained by writing $\Sigma M$ about the shear web

$$
\begin{gathered}
-5,000 \mathrm{~d}-28.2 \times 2 \times 392+27.42 \times 2 \times 990=0 \\
d_{0}=6.44 \mathrm{in}
\end{gathered}
$$

$A$ load $P$ located at $d=46$ inches. $W i l l$ therofore cause a torque

$$
-P\left(d-d_{0}\right)=-5,000(46-6.44)=-197,800 \text { in. }-1 b
$$

The stresses $T_{B}$ due to bonding are obtained from $X \quad$ and Y as before.

The strosses $T_{T}$ due to the torque of--I97,800 in.-Ib. are obtained by multiplying the stresses from example 7 by $\frac{1}{2} \frac{97800}{50000}=0.791$.

- The final stresses $T$ are obtained by superposition as shown in the following table.

| Tall | $\begin{gathered} \mathrm{T}_{\mathrm{B}} \\ \left(\mathrm{Ib} \cdot / \mathrm{sq}_{\mathrm{i}} \mathrm{in}_{0}\right) \end{gathered}$ | $\left(10 . / \mathrm{sq} \cdot \mathrm{~T}_{\mathrm{m}}\right)$ | $\left(1 \mathrm{~b}, / \mathrm{sq}_{\mathrm{q}}^{\mathrm{T}} \mathrm{in},\right)$ |
| :---: | :---: | :---: | :---: |
| ac | $-1,410$ | 2,190 | 780 |
| bd | 376 | 1,132 | 1,508 |
| be | 762 | 2,297 | 3,059 |
| bi | 914 | 2,757 | 3,671 |
| ab | 2,990 | -764 | 2,226 |

Example 10
For the section shown in figure 24, find. the shear stresses. The section is identical with that shown in figure 22 except for the flanges. The load is assumed to be perpendicular to the neutral axis. The inclination of the shear webs is neglected.

(b)

Figure 24.

Imagine the cover to be slotted in both cells as indicated, leaving a structurc consisting of two spars. The vortical shear is dividod between these two spars in the ratio of their momenta of inertia.

$$
\begin{aligned}
& I_{1}=\frac{1}{2} \times 1.85 \times 24^{2}=533 \operatorname{In} .4 \\
& I_{a}=\frac{1}{2} \times 1.65 \times 20^{2} .330 . \operatorname{In} .4 \\
& I=I_{1}+I_{2}=863 \text { in. } .^{4}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& S_{1}=P \times \frac{I_{1}}{I}=5,000 \times \frac{533}{863}=3,0901 \mathrm{~b} \\
& q_{1}=\frac{S_{1}}{h_{1}}=\frac{3090}{24}=128.81 \mathrm{~b} . / \mathrm{in} . \\
& S_{2}=P \times \frac{I_{2}}{I}=5,000 \times \frac{330}{863}=1,9101 \mathrm{~b} \\
& q_{a}=\frac{S_{a}}{h_{2}}=\frac{1910}{20}=95.51 \mathrm{~b} . / \mathrm{in} .
\end{aligned}
$$

Procooding as in tho previous cases, write the expressions for $\theta_{a}$ and $\theta_{b}$.

$$
\begin{aligned}
& \theta_{\theta}=\frac{1}{2 \times 392}[2,550 X+470(X-\dot{Y}+128.8)] \\
& =3.85 \mathrm{X}-0.600 \mathrm{Y}+77.3 \\
& \theta_{b}=\frac{1}{2 \times 990}[602.5 \Psi+555(Y+95.5)+I, 467 Y+470(Y-X-128.8)] \\
& =1.562 Y-0.2376 X-3.8
\end{aligned}
$$

Equating $\overline{\theta_{a}}$ to $\theta_{b}$, obtain

$$
4.09 X-2.162 Y+81.1=0
$$

The equation of moment equilibrium gives

$$
-5,000 \times 46+I, 910 \times 44+2 \times 392 X+2 \times 990 \times Y=0
$$

Solving these trod equations, obtain

$$
X=15.9 \mathrm{lb} . / \mathrm{in} . \quad Y=67.5 \mathrm{lb} . / \mathrm{in}
$$

The shear stresses are therefore

$$
\begin{aligned}
& \tau_{\mathrm{ac}}=\frac{\mathrm{X}}{0.020}=795 \mathrm{Ib} \cdot / \mathrm{sq} \cdot \mathrm{in} \\
& \tau_{\mathrm{bd}}=\frac{Y}{0.073}=9251 \mathrm{~b} . / \mathrm{sq} . \operatorname{in}
\end{aligned}
$$

$$
\begin{aligned}
& T_{b e}=\frac{Y+q_{2}}{0.036}=4,5301 \mathrm{~b} . / \mathrm{sq} . \operatorname{in} . \\
& T_{b f}=\frac{Y}{0.030}=2,2501 b . / \mathrm{sq} . \operatorname{in} . \\
& \tau_{a b}=\frac{X .+q_{1}-Y}{0.05 I}=1,5131 \mathrm{Y} . / \mathrm{sq.in} .
\end{aligned}
$$

Example 11
Find the shear center; and the shear stresses assodiate with torsion-free bending, for the section shown in figure 24.

Take the expressions for $\theta_{a}$ and. $\theta_{b}$ from the prem ceding example and equate each one to zero.

$$
\begin{aligned}
& 3.85 X-0.600 Y+77.3 \div 0 \\
& -0.2376 X+1.562 Y-3.8=0
\end{aligned}
$$

Solving

$$
X=-20.2^{\prime} 1 \mathrm{~b} . / \mathrm{in} . \quad Y=-0.635 \mathrm{Ib} . / \mathrm{in} .
$$

The shear stresses are therefore

$$
\begin{aligned}
T_{a c} & =-\frac{20.2}{0.020}=-1,0101 \mathrm{~b} . / \mathrm{sq.in} \\
T_{b d} & =-\frac{0.635}{0.073}=-91 \mathrm{~b} . / \mathrm{sq.in} \\
T_{b e} & =\frac{-0.635+95.5}{0.036}=2,6401 \mathrm{~b} . / \mathrm{sq.in} \\
T_{b f} & =-\frac{0.635}{0.030}=-211 \mathrm{b.} / \mathrm{sq.in} \\
T_{a b} & =-20.2+1
\end{aligned}
$$

Leaving the location. d. of. the load $P$ undetermined, write the moment equation

$$
\Sigma M=-5,000 d+1,910 \times 44-2 \times 392 \times 20.2-2 \times 990 \times 0.635=0
$$

Which gives as location of the shear center

$$
d=13.39 \text { in. behind front shear web }
$$

Fxample 12
For the section analyzed in examples 10 and ll, find the stressea if the section is a root section that is rigidly built in. The length $L$ of the beam is 200 inches; the load $P$ is applied at the tip.

The first step is to seperate the load on the ontire section into bending moment and torque. The bending moment is

$$
M=P I=5,000 \times 200=1,000,000 \text { in. }-1 \mathrm{lb}
$$

According to exemple 9 , the ghear center is located at $d_{0}=6.44$ inches, and the torque is

$$
T=P\left(a-d_{0}\right)=-197,800 \text { in. }-1 b
$$

The effects of restraint against marping will be calculated under the assumption that only tho approximatoly rectangular cell $b$ between the four main fittings is row strained against warping and that the nose part hes no influenco on theso warping stresses.

According to example 7 , a torque of 250,000 in. $n l b$. creates a shoar intensity $Y=104.5$ lb./in. incoll b. The oxisting torquo of 197, 800 in.wlb. Will thorefore givo a shoar intonsity

$$
q_{b}=104.5 \cdot \frac{197800}{250000}=82.811 \mathrm{~b} . / \mathrm{in} .
$$

The torque carried by cell b is therefore (approximately)

$$
T_{0}=82.8 \times 2 \times 990=164,000 \text { in. }-10
$$

With the average values

$$
\begin{array}{ll}
b=.44 \mathrm{in} . & t_{b}=\frac{0.073+0.030}{2}=0.0515 \mathrm{in} . \\
c=\frac{24+20}{2}=22 \mathrm{in} . & t_{c}=\frac{0.051+0.036}{2}=0.0435 \mathrm{in} .
\end{array}
$$

$$
A_{\text {F. }}=\frac{1.85+1.65}{2}=1.75 \cdot \mathrm{sq.in.}
$$

The normal force on the flange due to torque becomes, by formula (22)

$$
X=0.56 \frac{164000}{44 \times 22}(855-506) \sqrt{\frac{1.75}{855+506}}=1,190 \mathrm{Ib} .
$$

and the correction for shear intensity becomes, by formila (23)

$$
\Delta \mathrm{q}=\frac{164000}{2 \times 44 \times 22} \frac{(855-506)}{(855+506)}=21.7515 . / \mathrm{in}
$$

The bending stresses due to the bending moment are, in the front flanges,

$$
\sigma_{I B}= \pm \frac{1000000 \times 12}{863}= \pm 13,900 \mathrm{Ib} . / \mathrm{sq} . \text { in. }
$$

and, in the rear flanges,

$$
\sigma_{2 B}= \pm \frac{1000000 \times 10}{863}= \pm 11,580 \mathrm{Ib} . / \mathrm{sq} . \mathrm{in} .
$$

the upper sign applying to the upper flange in each case. The bending stresses duo to torque are

$$
\sigma_{I T}=\frac{X}{A_{1}}=\frac{1190}{1.85}= \pm 643 \mathrm{Ib} . / \mathrm{sq.in} .
$$

and

$$
\sigma_{2 \mathbb{T}}=\frac{X}{A_{2}}=\frac{1190}{1.65}= \pm 721 \mathrm{Ib} . / \mathrm{sq}: \operatorname{in}
$$

the upper sign applying to the upper flanges. Tho final strosses are therefore.

$$
\begin{aligned}
& \sigma_{1 U}=-13,900+643=-13,257 \text { Ib./sq.in. } \\
& \sigma_{I I}=13,900-643=13,257 \mathrm{Ib} \cdot / \mathrm{sq} \cdot \text { in. } \\
& \sigma_{2 U}=-11,580-72 I=-12,30 I \cdot I \mathrm{~b} . / \mathrm{sq.in} . \\
& \sigma_{2 I}=11,580+721=12,30 I \text { Ib./sq.in. }
\end{aligned}
$$

The shear stresses for the section free to warp are obtained from example 10 . Superposing the corrections $\Delta t=\frac{\Delta q}{t}$ gives the final shear stresses

$$
\begin{aligned}
& \tau_{a c}=795+0=795 \mathrm{Ib} \cdot / \mathrm{sq} . \mathrm{In}_{\mathrm{a}} . \\
& \tau_{b d}=925-\frac{2 i .75}{0.073}=627 \mathrm{Ib} . / \mathrm{sq} . \mathrm{in} \text {. } \\
& T_{b e}=4,530+\frac{21.75}{0.036}=5,134 \mathrm{Ib} . / \mathrm{sq} \text {. in. } \\
& T_{b f}=2,250-\frac{21.75}{0.030}=1,5251 \mathrm{~b} . / \mathrm{sq} . \operatorname{in} . \\
& \tau_{b a}=-1,513+\frac{21.75}{0.051}=-1,0871 \mathrm{~b} . / \mathrm{sq} . \mathrm{in} .
\end{aligned}
$$

The corrections influence the design of the rear spar more critically than the front spar. The correction on the flange stress is somewhat over 6 percent; on the web shear stress, it is somewhat over 13 percent. An error of 25 percent on tho corroction would thoreforo causo an error of $1-1 / 2$ percent on the flange stress and an error of 3 percent on tho web shoar stress.

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