TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 691

SOME ELEMENTARY PRINCIPLES OF SHELL STRESS ANALYSIS WITH NOTES ON THE USE OF THE SHEAR CENTER

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Washington
March 1939

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SUMMARY

The analysis of various types of shell under combined bending and torsion is discussed. The calculation and the use of the shear center are touched upon as incidental problems. Twelve fully worked numerical examples are given in an appendix.

INTRODUCTION

The literature on shell analysis is quite scattered and some of it is not easily available. A definite need therefore exists for setting down in a reasonably comprehensive, but concise, manner the principles and the methods used in the various phases of shell analysis. The present paper deals with the distribution of the stresses, chiefly the shearing stresses, over the cross sections of cantilever shells of constant cross section subjected to combined bending and torsion.

The subject matter is far from being new; as the appended bibliography indicates, the main principles were well established in 1931. Continued discussion in the technical literature indicates, however, that the knowledge of the subject is not so widely disseminated as it needs to be. It is hoped that the manner of presentation chosen for this paper, in particular the collection of fully worked numerical examples, will help materially to achieve the ultimate goal, namely, to provide all practicing engineers with a working knowledge of the subject.

THE ANALYSIS OF SHELLS

Basic Assumptions and Theories

Typical cases of the problem to be treated are shown in figure 1(a). As indicated by the figure, the shell will be assumed to have a constant cross section. It will also be assumed, in general, that the material effective in bending is disposed symmetrically about the horizontal axis. The horizontal and the vertical axes will then be principal axes, and the load will be assumed to act vertically.

For purposes of stress analysis, the structures are idealized in the usual manner. A certain effective width of skin is added to each actual longitudinal or flange to obtain the cross-sectional area effective in bending; this effective area is assumed to be concentrated at the centroid. The skin itself is assumed to carry only shear. Idealized sections are represented as in figure 1(b).

Stresses caused by bending are obtained in first approximation by applying the engineering theory of bending to the idealized cross sections. This theory is based on the assumptions that plane cross sections remain plane and that Hooke's law applies. The theory leads to the formulas, for the normal stresses due to bending,

$$\sigma = \frac{My}{I} \tag{1}$$

and, for the shear stresses due to bending,

$$\tau = \frac{PQ}{bI} \tag{2a}$$

The derivation of these formulas can be found in any textbook on strength of materials.

It should be noted that, in the computation of the static moments Q as well as of the moments of inertia I, consideration is given only to the material assumed to be effective in bending.

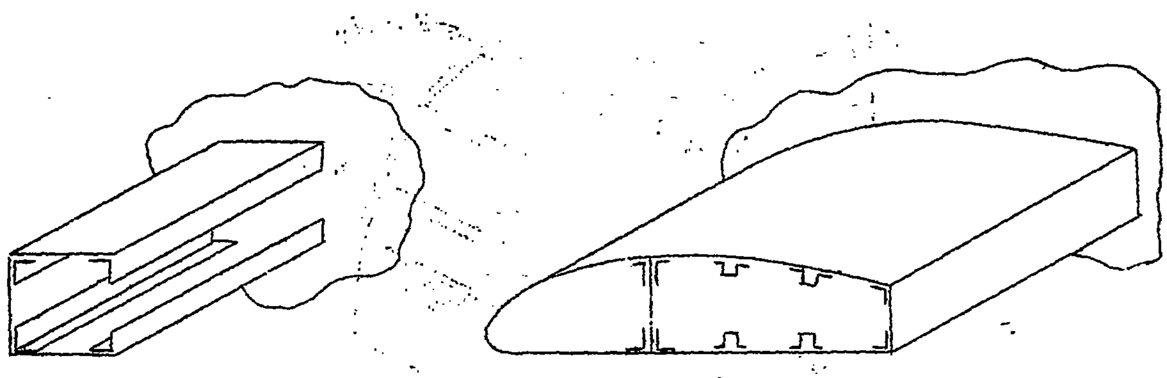
In shell structures, it is often convenient to use, not the shear stress τ, but the shear force per inch length of sheet, which will be designated by

$$q = \tau t$$

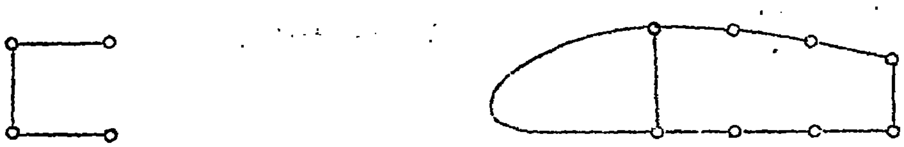
and which will be called the "shear-force intensity" or, briefly, "shear intensity." Formula (2a) then becomes

$$q = \frac{PQ}{I} \qquad (2)$$

for open sections, where $b = t$.



(a)



(b)

Figure 1.

In the case of a closed cross section, an equivalent formula is obtained by considering the equilibrium of horizontal forces on a piece of the cross section as shown in figure 2

$$q_m - q_r = \frac{PQ}{I} \qquad (2b)$$

where Q is now the static moment about the neutral axis of the (effective) areas of the longitudinals lying between the skin panels r and m where the shear intensities are measured.

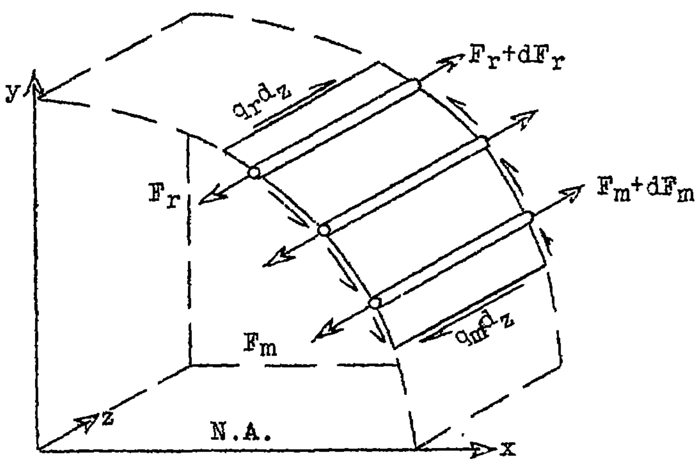


Figure 2.

Shear stresses caused by torsion in a tube (fig. 3) are obtained by the well-known formula

$$q = \tau t = \frac{T}{2A} \qquad (3)$$

applicable to thin-wall tubes. The angle of twist per unit length of tube is given by

$$\theta = \frac{T}{GJ}$$

where the torsion constant J is defined by

$$J = \frac{4A^2}{\oint \frac{ds}{t}}$$

the symbol $\oint$ denoting an integration around the entire circumference of the tube.

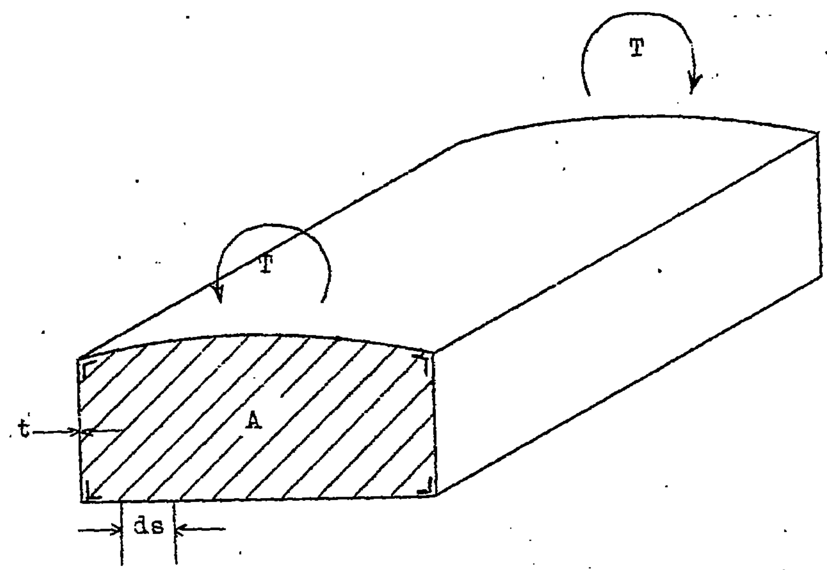


Figure 3.

In practical cases, the thickness is constant over large parts of the circumference; the calculation of the line integral therefore reduces to the addition of a few fractions of the type s/t. Substitute (3) into the formula for angle of twist, and there is obtained

$$\theta = \frac{q}{2GA} \oint \frac{ds}{t}$$

In the most general case, G and q may be variable along the circumference. Variation of G may be due to the use of different materials or to the formation of diagonal-tension fields. Variation of q may be caused by attaching other torsion tubes to form multicellular tubes. In the general case, the formula for twist becomes

$$\theta = \frac{1}{2A} \oint q \frac{ds}{G_e t} \qquad (4a)$$

or

$$\theta = \frac{1}{2A} \oint \frac{\tau}{G_e} ds \qquad (4b)$$

where G_e is the effective shear stiffness. It should be noted that longitudinals have no influence in the simple torsion problem.

The derivation of these torsion formulas, which may be found in a number of standard textbooks, is based on the assumptions that the torques are applied as shear stresses distributed over the end faces according to the theory and that the cross sections are free to follow the tendency to warp that exists in most cases. In practical structures, it is usually not possible to comply with these assumptions. The root section is usually built in more or less completely, and the resulting restraint on the warping causes normal stresses, or bending stresses, and a redistribution of shearing stresses (reference 1). These effects disappear quite rapidly with increasing distance from the root and are usually negligible at a distance from the root equal to, or greater than, the width of the box. At the root, however, they may be quite appreciable.

The commonly accepted theory of shells in combined bending and torsion uses the simple theories of bending and torsion. Corrections may therefore be necessary near the root to account for the effects of restraint against warping just mentioned, which modify the simple theory of torsion. Corrections may also be necessary to account for the effects of shear deformation, which modify the simple bending theory (reference 2).

An emphatic word of warning must be given relating to the use of the theories of bending and torsion. These theories give fairly reliable results, if they are used with judgment. The theory that the entire cross section acts as a unit naturally cannot be expected to hold very well if the joints are not perfect or if the changes in dimensions and shape are too sudden. Nose covers attached with piano hinges and trailing edges with their acute angles at the tips are the most usual examples of structural components that cannot be expected to be fully effective either in bending or in torsion.

Sign Conventions

External forces will be taken as positive when acting upward. External torques will be taken as positive when clockwise.

Bow's notation will be used to designate cells and walls. The cells will be designated by letters from left to right, starting with "a."

Shear stresses and forces in the walls of a cell will be taken as positive when going clockwise around the cell. If a wall belongs to two cells, the sign will be established by assuming the wall to belong to the left-hand cell.

Line integrations will be performed in a clockwise direction. It should be noted that, in a wall belonging to two cells, the sign of the shear must be reversed in the right-hand cell when performing a line integration, because the arrow of the positive shear direction as established will oppose the sense of positive direction of integration. In doubtful cases, and preferably in all cases, free-body diagrams should be drawn indicating the directions of all forces. The use of such diagrams will materially reduce the chances of errors in sign and will do away with the necessity of adhering rigidly to a set convention of signs, provided that care is taken in writing the equations of equilibrium.

Note that, in cases where the sign convention is adhered to rigidly, the basic equation (3) must be written

$$q = -\frac{T}{2A}$$

The Open Shell

Open shells (fig. 4) can be analyzed by applying formulas (1) and (2). After the shear forces in each part of the cross section have been found, the resultant of the internal shear forces can be found by ordinary statics. This resultant lies on a vertical line distant e from the open wall of the shell. The point where this resultant intersects the horizontal axis is called the "shear center." The external load P must pass through the shear center if there is to be no torsion. The torsional stiffness and strength of an open section being extremely small, it is necessary to keep the external load very close to the shear center. The knowledge of the shear center is therefore important for an open section.

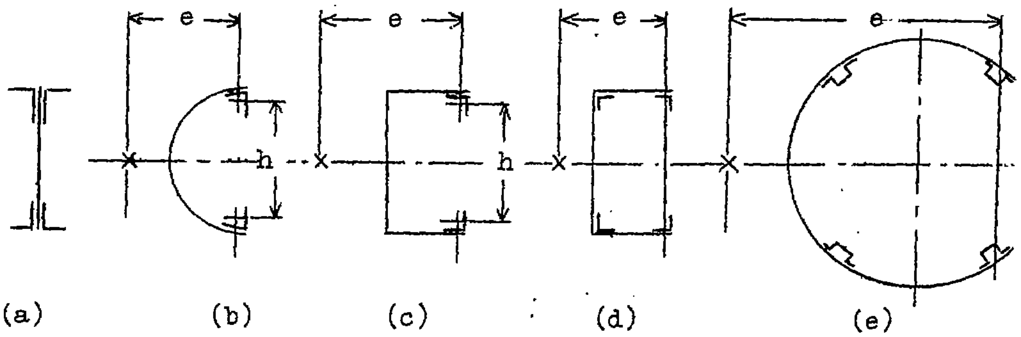


Figure 4.

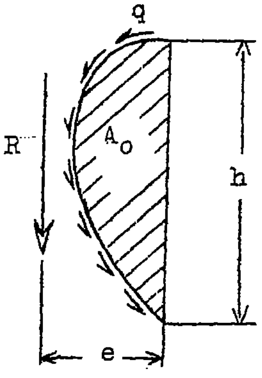


Figure 5.

In curved sheets with a constant shear intensity q, such as the webs of the sections shown in figures 4(b) and 4(c), it is often convenient to replace the shear stresses acting on the curved cross section by a resultant force. Integrating horizontal components, vertical components, and moments of the elementary shear forces, there is obtained (fig. 5) the horizontal resultant

$$H = 0$$

the vertical resultant

$$V = qh \tag{5}$$

and the torque moment about any point

$$T = 2qA_o \tag{6}$$

which gives as the location of the resultant force $R = V$ the distance

$$e = \frac{2A_o}{h} \tag{7}$$

from the open face of the shell. In these formulas, A_o is the area included between the contour of the sheet and the open face. It should be noted that the formulas do not apply to the entire sections shown in figures 4(d) and 4(e) subjected to bending loads, because the shear intensity would not be constant. The formulas would apply only to the part of the section included between the two longitudinals next to the neutral axis.

Numerical examples 1 to 5 illustrate the analysis of open sections.

The Two-Flange, Single-Cell Box (D-Section)

The two-flange, single-cell section (fig. 6) may be considered as a combination of a beam and a torsion tube. The beam can take bending moments only in a plane parallel to the plane of the two flanges, so that the load P producing the bending must be parallel to this plane. The torsion tube can take care of the torsion existing if the load P is not applied at the shear center of the shell.

The total shear force acting on any cross section may be resolved into two components (fig. 6(b)): the shear force S_W acting in the plane web, and the shear force S_N in the nose sheet acting at the shear center of the nose sheet; these forces are known as to location and direction but unknown as to magnitude. There are available two equations of static equilibrium to find them:

$$\Sigma V = P - S_W + S_N = 0$$

$$\Sigma M = - Pd + S_N e = 0$$

giving

$$S_N = P \frac{d}{e} = P \frac{dh}{2A_a}$$

$$S_W = P + S_N = P \left(1 + \frac{dh}{2A_a}\right)$$

and, finally

$$\tau_N = \frac{S_N}{ht_N} = P \frac{d}{2A_a t_N} \tag{8}$$

$$\tau_W = \frac{S_W}{ht_W} = \frac{P}{t_W}\left(\frac{1}{h} + \frac{d}{2A_a}\right) \tag{9}$$

The flange stresses are found by using equation (1), which simplifies to

$$\sigma = \frac{M}{hA_F}$$

A_F being the effective area of the flange.

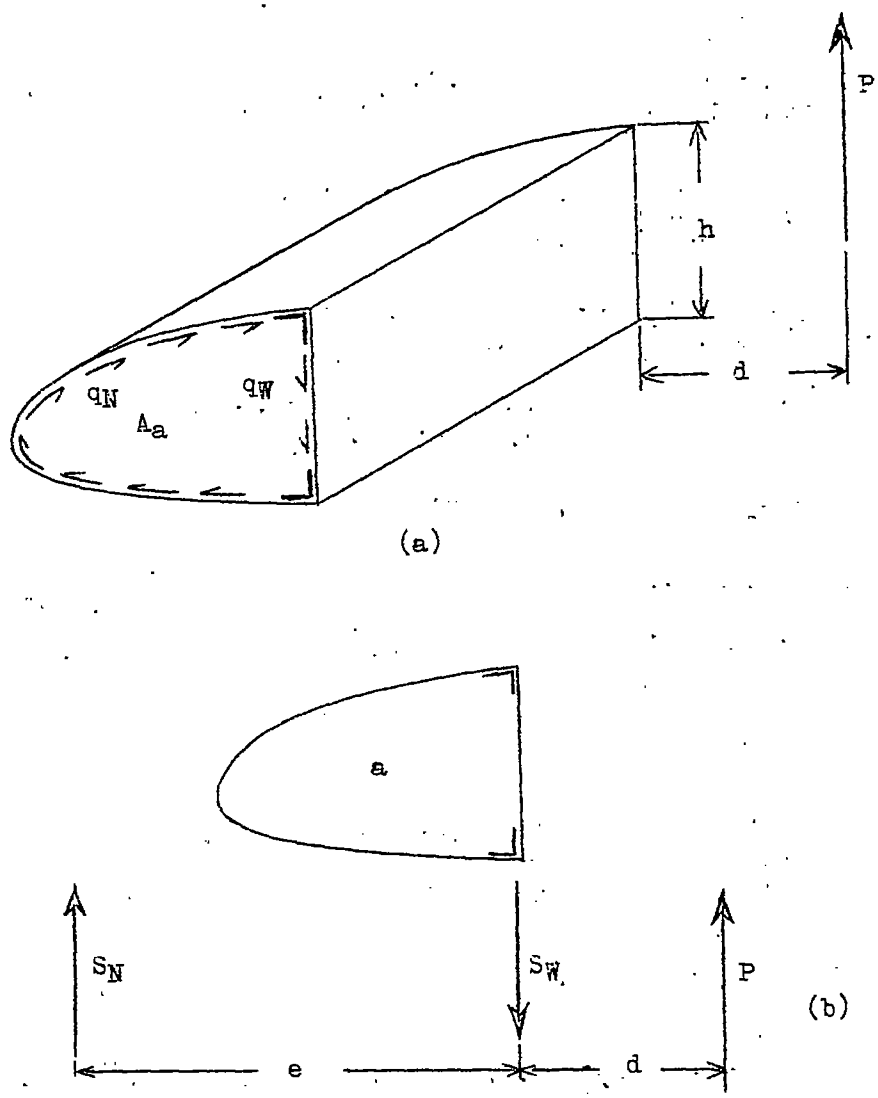


Figure 6.

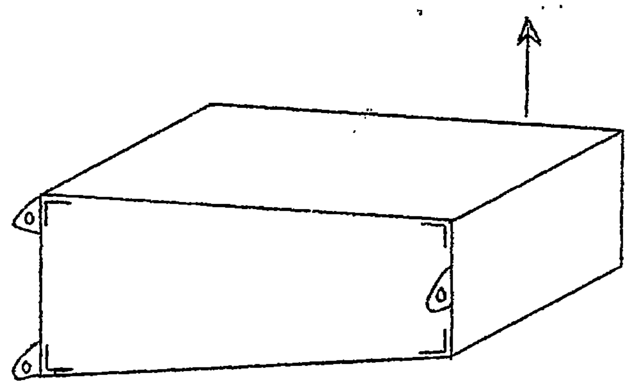


Figure 7.

If the angle of twist per unit length d_z is desired, it can be found by substituting (8) and (9) into formula (4b)

$$\theta = \frac{1}{2AG}\left[\frac{Pdp}{2A_a t_N} + \frac{P}{t_W}\left(1 + \frac{dh}{2A_a}\right)\right] \tag{10}$$

assuming $G_\theta = G$ for both webs and denoting by p the perimeter, or developed length, of the nose contour.

From equation (9) it can be concluded that the shear stress in the plane web becomes zero if P is located at

$$d = -\frac{2A}{h}$$

which is the location of the shear center of the nose web alone. Vice versa, if P is located at the shear center of the plane web, i.e., at the plane web or $d = 0$, the shear stress in the nose web becomes zero.

Although the section shown in figure 6 is the most common example of the two-flange single-cell section, it is not the only one. Figure 7 shows another example, a two-spar box with three-point attachment. In this case, the spar attached with a single bolt cannot act as a beam, and the box might be termed a "rectangular D-section."

Analysis of the D-Section by the Shear-Center Method

If the distance d is chosen so that the angle of twist θ becomes zero, the condition of bending without torsion is obtained, and the distance d_o thus found locates the shear center, or the elastic center, of the entire section. Letting $\theta = 0$ in equation (10),

$$d_o = -\frac{2At_N}{ht_N + pt_W} \tag{11}$$

A load P applied at d_o will cause only bending, the associated shear stresses being

$$\tau_{NB} = -\frac{P}{ht_N + pt_W} \tag{12a}$$

$$\tau_{WB} = \frac{Pp}{h(ht_N + pt_W)} \tag{12b}$$

A load P applied at a distance d can be replaced by an equal load P at d_o and a torque

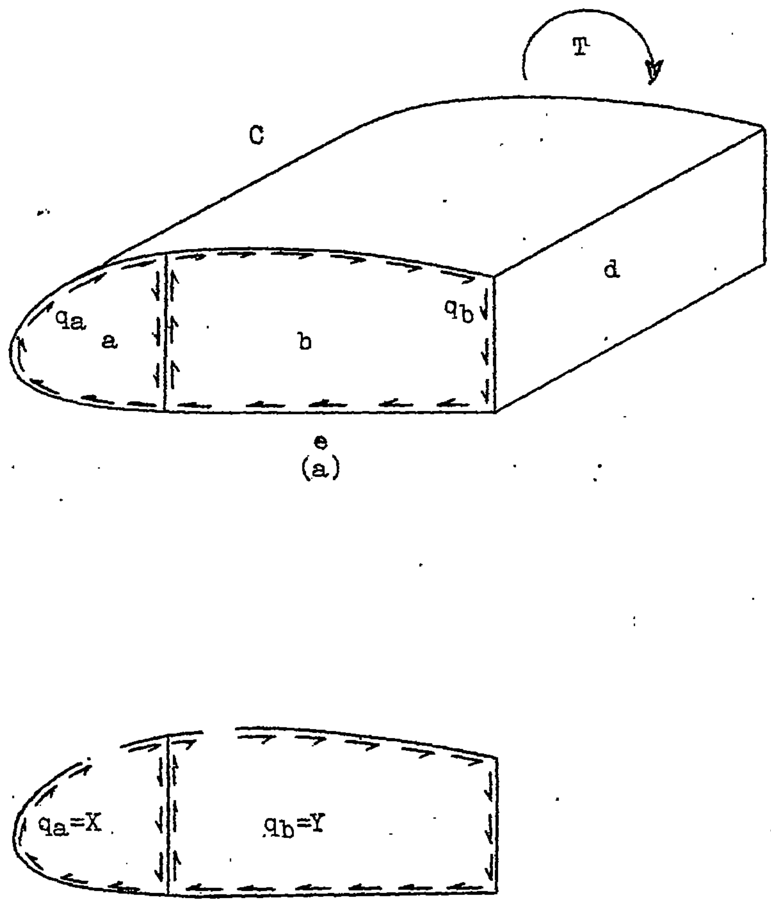
$$T = P(d - d_o)$$

The shear stresses caused by the torque can be calculated by equation (3) and added to the stresses given by equations (12a) and (12b). Obviously this method of analysis using the shear center is much more laborious than the direct method of analysis.

Example 6 in the appendix illustrates the analysis of D-sections.

The Two-Cell Torsion Tube

Since a single-cell tube is sufficient to take torsion, a two-cell tube (fig. 8) is statically indeterminate.



(a)

(b)

Figure 8.

In order to analyze it, imagine the two cells to be split open and unknown shears of the intensities $q_a = X$ and $q_b = Y$ applied to the cells. One equation for these two unknowns is furnished by the static equation stating that the sum of the torques must be zero; using equation (6),

$$X \times 2A_a + Y \times 2A_b + T = 0 \qquad (13)$$

An additional equation is obtained from the condition that elastic continuity must be preserved, namely, that the angle of twist of cell a relative to cell b must be zero, or that the twist of cell a must equal that of cell b.

$$\theta_a = \theta_b \qquad (14)$$

Expressing θ_a and θ_b as functions of X and Y, by using equation (4b), as

$$\theta_a = \frac{1}{2A_a} \oint_a \frac{T}{G}\, ds \quad \text{and} \quad \theta_b = \frac{1}{2A_b} \oint_b \frac{T}{G}\, ds$$

The second relation needed to find X and Y is obtained by equating these two expressions in conformance with (14).

If the angle of twist is desired, it is found by substituting the values for X and Y in the expression for θ_a or for θ_b.

Example 7 illustrates the numerical procedure.

The Two-Flange Two-Cell Shell

The analysis of the two-flange two-cell shell in combined bending and torsion (fig. 9(a)) is closely analogous to the analysis of the two-cell torsion tube. Imagine the two cells to be split open (fig. 9(b)) and the shears of intensities X and Y applied. The load P located at d is replaced by a load P located in the plane of the flanges and a torque Pd. The shear intensity in the shear web is then

$$q_W = X - Y + \frac{P}{h} \qquad (15)$$

Equations (13) and (14) are again used to find the shear intensities X and Y, as in the case of the torsion tube; the only difference between the two cases lies in the appearance of the term P/h in the web shear intensity.

It should be noted in figure 9(a) that there is only a single bolt attaching the auxiliary rear spar and that the flanges of the rear spar are dotted, indicating that they do not enter into the calculation.

Example 8 illustrates the analysis of a two-flange two-cell shell.

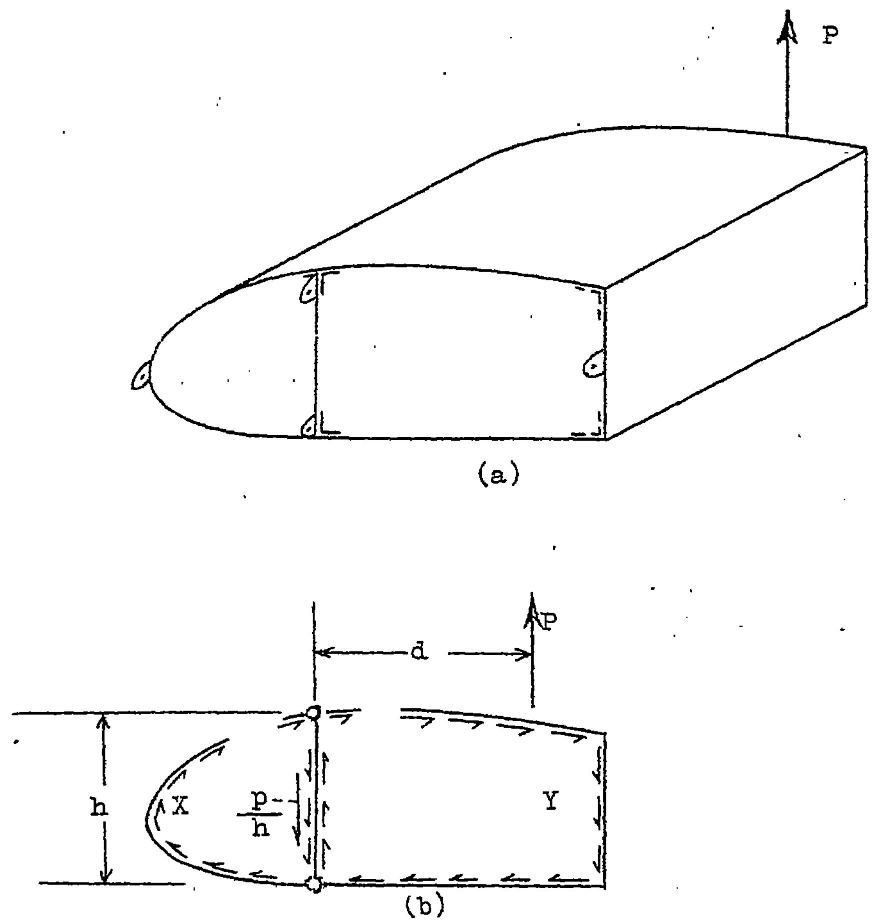


Figure 9.

Analysis of the Two-Flange Two-Cell Shell by the Shear-Center Method

The location of the shear center is found as before from the condition that the angle of twist must be zero.

Equating ϵ_a and ϵ_b each to zero, two equations are obtained instead of the single equation (14). These two equations together with (13) are sufficient to find the unknown location d_o at which P must be placed to produce bending without torsion as well as the shear stresses associated with this special case of bending.

After the shear center has been found and the solutions for bending only and for torque only have been completed, any additional loading case to be investigated may be broken up into a combination of bending only and torsion only, as discussed for the two-flange, two-web shell. The analysis of any given case then consists merely in multiplying the stresses from the two basic solutions by appropriate factors and adding them. This method requires less numerical work than setting up and solving equations (13) and (14) for each case. Consequently, the shear-center method of analysis saves time if a sufficient number of cases are investigated, so that the total time saved on individual cases overbalances the time required for finding the shear center and making the basic solutions.

It might be pointed out that the same advantages can be had by using any arbitrary-load case and the pure-torsion case as basic cases. If the arbitrary case chosen as basic is for a load P_1 located at d_1, then a load P_2 located at d_2 can be replaced by a load P_2 at d_1 and a torque P_2 (d_2-d_1). The analysis of additional cases is therefore just as simple as if the shear-center method had been used.

The shear-center method is illustrated by example 9.

The Three-Flange Single-Cell Shell

The single-cell shell with three flanges is of interest as the practical example of a D-section capable of taking bending in any plane (fig. 10). This section can be easily analyzed for the general case of a section without symmetry. The location of the resultant shear in each web is known from formula (7), and the equilibrant of the load P can be resolved into three forces along these lines by statics. If the load is parallel to the plane of two flanges, the third flange is unstressed, and the shear intensity q is constant for the two webs joining the third flange. The analysis is then analogous to that of a two-flange shell.

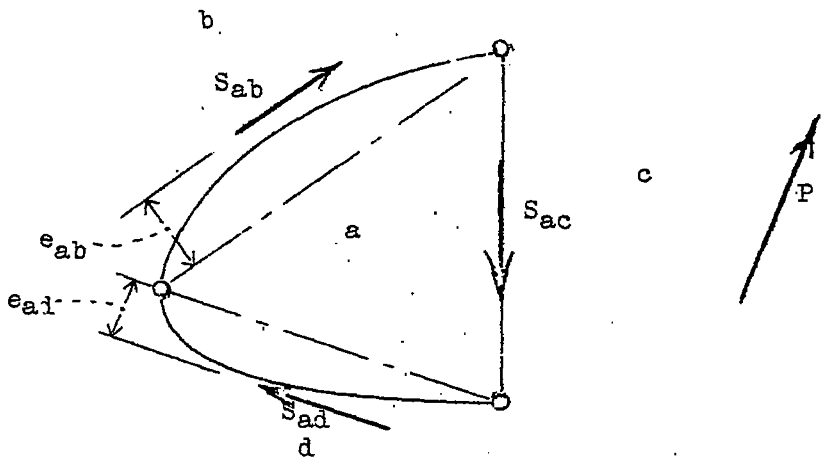


Figure 10

The Multiflange Single-Cell Shell

The four-flange box (fig. 11) may be considered from two points of view. If the upper cover is cut, the lower cover will also drop out of action. The structure is then the familiar two-spar wing. This structure is statically determinate (for vertical loads), torsion being taken care of by one spar bending down and the other one bending up. With the cover intact, the structure is statically indeterminate.

In the commonly accepted shell theory, the torsion taken by opposite bending of the spars is neglected. Torsion is assumed to be absorbed entirely by the four walls acting together as a torsion tube. All flanges are assumed to act as a unit, namely, a single beam obeying the engineering theory of bending. The shell is then, in principle, analogous to the D-section, consisting of a combination of a torsion tube and a beam, and is statically determinate. This conclusion remains valid if there are longitudinals attached to the cover sheets. Each longitudinal introduces one more unknown shear stress in the sheet and also one additional equation of equilibrium of forces along the z axis.

The justification for using this theory in preference to the one first mentioned lies in the fact that, for all-metal stressed-skin wings, the torque taken by differential bending of the spars is very small compared with that taken by the torsion tube except near the root. In the region of the root, corrections must be made to allow for this effect, as will be discussed later.

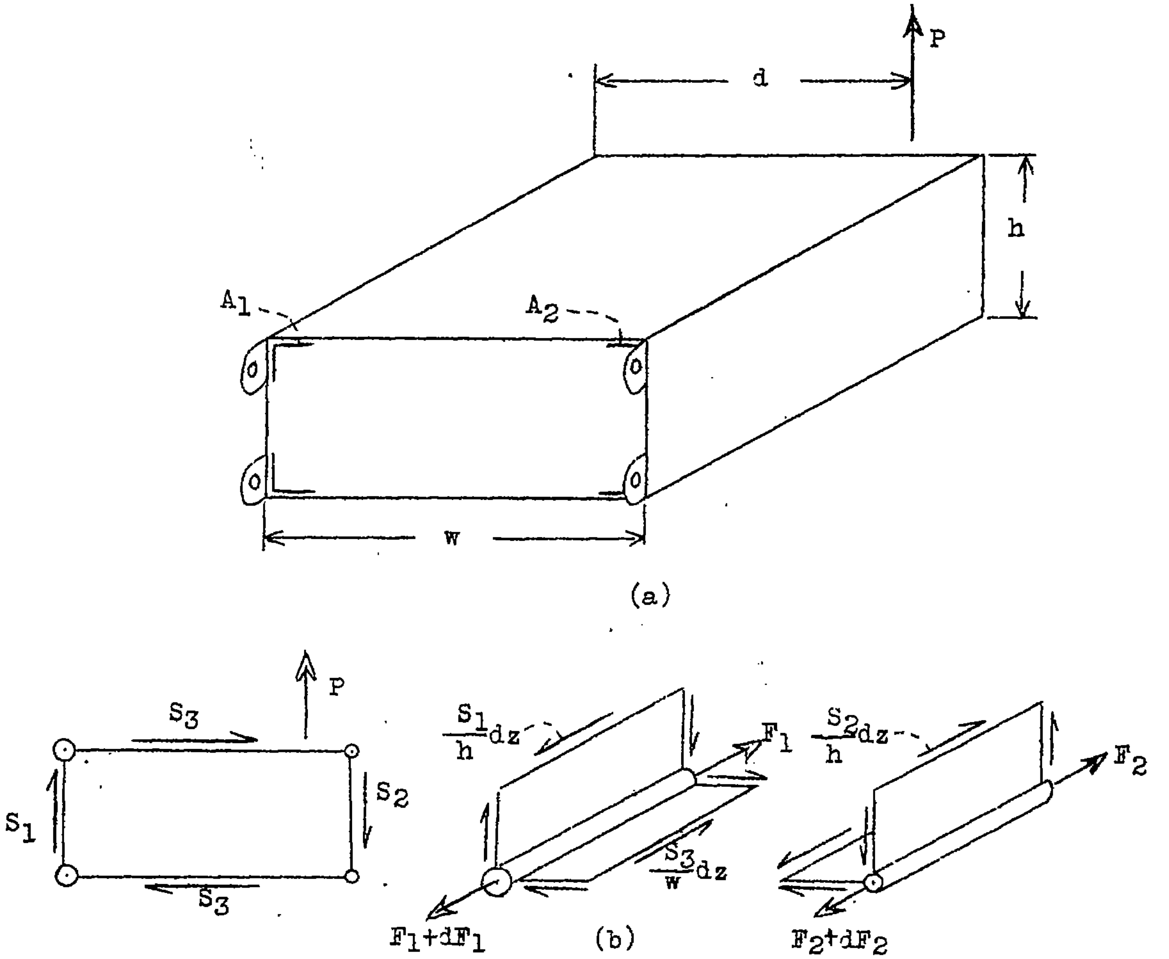


Figure 11.

As example, the equation for the simple case of figure 11 will be developed.

$$\Sigma V: \quad P + S_1 - S_2 = 0 \qquad (a)$$

$$\Sigma M: \quad -Pd + S_3 h + S_2 w = 0 \qquad (b)$$

$$\Sigma L \text{ on flange 1: } \quad dF_1 + \frac{S_1}{h}\,dz - \frac{S_3}{w}\,dz = 0 \qquad (c)$$

$$\Sigma L \text{ on flange 2: } \quad dF_2 + \frac{S_3}{w}\,dz - \frac{S_2}{h}\,dz = 0 \qquad (d)$$

Now
$$\frac{d}{dz}(F_1 + F_2) = \frac{d}{dz}\left(\frac{M}{h}\right) = \frac{P}{h}$$

Since the stresses in the two flanges are assumed to be equal

$$\frac{dF_1}{dz} = \frac{P}{h}\frac{A_1}{A} \quad \text{and} \quad \frac{dF_2}{dz} = \frac{P}{h}\frac{A_2}{A} \tag{e}$$

where $A = A_1 + A_2$

Substituting (e) into (c) and (d)

$$\frac{PA_1}{hA} + \frac{S_1}{h} - \frac{S_3}{w} = 0 \tag{f}$$

$$\frac{PA_2}{hA} + \frac{S_3}{w} - \frac{S_2}{h} = 0 \tag{g}$$

From (g) $\quad S_3h = S_2w - \frac{PA_2w}{A}$

Substituting into (b)

$$-Pd + S_2w - \frac{PA_2w}{A} + S_2w = 0$$

$$S_2 = \frac{P}{2}\left(\frac{d}{w} + \frac{A_2}{A}\right) \tag{16}$$

$$S_1 = \frac{P}{2}\left(\frac{d}{w} + \frac{A_2}{A} - 2\right) \tag{17}$$

$$S_3 = \frac{P}{2h}\left(d - w\frac{A_2}{A}\right) \tag{18}$$

In the more general case of the trapezoidal box (fig. 12(a)), the equations become

$$\Sigma V: \quad P + S_1 - S_2 - S_3\frac{h_1 - h_2}{w'} = 0$$

$$\Sigma M: \quad -Pd + S_3h_1' + S_2w = 0$$

$$\Sigma L_1: \quad dF_1 + \frac{S_1}{h_1}dz - \frac{S_3}{w'}dz = 0$$

$$\Sigma L_2: \quad dF_2 + \frac{S_3}{w'}dz - \frac{S_2}{h_2}dz = 0$$

In this case

$$\frac{dF_1}{dz} = \frac{h_1A_1}{2I}P \quad \text{and} \quad \frac{dF_2}{dz} = \frac{h_2A_2}{2I}P$$

so that

$$S_3 = S_2 \frac{w'}{h_2} - \frac{h_2 w'}{2I} A_2 P$$

and finally

$$S_2 = P \frac{h_2}{h_1 + h_2} \left[\frac{d}{w} + \frac{h_1 h_2 A_2}{2I}\right] \tag{19}$$

Substituting S_2 into the preceding equation gives S_3, and the first equation then yields S_1.

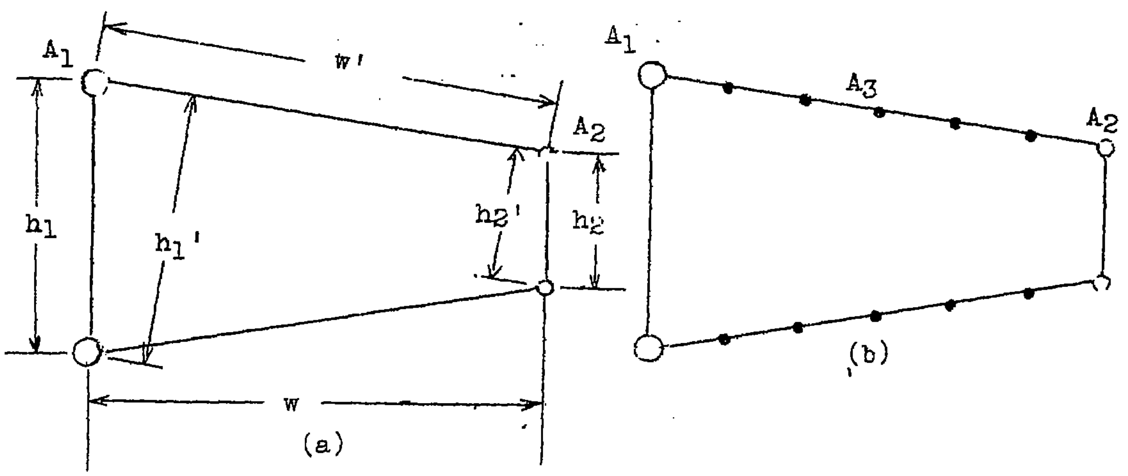


Figure 12.

If a number of stringers with a total cross-sectional area A_3 are uniformly distributed along the width of the cover sheets (fig. 12(b)), the formula for S_2 becomes

$$S_2 = P \frac{h_2}{h_1 + h_2} \left[\frac{d}{w} + \frac{h_1 h_2 A_2}{2I} + \frac{h_1 (h_1 + 2h_2) A_3}{12 I}\right] \tag{19a}$$

Since the analysis of a given case consists merely in substituting numerical values into equations (16), (17), and (18), no examples will be given here. If examples are desired, they may be found in reference 3, which covers in detail the analysis of the four-flange box by the shear-center method.

The Multicellular Shell

Inasmuch as the multiflange single-cell shell is statically determinate, the multicell shell is statically indeterminate. The method of analysis is analogous to that used for the two-flange, multicell shell and will be illustrated by the example of a four-flange, two-cell shell (fig. 13(a)).

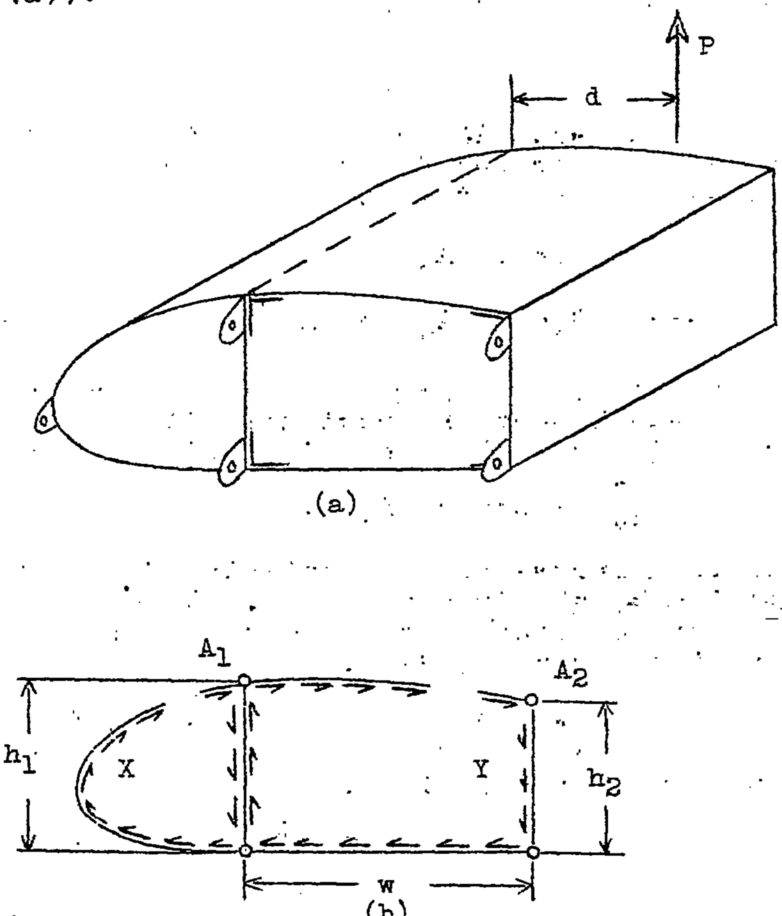


Figure 13.

Imagine each cell cut open and shears of intensity X and Y applied. The transverse load P causes shear loads S_1 and S_2 in the two-spar webs, which are proportional to the moments of inertia of these spars

$$S_1 = P\,\frac{I_1}{I} \quad \text{and} \quad S_2 = P\,\frac{I_2}{I} \tag{20}$$

with

$$I_1 = \frac{1}{2}\,A_1 h_1^2 \qquad I_2 = \frac{1}{2}\,A_2 h_2^2 \qquad I = I_1 + I_2$$

The equations (4b) for angle of twist are written down, and the condition of continuity

$$\theta_a = \theta_b$$

furnishes one equation. The second equation is found, as before, from the static condition that the internal torque must balance the external torque.

Taking moments about A_1

$$-Pd + S_2 w + 2A_a X + 2A_b Y = 0 \tag{21}$$

The analysis is closely analogous to the analysis of the two-flange, two-cell section previously discussed. The only difference is that the shear intensity in the rear spar is now $Y + \frac{S_2}{h_2}$ instead of Y.

Example 10 illustrates the analysis of a four-flange, two-cell section by the direct method. Example 11 illustrates the method of finding the shear center and the shear stresses associated with bending.

The procedure for more complicated cases (fig. 14) is merely a simple extension of the procedure discussed, so that no example will be required.

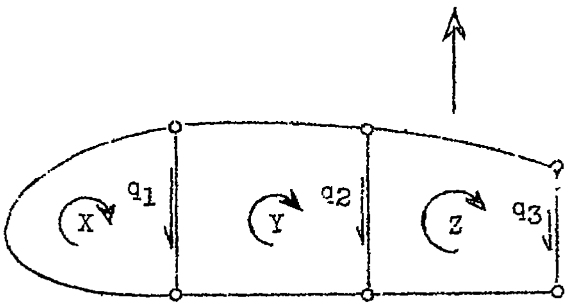


Figure 14.

Corrections to Simple Theories of Bending and Torsion

As mentioned before, the simple theories of bending and of torsion used thus far may require corrections. These corrections are important only in the inboard region near the root, if they are important at all. Whenever they are to be made, it is very advisable to separate bending from torsion at the outset. Such a procedure will make the calculations much clearer and will materially reduce the danger of committing errors in sign.

The flange material in box beams is usually distributed across the cover in the form of individual stringers or corrugated sheet. The bending action of such beams differs from that assumed by the simple theory of bending, because the sheet deforms under the shear stresses imposed on it. The analysis of bending action under such circumstances is discussed in reference 2 and no discussion will be given here.

In tubes subjected to torsion, the cross sections usually have a tendency to warp out of their original planes. If this warping is prevented by attaching the tube to a rigid support, or by conditions of symmetry in the middle of the span, then longitudinal (normal) stresses will arise, and the shear stresses will be redistributed. Reference 4 gives a method of calculating the effects for cross sections of arbitrary shape, but the method is of limited usefulness. Methods of analyzing rectangular tubes have been developed by a number of writers; the most important reports dealing with the methods are summarized in reference 1. This reference also gives what appears to be the only published experimental data. They do not agree very well with the theory; fortunately, the effects of end restraint are small in most practical cases, so that they need not be very accurately calculated.

For a rectangular tube symmetrical about both axes, such as shown in figure 15, the normal forces on the corner flanges caused by complete restraint may be calculated by the formula

$$X = \pm 0.56 \frac{T}{A} \left(\frac{b}{t_b} - \frac{c}{t_c}\right) \sqrt{\frac{A_F}{\frac{b}{t_b} + \frac{c}{t_c}}} \qquad (22)$$

(reference 1, equation (3c) with $G/E = 0.4$). The sign of the stresses is determined from the rule that the walls with the smaller ratio of width to thickness (in general, the vertical walls) act like independent spars, absorbing the torque by bending in opposite directions.

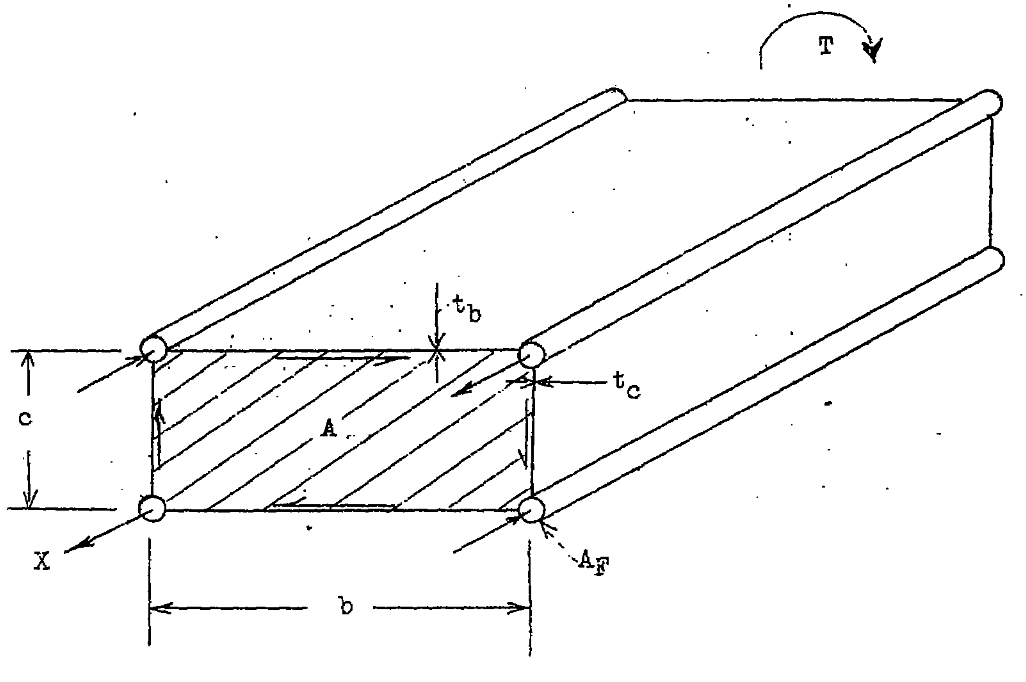


Figure 15.

The effect of the end restraint on the shear intensities is written most conveniently in the form

$$\Delta q = \pm \frac{T}{2A} \frac{\frac{b}{t_b} - \frac{c}{t_c}}{\frac{b}{t_b} + \frac{c}{t_c}} \tag{23}$$

obtained by using equations (3c), (9), and (10) of reference 1; Δq is the correction to be applied to the shear intensity calculated on the assumption of no restraint, namely,

$$q = \frac{T}{2A}$$

The negative sign in (23) is used for the walls with the larger ratio of width to thickness, in general the horizontal walls.

In actual cases, the box will seldom be symmetrical about both axes, as assumed in the derivation of formulas (22) and (23). The simplest procedure in such a case will be to use average values for b, c, t_b, and t_c. This procedure is somewhat unconservative, but formulas (22) and (23) are basically conservative because they assume infinitely closely spaced bulkheads. Furthermore, except in such cases as wings continuous across the center line of the airplane, the root section will not be rigidly built in, because there will be play in the fittings and elastic yielding in the fittings and in the center section.

The simple procedure outlined here and used in example 12 may, of course, be insufficient in some cases; a more detailed treatment, however, is beyond the scope of this paper.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., January 20, 1939.

APPENDIX

Example 1

Find the shear intensity and the shear center of the section shown in figure 16 for a vertical load P.

$$I = 2A\bar{x}^2 = 2 \times \frac{1}{4} \times 4 = 2 \text{ in.}^4$$

$$Q = A\bar{x} = \frac{1}{4} \times 2 = \frac{1}{2} \text{ in.}^3$$

By formula (2), the shear intensity is

$$q = \frac{P \times \frac{1}{2}}{2} = \frac{1}{4} P \text{ lb./in.}$$

In this simple case, the correct direction of the arrow can be found by inspection, and the sign convention is not needed.

The force H in the leg (fig. 16(b)) is

$$H = qw = \frac{1}{4} P \times 3 = \frac{3}{4} P$$

$$H \times 4 = x \times P \qquad x = 3 \text{ in.}$$

or the location e of the shear center e=x+w=6 in. = 2w, which agrees with formula (7).

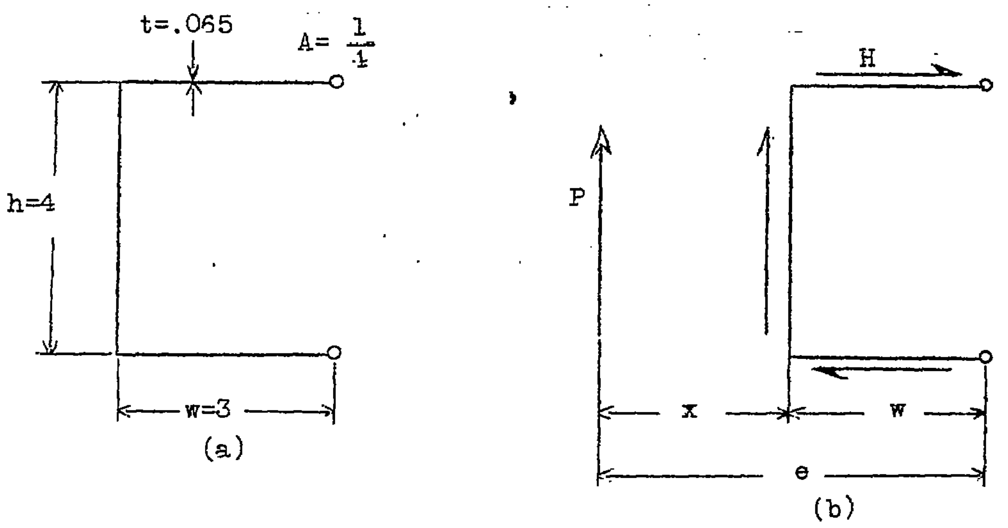


Figure 16.

Example 2

Find the shear intensity and the shear center for the section shown in figure 17.

Bow's notation is used as indicated:

$$I = 2 \times \frac{1}{4} \times 4 + 2 \times \frac{1}{4} \times 4 = 4 \text{ in.}^4$$

$$Q_{ac} = \frac{1}{4} \times 2 = \frac{1}{2}$$

$$Q_{ab} = \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 1$$

The shear intensities are therefore

$$q_{ac} = \frac{P \times \frac{1}{2}}{4} = \frac{1}{8} P \qquad q_{ab} = \frac{P \times 1}{4} = \frac{1}{4} P$$

The horizontal force H is

$$H = q_{ac} \times w = \frac{1}{8} P \times 3 = \frac{3}{8} P$$

Taking moments about the lower left corner of the channel as before,

$$Px = Hh = \frac{3}{8} P \times 4 \qquad x = \frac{3}{2} \text{ in.}$$

$$e = \frac{3}{2} + 3 = 4\frac{1}{2} \text{ in.} = \frac{3}{2} w$$

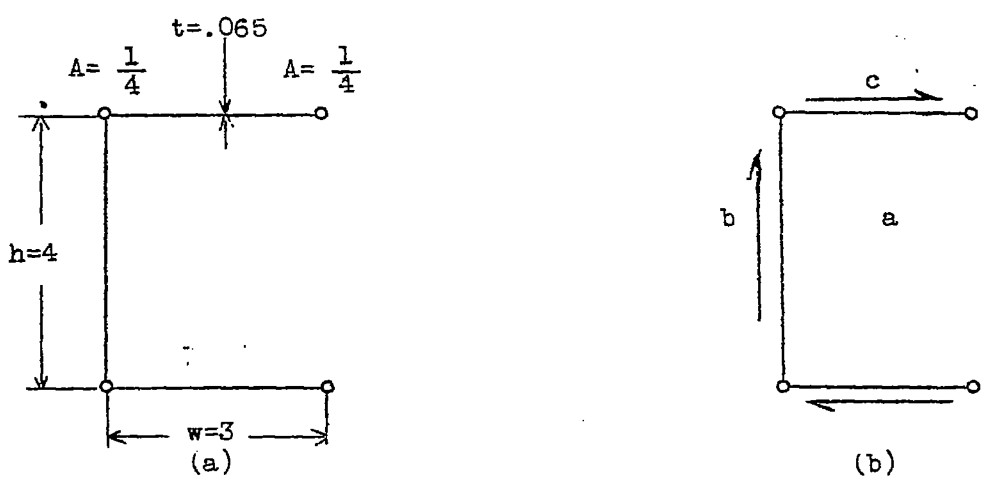


Figure 17.

Example 3

Find the shear intensities and the shear center of the section shown in figure 18.

$$I = 6 \text{ in.}^4$$

$$Q_{ad} = \frac{1}{2} \text{ in.}^3 \qquad q_{ad} = P \times \frac{1}{12} \text{ lb./in.}$$

$$Q_{ac} = 1 \text{ in.}^3 \qquad q_{ac} = P \times \frac{1}{6} \text{ lb./in.}$$

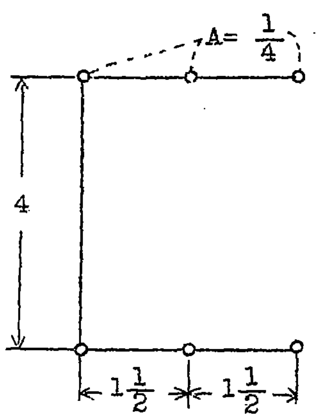
$$Q_{ab} = 1\tfrac{1}{2} \text{ in.}^3 \qquad q_{ab} = P \times \frac{1}{4} \text{ lb./in.}$$

$$H = q_{ad} \times 1.5 + q_{ac} \times 1.5 = \frac{P}{12} \times 1.5 + \frac{P}{6} \times 1.5 = \frac{3}{8} P$$

$$Px = Hh = \frac{3}{8} P \times 4 = \frac{3}{2} P$$

$$x = \frac{3}{2} \text{ in.}$$

as in the preceding case. The relation $e = \frac{3}{2} w$ holds for all channels, if the effective material is uniformly distributed along the legs of the channel.



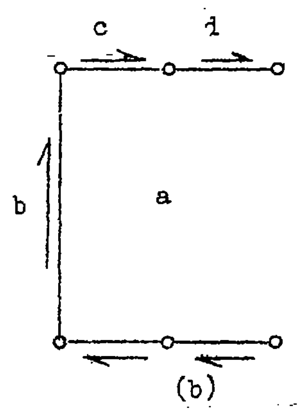


Figure 18.

Example 4

Find the shear intensity and the shear center for the section shown in figure 19.

$$I = 2\ A_F R^2 \text{ in.}^4$$

$$Q = A_F R \quad \text{in.}^3$$

$$q = P\ \frac{A_F R}{2A_F R^2} = \frac{P}{2R} \text{ lb./in.}$$

Taking moments about the center of the circle

$$Pe = q \times \pi R \times R = \frac{P}{2R} \times \pi R \times R$$

$$= \frac{\pi P R}{2}$$

$$e = \frac{\pi}{2} R \quad \text{in.}$$

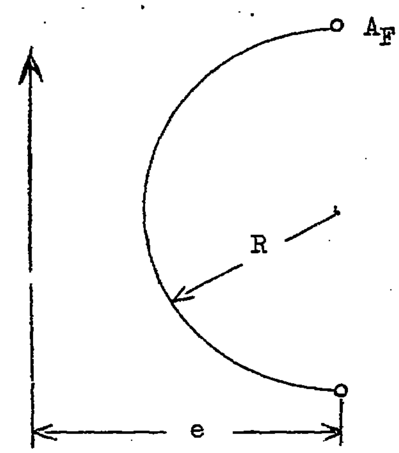


Figure 19.

Example 5

Find the shear center of the section shown in figure 20. In this case, all the sheet is effective in bending.

$$I = \frac{1}{2}\ \pi R^3 t \quad \text{in.}^4$$

$$Q_\theta = \int_0^\theta Rt d\theta \times R \cos\theta = R^2 t \sin\theta \text{ in.}^3$$

$$q_\theta = P \times \frac{R^2 t \sin\theta}{\frac{1}{2}\ \pi R^3 t} = P\ \frac{2}{\pi}\ \frac{\sin\theta}{R} \text{ lb./in.}$$

Taking moments about the center of the circle

$$Pe = \int_0^\pi P\ \frac{2}{\pi}\ \frac{\sin\theta}{R}\ R d\theta \times R$$

$$e = \frac{4}{\pi} R \quad \text{in.}$$

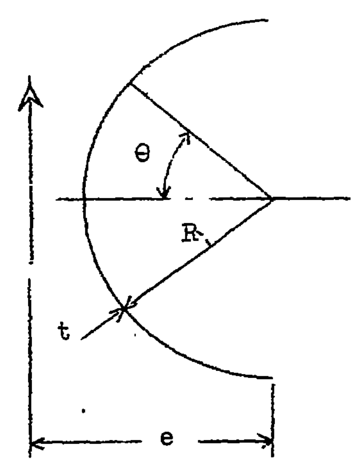


Figure 20.

Example 6

Given the D-section shown in figure 21.

(a) By direct analysis, find the stresses in the section and the angle of twist, assuming that no buckling occurs.

(b) Find the shear center of the section, and make the analysis by the shear-center method.

(c) Find the changes caused by the flat sheet developing a full diagonal-tension field.

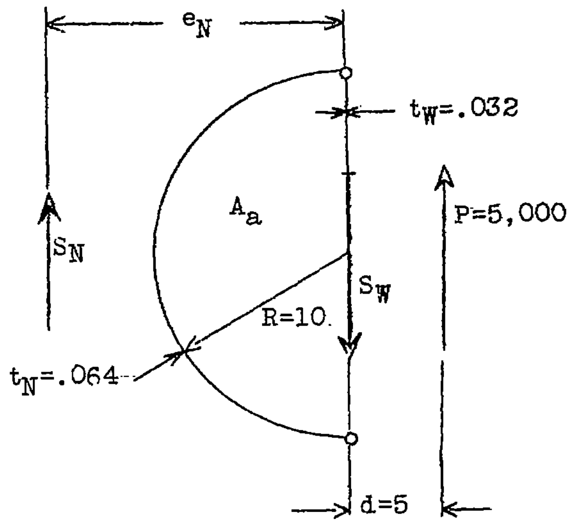


Figure 21.

(a) Direct analysis.- Find first the location e of the shear center of the nose by formula (7)

$$e_N = \frac{2A_a}{h} = \frac{2\,\frac{\pi}{2}\,R^2}{2R} = \frac{\pi}{2}\,R = 15.71 \text{ in.}$$

Taking moments about the plane web

$$S_N \times 15.71 = Pd = 5{,}000 \times 5 = 25{,}000 \text{ in.-lb.}$$

$$S_N = 1{,}591 \text{ lb.}$$

which gives

$$S_W = 6{,}591 \text{ lb.}$$

The shear stresses are therefore

$$\tau_N = \frac{1591}{0.064 \times 20} = 1{,}244 \text{ lb./sq.in.}$$

$$\tau_W = \frac{6591}{0.032 \times 20} = 10{,}300 \text{ lb./sq.in.}$$

The angle of twist is obtained from the basic formula (4b)

$$\theta = \frac{1}{2A_aG} \oint \tau ds$$

$$= \frac{1}{2 \times \frac{\pi}{2} \times 100\ G} \left[1,244 \times \pi \times 10 + 10,300 \times 20\right]$$

$$= \frac{1}{G} \times 780$$

With $G = 4 \times 10^6$, this value becomes

$$\theta = 195 \times 10^{-6} \text{ radian per inch length.}$$

(b) Shear-center analysis.- In this simple case, the location of the shear center can be found from formula (11)

$$d_o = -\frac{2 \times \frac{\pi}{2} \times 100 \times 0.064}{20 \times 0.064 + \pi \times 10 \times 0.032} = -8.80 \text{ in.}$$

In order to illustrate the procedure used in general cases, the solution will be carried through starting from fundamental principles.

The load P is assumed to act at an unknown distance d. There are then three unknowns: τ_N, τ_W, and d. To find these unknowns, there are available the static equations $\Sigma V = 0$ and $\Sigma M = 0$ and the elastic equation $\theta = 0$.

$$\Sigma V = q_N h - q_W h + P = 0$$

$$\Sigma M \text{ (about the web)} = q_N he - Pd_o = 0$$

$$\theta = \frac{1}{2A_aG}\left[\frac{q_N}{t_N} \times P + \frac{q_W}{t_W} \times h\right] = 0 \quad \text{(by equation (4a))}$$

Numerically

$$q_N \times 20 - q_W \times 20 + 5,000 = 0$$

$$q_N \times 20 \times 15.71 - 5,000\ d_o = 0$$

$$\theta G = \frac{1}{2 \times 157.1}\left[\frac{q_N}{0.064} \times \pi \times 10 + \frac{q_W}{0.032} \times 20\right] = 0$$

These three equations are solved and yield

$$q_N = -140.1 \text{ lb./in.}; \quad q_W = 110.0 \text{ lb./in.}$$

$$d_o = -8.80 \text{ in} \qquad \text{(a)}$$

The shear intensities just obtained are those associated with bending caused by a load P applied at the shear center. The actual load is applied at $d = 5$, so that there is a torque

$$T = -5,000\ (5 + 8.80) = -69,000 \text{ in.-lb.}$$

giving a shear intensity

$$q = \frac{69000}{2 \times 157.1} = 219.8 \qquad \text{(b)}$$

The total shear intensities are then

$$q_N = -140.1 + 219.8 = 79.7$$

and

$$\tau_N = \frac{q_N}{0.064} = 1,245 \text{ lb./sq.in.}$$

$$q_W = 110.0 + 219.8 = 329.8$$

and

$$\tau_W = \frac{q_W}{0.032} = 10,300 \text{ lb./sq.in.}$$

(c) Changes caused by buckling in flat web.- When the flat web is allowed to bucklo into a full diagonal-tension field, the effective thickness becomes

$$t_e = \frac{5}{8}\,t = \frac{5}{8} \times 0.032 = 0.020 \text{ in.}$$

In order to evaluate the angle of twist θ, it is necessary to obtain an effective shear stress

$$\tau_e = \frac{8}{5}\,\tau = \frac{8}{5} \times 10,300 = 16,500 \text{ lb./sq.in.}$$

Substituting this value into the expression for θ

$$\theta = \frac{1}{100\ \pi\ G}\left[1,244\ \pi \times 10 + 16,500 \times 20\right]$$

$$= \frac{1}{G} \times 1,174 = 293.5 \times 10^{-6} \text{ radian per inch}$$

The changed location d_o of the elastic center is obtained by substituting t_e instead of t into formula (11)

$$d_o = -\frac{2 \times \frac{\pi}{2} \times 100 \times 0.064}{20 \times 0.064 + \pi \times 10 \times 0.020} = -10.55 \text{ in.}$$

Example 7

Find the shear stresses and the angle of twist in the torsion tube shown in figure 22. Assume $G_o = G$ for all walls.

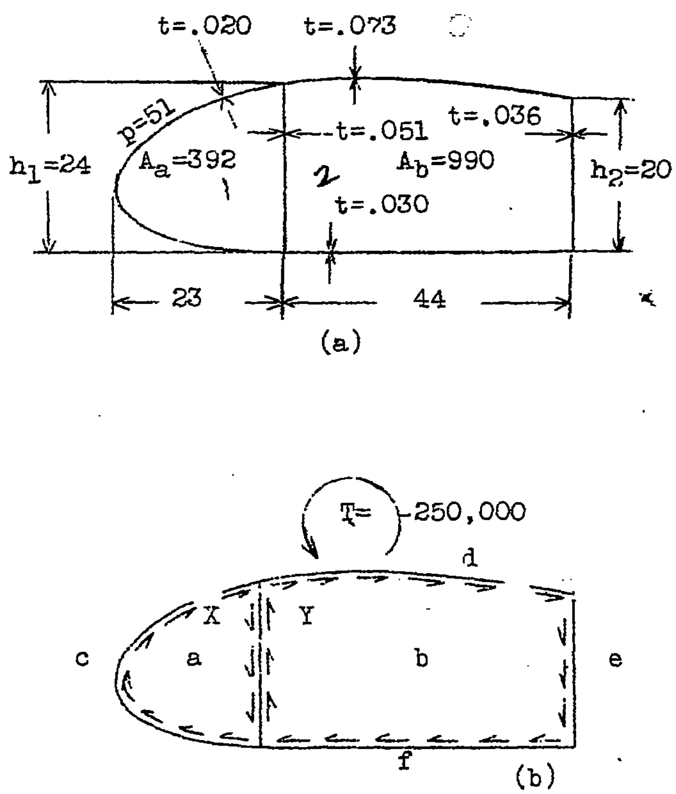


Figure 22.

It will be necessary to set up expressions for the angles of twist θ in terms of the shear intensities X and Y by using formula (4a). As a preliminary step, the auxiliary parameters

$$a = \int \frac{ds}{t}$$

will be computed, so that formula (4a) will be used in the form

$$\theta G = \frac{1}{2A} \oint aq$$

Bow's notation is used as indicated.

$$a_{ac} = \frac{51}{0.020} = 2{,}550$$

$$a_{bd} = \frac{44}{0.073} = 602.5$$

$$a_{be} = \frac{20}{0.036} = 555$$

$$a_{bf} = \frac{44}{0.030} = 1{,}467$$

$$a_{ba} = \frac{24}{0.051} = 470$$

(Note that $a_{ba} = a_{ab}$.)

The expressions for the angles of twist are

$$\theta_a G = \frac{1}{2 \times A_a}\left[X \times a_{ac} + (X - Y)\, a_{ab}\right]$$

$$= \frac{1}{2 \times 392}\left[X \times 2{,}550 + (X - Y)\, 470\right]$$

$$= 3.85\ X - 0.600\ Y$$

$$\theta_b G = \frac{1}{2A_b}\left[Ya_{bd} + Ya_{be} + Ya_{bf} + (Y - X)\, a_{ba}\right]$$

$$= \frac{1}{2 \times 990}\left[Y \times 602.5 + Y \times 555 + Y \times 1{,}467 + (Y - X)\, 470\right]$$

$$= 1.562\ Y - 0.2376\ X$$

Equating $\theta_a = \theta_b$, obtain

$$4.088\ X = 2.162\ Y \qquad \text{(a)}$$

The equation of moment equilibrium (13) is

$$X \times 2 \times 392 + Y \times 2 \times 990 - 250{,}000 = 0 \qquad \text{(b)}$$

The solution of these two equations is

$$X = 55.3 \text{ lb./in.} \qquad Y = 104.5 \text{ lb./in.}$$

The shear stresses are therefore

$$\tau_{ac} = \frac{X}{t} = \frac{55.3}{0.020} = 2,765 \text{ lb./sq.in.}$$

$$\tau_{bd} = \frac{Y}{t} = \frac{104.5}{0.073} = 1,431 \text{ lb./sq.in.}$$

$$\tau_{be} = \frac{Y}{t} = \frac{104.5}{0.036} = 2,900 \text{ lb./sq.in.}$$

$$\tau_{bf} = \frac{Y}{t} = \frac{104.5}{0.030} = 3,480 \text{ lb./sq.in.}$$

$$\tau_{ab} = \frac{X - Y}{t} = - \frac{49.2}{0.051} = - 965 \text{ lb./sq.in.}$$

The angle of twist is obtained by substituting into the expression for θ_a

$$\theta = \theta_a = \frac{1}{G}\left[3.85\ X - 0.600\ Y\right] = \frac{1}{G} \times 150.3$$

With $G = 4 \times 10^6$ this value becomes

$$\theta = 37.6 \times 10^{-6} \text{ radian per inch length,}$$

Example 8

Find the shear stresses in the section shown in figure 23. This section is identical with the one used in example 7, except that two flanges have been added to take care of beam action.

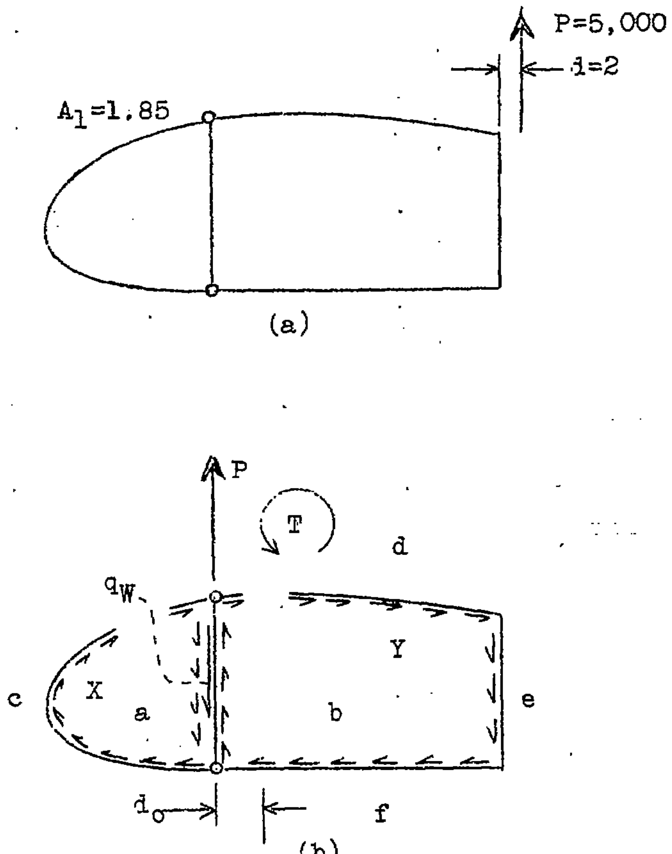


Figure 23.

The parameters a can be taken from the preceding example.

The shear intensity caused by the load P acting on the shear web is

$$q_W = \frac{P}{h} = \frac{5000}{24} = 208.2 \text{ lb./in.}$$

The expressions for θ are now written exactly as in example 7 except for the addition of q_W.

$$\theta_a G = \frac{1}{2 \times 392}\left[X \times 2{,}550 + (X - Y + 208.2) \times 470\right]$$

$$= 3.85\ X - 0.600\ Y + 124.96\ .$$

$$\theta_b G = \frac{1}{2 \times 990}\left[Y \times 602.5 + Y \times 555 + Y \times 1467 + (Y - X - 208.2) \times 470\right]$$

$$= 1.562\ Y - 0.2376\ X - 49.5$$

Equating $\theta_a G$ to $\theta_b G$ gives

$$4.088\ X - 2.162\ Y + 174.46 = 0 \qquad \text{(a)}$$

The equation of moment equilibrium is taken around the shear web, to eliminate one term

$$X \times 2 \times 392 + Y \times 2 \times 990 - 5{,}000 \times 46 = 0 \qquad \text{(b)}$$

Solving equations (a) and (b), obtain

$$X = 15.5 \qquad Y = 110.1$$

The shear stresses are therefore

$$\tau_{ac} = \frac{15.5}{0.020} = 775 \text{ lb./sq.in.}$$

$$\tau_{bd} = \frac{110.1}{0.073} = 1{,}510 \text{ lb./sq.in.}$$

$$\tau_{be} = \frac{110.1}{0.036} = 3{,}060 \text{ lb./sq.in.}$$

$$\tau_{bf} = \frac{110.1}{0.030} = 3{,}670 \text{ lb./sq.in.}$$

$$\tau_{ab} = \frac{15.5 - 110.1 + 208.2}{0.051} = 2{,}226 \text{ lb./sq.in.}$$

Example 9

For the section used in example 8, find the shear center, and analyze the load case of example 8 by the shear-center method.

Two equations are obtained by equating to zero the expressions for θ_a and θ_b, which are taken from example 8.

$$3.85\ X - 0.600\ Y + 124.96 = 0$$

$$-0.2376\ X + 1.562\ Y - 49.5 = 0$$

The solution of these equations gives the shear intensities associated with torsion-free bending.

$$X = -28.2 \text{ lb./in.} \qquad Y = 27.42 \text{ lb./in.}$$

The distance d_o of the shear center from the shear web is obtained by writing ΣM about the shear web

$$-5{,}000\ d - 28.2 \times 2 \times 392 + 27.42 \times 2 \times 990 = 0$$

$$d_o = 6.44 \text{ in.}$$

A load P located at $d = 46$ inches will therefore cause a torque

$$-P(d - d_o) = -5{,}000(46 - 6.44) = -197{,}800 \text{ in.-lb.}$$

The stresses τ_B due to bending are obtained from X and Y as before.

The stresses τ_T due to the torque of -197,800 in.-lb. are obtained by multiplying the stresses from example 7 by $\frac{197800}{250000} = 0.791$.

The final stresses τ are obtained by superposition as shown in the following table.

Wall	τ_B (lb./sq.in.)	τ_T (lb./sq.in.)	τ (lb./sq.in.)
ac	-1,410	2,190	780
bd	376	1,132	1,508
be	762	2,297	3,059
bf	914	2,757	3,671
ab	2,990	-764	2,226

Example 10

For the section shown in figure 24, find the shear stresses. The section is identical with that shown in figure 22 except for the flanges. The load is assumed to be perpendicular to the neutral axis. The inclination of the shear webs is neglected.

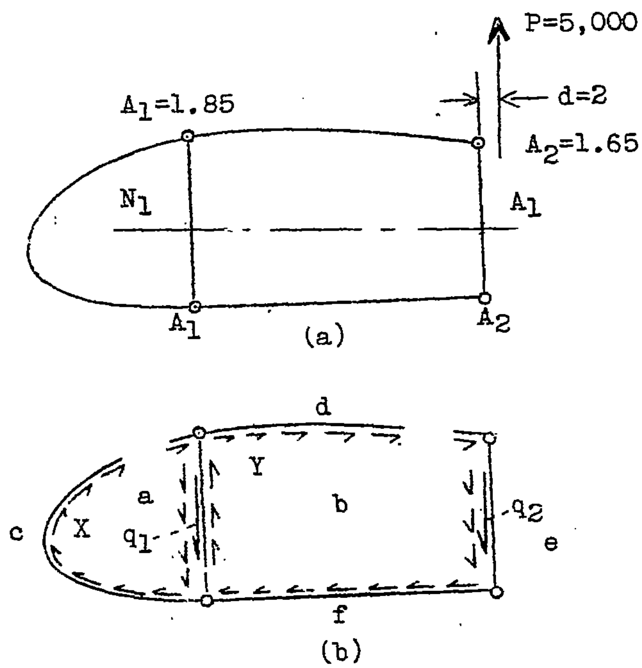


Figure 24.

Imagine the cover to be slotted in both cells as indicated, leaving a structure consisting of two spars. The vertical shear is divided between these two spars in the ratio of their moments of inertia.

$$I_1 = \frac{1}{2} \times 1.85 \times 24^2 = 533 \text{ in.}^4$$

$$I_2 = \frac{1}{2} \times 1.65 \times 20^2 = 330 \text{ in.}^4$$

$$I = I_1 + I_2 = 863 \text{ in.}^4$$

Therefore

$$S_1 = P \times \frac{I_1}{I} = 5{,}000 \times \frac{533}{863} = 3{,}090 \text{ lb.}$$

$$q_1 = \frac{S_1}{h_1} = \frac{3090}{24} = 128.8 \text{ lb./in.}$$

$$S_2 = P \times \frac{I_2}{I} = 5{,}000 \times \frac{330}{863} = 1{,}910 \text{ lb.}$$

$$q_2 = \frac{S_2}{h_2} = \frac{1910}{20} = 95.5 \text{ lb./in.}$$

Proceeding as in the previous cases, write the expressions for θ_a and θ_b.

$$\theta_a = \frac{1}{2 \times 392}\left[2{,}550\ X + 470\ (X - Y + 128.8)\right]$$

$$= 3.85\ X - 0.600\ Y + 77.3$$

$$\theta_b = \frac{1}{2 \times 990}\left[\overset{\tau_{bd}}{602.5}\ Y + \overset{a_{be}}{555}(Y + \overset{q_2}{95.5}) + \overset{a_{jf}}{1{,}467}\ Y + \overset{a_{hk}}{470}(Y - X - \overset{q_1}{128.8})\right]$$

$$= 1.562\ Y - 0.2376\ X - 3.8$$

Equating θ_a to θ_b, obtain

$$4.09\ X - 2.162\ Y + 81.1 = 0$$

The equation of moment equilibrium gives

$$-5{,}000 \times 46 + 1{,}910 \times 44 + 2 \times 392\ X + 2 \times 990 \times Y = 0$$

Solving these two equations, obtain

$$X = 15.9 \text{ lb./in.} \qquad Y = 67.5 \text{ lb./in.}$$

The shear stresses are therefore

$$\tau_{ac} = \frac{X}{0.020} = 795 \text{ lb./sq.in.}$$

$$\tau_{bd} = \frac{Y}{0.073} = 925 \text{ lb./sq.in.}$$

$$\tau_{be} = \frac{Y + q_2}{0.036} = 4{,}530 \text{ lb./sq.in.}$$

$$\tau_{bf} = \frac{Y}{0.030} = 2{,}250 \text{ lb./sq.in.}$$

$$\tau_{ab} = \frac{X + q_1 - Y}{0.051} = 1{,}513 \text{ lb./sq.in.}$$

Example 11

Find the shear center, and the shear stresses associated with torsion-free bending, for the section shown in figure 24.

Take the expressions for θ_a and θ_b from the preceding example and equate each one to zero.

$$3.85\ X - 0.600\ Y + 77.3 = 0$$

$$-0.2376\ X + 1.562\ Y - 3.8 = 0$$

Solving

$$X = -20.2 \text{ lb./in.} \qquad Y = -0.635 \text{ lb./in.}$$

The shear stresses are therefore

$$\tau_{ac} = -\frac{20.2}{0.020} = -1{,}010 \text{ lb./sq.in.}$$

$$\tau_{bd} = -\frac{0.635}{0.073} = -9 \text{ lb./sq.in.}$$

$$\tau_{be} = \frac{-0.635 + 95.5}{0.036} = 2{,}640 \text{ lb./sq.in.}$$

$$\tau_{bf} = \frac{-0.635}{0.030} = -21 \text{ lb./sq.in.}$$

$$\tau_{ab} = \frac{-20.2 + 128.8 + 0.635}{0.051} = 2{,}140 \text{ lb./sq.in.}$$

Leaving the location d of the load P undetermined, write the moment equation

$$\Sigma M = -5{,}000d + 1{,}910\times 44 - 2\times 392\times 20.2 - 2\times 990\times 0.635 = 0$$

which gives as location of the shear center

$$d = 13.39 \text{ in. behind front shear web}$$

Example 12

For the section analyzed in examples 10 and 11, find the stresses if the section is a root section that is rigidly built in. The length L of the beam is 200 inches; the load P is applied at the tip.

The first step is to separate the load on the entire section into bending moment and torque. The bending moment is

$$M = PL = 5,000 \times 200 = 1,000,000 \text{ in.-lb.}$$

According to example 9, the shear center is located at $d_o = 6.44$ inches, and the torque is

$$T = P(d - d_o) = -197,800 \text{ in.-lb.}$$

The effects of restraint against warping will be calculated under the assumption that only the approximately rectangular cell b between the four main fittings is restrained against warping and that the nose part has no influence on these warping stresses.

According to example 7, a torque of 250,000 in.-lb. creates a shear intensity $Y = 104.5$ lb./in. in cell b. The existing torque of 197,800 in.-lb. will therefore give a shear intensity

$$q_b = 104.5 \times \frac{197800}{250000} = 82.8 \text{ lb./in.}$$

The torque carried by cell b is therefore (approximately)

$$T_b = 82.8 \times 2 \times 990 = 164,000 \text{ in.-lb.}$$

With the average values

$$b = 44 \text{ in.} \qquad t_b = \frac{0.073 + 0.030}{2} = 0.0515 \text{ in.}$$

$$c = \frac{24 + 20}{2} = 22 \text{ in.} \qquad t_c = \frac{0.051 + 0.036}{2} = 0.0435 \text{ in.}$$

$$A_F = \frac{1.85 + 1.65}{2} = 1.75 \text{ sq.in.}$$

The normal force on the flange due to torque becomes, by formula (22)

$$X = 0.56 \frac{164000}{44 \times 22} (855 - 506) \sqrt{\frac{1.75}{855 + 506}} = 1,190 \text{ lb.}$$

and the correction for shear intensity becomes, by formula (23)

$$\Delta q = \frac{164000}{2 \times 44 \times 22} \frac{(855 - 506)}{(855 + 506)} = 21.75 \text{ lb./in.}$$

The bending stresses due to the bending moment are, in the front flanges,

$$\sigma_{1B} = \pm \frac{1000000 \times 12}{863} = \pm 13,900 \text{ lb./sq.in.}$$

and, in the rear flanges,

$$\sigma_{2B} = \pm \frac{1000000 \times 10}{863} = \pm 11,580 \text{ lb./sq.in.}$$

the upper sign applying to the upper flange in each case. The bending stresses due to torque are

$$\sigma_{1T} = \frac{X}{A_1} = \frac{1190}{1.85} = \pm 643 \text{ lb./sq.in.}$$

and

$$\sigma_{2T} = \frac{X}{A_2} = \frac{1190}{1.65} = \pm 721 \text{ lb./sq.in.}$$

the upper sign applying to the upper flanges. The final stresses are therefore.

$$\sigma_{1U} = -13,900 + 643 = -13,257 \text{ lb./sq.in.}$$

$$\sigma_{1L} = 13,900 - 643 = 13,257 \text{ lb./sq.in.}$$

$$\sigma_{2U} = -11,580 - 721 = -12,301 \text{ lb./sq.in.}$$

$$\sigma_{2L} = 11,580 + 721 = 12,301 \text{ lb./sq.in.}$$

The shear stresses for the section free to warp are obtained from example 10. Superposing the corrections $\Delta t = \frac{\Delta q}{t}$ gives the final shear stresses

$$\tau_{ac} = 795 + 0 = 795 \text{ lb./sq.in.}$$

$$\tau_{bd} = 925 - \frac{21.75}{0.073} = 627 \text{ lb./sq.in.}$$

$$\tau_{be} = 4,530 + \frac{21.75}{0.036} = 5,134 \text{ lb./sq.in.}$$

$$\tau_{bf} = 2,250 - \frac{21.75}{0.030} = 1,525 \text{ lb./sq.in.}$$

$$\tau_{ba} = -1,513 + \frac{21.75}{0.051} = -1,087 \text{ lb./sq.in.}$$

The corrections influence the design of the rear spar more critically than the front spar. The correction on the flange stress is somewhat over 6 percent; on the web shear stress, it is somewhat over 13 percent. An error of 25 percent on the correction would therefore cause an error of 1-1/2 percent on the flange stress and an error of 3 percent on the web shear stress.

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