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14. ABSTRACT
This Test Operating Procedure (TOP) is to be used as a guideline for development of Laboratory Vibration Test Schedules (LVTS) for Multiple Degree-of-Freedom (MDOF) scenarios.

15. SUBJECT TERMS
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US ARMY TEST AND EVALUATION COMMAND
TEST OPERATIONS PROCEDURE

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LABORATORY VIBRATION TEST SCHEDULE DEVELOPMENT FOR MULTI-EXCITER APPLICATIONS

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1. SCOPE.

This Test Operating Procedure (TOP) presents considerations and techniques for developing Laboratory Vibration Test Schedules (LVTS) that can be utilized to simulate field vibration environments on a vibration table. Laboratory vibration tests are used extensively in lieu of more time-consuming and less cost effective field exposure tests. This TOP specifically addresses random vibration testing controlled to frequency-domain vibration spectra and is intended to address multiple "exciter" (also referred to as "shaker" or "actuator") scenarios with the emphasis on mechanical multiple degree-of-freedom (MDOF) scenarios. There is a significant increase in complexity between single-exciter/single-axis (SESA) and multiple-exciter/multiple-axis (MEMA) testing in terms of both mechanics and control. MEMA specific issues ranging from definitions and nomenclature consistency, to data analysis techniques, will be addressed.

2. FACILITIES AND INSTRUMENTATION.

2.1 Facilities.

The development of a LVTS will require access to the test item of interest (or a dynamically equivalent surrogate), access to the carrier vehicle, appropriately placed transducers, signal conditioning and data acquisition hardware, and a controlled environment for collecting input data (e.g., a road course for wheeled and/or tracked vehicles, waterway for watercraft, airspace for aircraft, rotorcraft, and/or spacecraft).

2.2 Instrumentation.

a. LVTSs are generally defined in terms of acceleration units. The transducer of choice for making acceleration measurements is an accelerometer. This TOP will address LVTS development in terms of acceleration.

b. It is strongly recommended that the same model of accelerometer and signal conditioning is employed at all instrumented locations to preserve phase characteristics during both the field acquisition and laboratory test phase of any MDOF test. Refer to the guidelines in Military Standard (MIL-STD)-810G[1] and Institute of Environmental Sciences and Technology (IEST) Recommended Practice IEST-RP-DTE012.2[2] for recommended accuracy of the transducers and associated signal conditioning.

3. REQUIRED TEST CONDITIONS.

The primary function of Vibration Schedule Development (VSD) is to combine vibration measurements of numerous events that collectively represent an item's lifetime vibration exposure (or some predefined subset thereof) into a manageable set of LVTS representing the equivalent exposure. The most dynamically accurate method to reproduce the full exposure would be to sequentially vibrate the system to all the individual, uncompressed events

*Superscript numbers correspond to those in Appendix E, References.

representing its full lifecycle. However, such an approach is generally not feasible from both schedule and economic perspectives and some compromises must be made to realize the benefits of testing in the laboratory. Time compression techniques based on fatigue equivalency are typically employed such that vibration testing can be performed in a timely and economic manner. North Atlantic Treaty Organization (NATO) Allied Environmental Conditions Test Publication (AECTP) 240, Leaflet 2410[3], provides general guidance for developing accurate representations, and issues that should be considered during the VSD process for the SESA scenario. This TOP expands upon the discussion in Leaflet 2410 to address the general multiple exciter test scenario. Discussions will be limited to random LVTS development. At the time of this publication, no commercially available multiple-input multiple-output (MIMO) solutions exist for swept narrowband random on random (NBROR) or sine-on-random (SOR). Refer to MIL-STD-810G, Method 525, Procedure I, for a Time Waveform Replication (TWR) based MDOF solution for swept NBROR and SOR requirements.

3.1 Test Configurations.

The MIMO random vibration test problem can refer to several configurations. One configuration is multiple exciters driving a single test item in one axis. This configuration is often used for large test items too large for a single exciter. A second configuration is the excitation of a single test item with multiple exciters in more than one axis. Linear displacements along defined directions are referred to as translation degree-of-freedom (DOF) and angular displacements along those same directions are referred to as rotation DOFs. Up to six DOFs exist for a rigid body (i.e., X-, Y-, Z-translations and roll, pitch, yaw rotations). In some cases, additional DOFs can be excited due to elastic deformations of the test article.

3.1.1 Basic Representation of a MIMO System.

All MIMO test systems are discussed using a common description in terms of matrix equations[2,4,5]. A simplified version of the general MIMO random vibration test problem can be generalized in Figure 1. The complete mechanical system is characterized by the power amplifiers and a system of several exciters, on which is mounted a single test article. The response of the test article is monitored by a vector of response channels (represented as {c}). Each element in the vector is typically the acceleration time history from a single accelerometer. In theory, other types of sensors could be used. The power amplifiers are driven by a vector of electrical drives (represented as {d}), generated by a control system. Each element in the vector is a time history driving a single shaker. The control system monitors the response of the test item {c}, and attempts to produce drive signals {d}, such that the statistics of the control signals meet some criteria as specified in the test specifications.

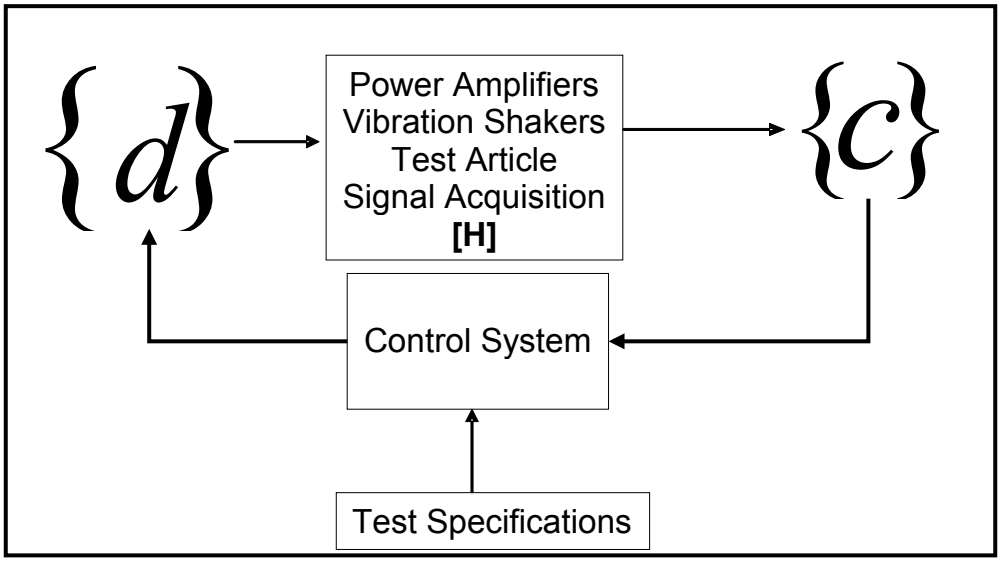

Figure 1. Basic representation of a MDOF system.

3.1.2 Generalized Representation of a MIMO System.

A more generalized MIMO system is shown in Figure 2. A system under test is driven by N_s shakers resulting in the response of N_a control accelerometers. The accelerometer data are typically structured in blocks. Each of the acceleration records will then be a vector of time samples. Some control systems then provide for a transformation matrix, $\mathbf{T_a}$, to convert the block of N_a accelerometer time histories to N_c control variables. The Spectral Density Matrix (SDM) of the control variables is then estimated from the current block of data and previous data. The transformation matrix, $\mathbf{T_a}$, is typically a constant independent of frequency. In theory the transformation matrix could be applied before or after the estimation of the control SDM. The estimated control SDM, **C** is then compared with the reference SDM, **R**, and a correction is computed for the drive SDM, **D**. The drive time histories {**d**} are then computed from the drive SDM, **D**, using time domain randomization. A second transformation matrix, $\mathbf{T_s}$, is employed to transform the N_d drive variables into N_s shaker drive signals. In theory, $\mathbf{T_s}$ could be implemented before or after the transformation into the time domain. One advantage of placing the transformation in the frequency domain section of the control algorithm is that the matrix could then be made a function of frequency. Having the transformation matrix, $\mathbf{T_s}$, a constant assumes the shakers are matched and the desired transformation can be deduced.

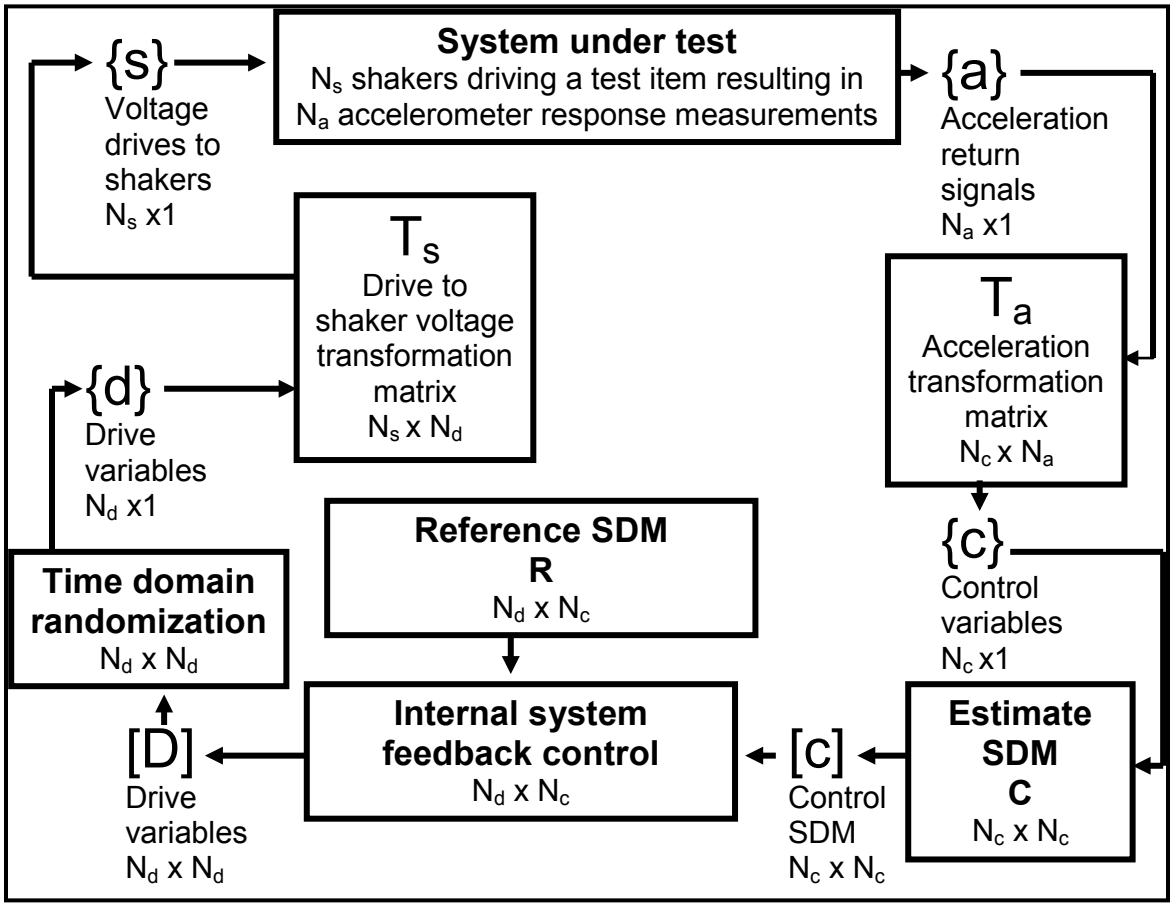

Figure 2. Generalized representation of a MDOF system.

3.2 Generalized MDOF Vibration Control Discussion.

a. A general discussion of the MDOF control process is provided for insight as to how the MDOF LVTS will serve as the reference in the control process. The correction can be computed in several ways. One method is to compute the drive from:

$$\mathbf{D} = \mathbf{Z}\hat{\mathbf{R}}\mathbf{Z}'$$

where the system impedance matrix, **Z**, is updated as new information is gathered, or a modified reference spectrum, $\hat{\mathbf{R}}$, is computed based on the error in the return spectrum. The initial drive vector is typically computed using the above equation and the reference SDM. A drive signal error can also be computed from:

$$\mathbf{D}_e = \mathbf{Z}(\mathbf{R} - \mathbf{C})\mathbf{Z}'$$

Sometimes an adaptive correction is used. Sometimes a combination of all methods is used.

b. The transformation matrices are often called the input and output transformation matrices. One should be careful with this nomenclature because of the confusion between input and output. The input to the system under test (voltages to the power amplifiers) is the output of the control system. The output of the system under test (the accelerometer measurements) is the input to the control system. Paragraphs 4.4.1 and 4.4.2 provide the nomenclature employed for input and output transformations, as they are applied within this document.

c. Minor errors in the matching of shakers can be corrected by the control algorithm, but major mismatches could be problematic. The time domain drive signals (represented by {s}), are sent to the shakers completing the control loop.

d. If $\mathbf{T_a}$ is not available, then $N_a = N_c$ and {a} = {c}. If $\mathbf{T_s}$ is not available, then $N_d = N_s$ and {d} = {s}. If $N_d = N_c$, the number of control variables and the number of drive variables are the same. This is referred to as square control. Square control is the most common control method. If $N_s > N_a$ the system is over-actuated and least squares approach using a pseudo inverse (pinv) is typically used to determine the drive signals. If $N_s < N_a$ the system is under-actuated and exact control of the control SDM is often not possible. In such cases, some kind of average control is usually implemented. Often when $N_s \neq N_a$ some combination of the transformation matrices are often used to force square control, $N_d = N_c$.

e. The entire mechanical system can be characterized by a matrix of frequency response functions [**H**]. For the typical case, these frequency response functions will have units of g/V (acceleration in gravitational units/volts of drive). For the typical case, the control signals are characterized by a SDM. The diagonal elements are the autospectral density (ASD or PSD) of the control signals. The off diagonal elements are the cross spectral densities (CSD) between pairs of control signals. The input to the system is characterized by the SDM of the voltage drive signals. The fundamental relationship between the drives and the control signals is given by:

$$\mathbf{C} = \mathbf{H}\mathbf{D}\mathbf{H}'$$

f. The complex conjugate transpose is denoted by [ ]'. All of the matrices in the equation are complex functions of frequency. The spectral density matrix is Hermitian[6], i.e. $D_{ij} = D_{ji}^*$ where D_{ji}^* is the complex conjugate of D_{ji}, and D_{ji} is an element from a spectral density matrix. Note that this requirement demands that the diagonal elements are real. Note that C and D are square matrices; they have the same number of rows and columns. C and D are the same size only if H is square, i.e. the same number of inputs and outputs. To be physically realizable, the SDM must also be positive semi-definite. This requirement will be discussed in paragraph 4.5.2.

g. The drive spectral density matrix is converted into the drive time histories using the method of time domain randomization[4]. The spectral density matrix is typically estimated using Welch's method[7].

4. TEST PROCEDURES.

VSD requires a thorough knowledge of the dynamic environment to which the test hardware will be exposed when fielded. This knowledge must include characterization of the exposure levels and durations for all relevant conditions.

4.1 Development of Mission or Lifetime Scenario.

The duration of the vibration environments can be derived from the item's Life Cycle Environment Profile (LCEP). The life cycle will include many different types of induced mechanical environments which may occur while the materiel is being handled, transported, deployed and operated. Although all the induced mechanical environments are not critical in terms of generating potential damaging response amplitudes, they contribute in varying degrees to the materiel's fatigue damage. All expected exposure conditions should be tabulated, along with corresponding durations, to form the items lifetime "scenario". The scenario is a key parameter in the development of any vibration schedule.

4.2 Limitations.

The mechanical degrees of freedom (DOFs) for which a VSD effort is capable of addressing, is a function of the number and placement of the transducers employed in the field data acquisition phase. Similarly, the maximum number of mechanical DOFs possible to reproduce in the laboratory environment is a function of the number and placement of actuators and coupling hardware. This TOP will consider the general case for VSD development in which the reference SDM will be defined in terms of the six classical (3-translational and 3-rotational) rigid body mechanical DOFs. In the event less than six mechanical DOFs are being considered, the generalized theory is easily configured to address the motion of interest.

4.3 Field Data Acquisition.

When in-service measurement data have been obtained, it is assumed that the data is processed in accordance with good data analysis procedures, as in Multi-Shaker Test and Control IEST-RP-DTE022.1[8] and Welch's method. In particular, an adequate number of statistical degrees of freedom (DOFs) have been obtained to provide information with acceptable statistical error. Consideration must be given to not only statistical error in auto-spectral density estimates, but also in cross-spectral density estimates (including transfer and coherence function estimates).

4.3.1 Instrumentation.

For the purpose of this TOP, all instrumentation related discussions will be limited to linear accelerometers and engineering units of g's, as was the case in the general control discussion provided in paragraph 3.1.1. Linear accelerometers have several advantages including familiarity to most users, low cost, wide bandwidth, small size and weight, and readily available low cost highly reliable signal conditioning options.

4.4 Use of Rigid Body Modes.

a. In single axis testing, the control input is often defined with a single accelerometer. This is satisfactory if the shaker and test fixtures are rigid within the frequency band of interest. If the shaker and test fixtures are not rigid, the technique of using a single accelerometer for control can sometimes lead to serious difficulty. To overcome these problems, methods using the average of several accelerometers and/or force limiting have come into common practice. In MEMA testing, the problem can be more serious as non-rigid body response is more common. When considering the special case of multiple shakers exciting a test item with multiple rigid body degrees of freedom, the use of the input transformation to define the response in terms of rigid body modes has several advantages. It is somewhat analogous to a generalization of the common practice for single axis testing. If there are more control channels than rigid body degrees of freedom, and an input transformation matrix is defined to transform the control accelerometers into rigid body modes, one essentially defines the motion of each rigid body mode as a weighted average of the accelerometers active for the mode. In many cases, given the control authority of the shakers, this is about the best viable solution. It is analogous to averaging accelerometers for a single axis test, which is common practice. The elastic modes are not controlled, since often the control authority over these modes does not exist. The system is driven with an equivalent rigid body motion in each of the rigid body modes. It is necessary to make sure that for any mode the transformation of the control accelerometers {a} does not result in zero for any of the rigid body modes. If higher flexural modes are present they will not be controlled. In theory the flexural modes can be controlled by adding control variables, but this requires knowledge of the modes in the test setup. This information can only be determined with materiel in the test configuration. For this reason, it is sometimes desirable to allow modification of the test requirements after this information is made available. Exactly how this will be accomplished in specification writing will have to be determined at a later date.

b. An advantage of using rigid body modes in the specification is that the field measurements used to define the environment can be made with the transducers in locations different from the locations of the transducers used in the laboratory test. The field measurements are reduced to equivalent rigid body modes using an acceleration transformation matrix (refer to paragraph 4.4.1), and the modes are controlled on the test using another transformation matrix for the laboratory test configuration. The two transformation matrices do not have to be the same. Use of alternate control points, while maintaining a full rank transformation matrix, provides a way of making the laboratory test “equivalent” in the sense of the rigid body modes.

c. A practical difficulty arises when more modes are attempted to be controlled. The general case of six (6) rigid body modes requires the specification of a 6 x 6 SDM (6 ASD’s and 15 CSD’s). Physical understanding of the SDM matrix associated with rigid-body motion by itself is difficult without the additional complications of elastic DOFs. Furthermore, it is difficult to assure that the specification results in a positive definite SDM, which is a physical requirement. (Additional discussion on positive definite matrices is the subject of paragraph 4.5.2.)

4.4.1 Acceleration (Input) Transformation.

The acceleration to control space transformation matrix, $\mathbf{T}_a$, commonly referred to as the "input transformation matrix" from the control system perspective, is defined in the article "Applying Coordinate Transformations to Multi-DOF Shaker Control"[9] and generalized in the article "Benefits and Challenges of Over-Actuated Excitation Systems"[10]. The acceleration transformation matrix transforms a set of accelerometer measurements into a set of control variables. Often these control variables are descriptions of rigid body modes. The acceleration transformation is usually performed in the time domain as:

$$\{\mathbf{c}\} = \mathbf{T}_\mathbf{a} \{\mathbf{a}\}$$

4.4.1.1 Acceleration (Input) Transformation Derivation.

One goal of this TOP is to define a standard nomenclature. The following summary has been restructured to the nomenclature defined by this TOP. Referring to the input transformation derivation[10], a generic acceleration measurement at the k^{th} position in orientation j is structured as Equation 4.1:

$$a_{k_j} = \begin{bmatrix} \underline{e}_j^T & -\underline{e}_j^T \left[{}^P\underline{r}_i^P \right]^\times \end{bmatrix} \begin{bmatrix} {}^P\underline{a}_o^P \\ {}^P\underline{\alpha}^P \end{bmatrix} \tag{4.1}$$

where a_0 is the linear acceleration at some reference point designated the "origin", α is the angular acceleration of the body (assuming it is rigid), $k \in (1, 2, ..., N_a)$, $i \in (1, 2, ..., n^*)$, $j \in (x, y, z)$, and $\underline{e}_x^T = [1\ 0\ 0]$, $\underline{e}_y^T = [0\ 1\ 0]$, and $\underline{e}_z^T = [0\ 0\ 1]$ are row selection vectors (as shown assuming accelerometer orientation is aligned per a traditional right hand Cartesian system). Parameter N_a represents the number of accelerometer measurements (as previously defined) and $n^* \le N_a$ the number of measurement locations; e.g., utilization of multi-axis accelerometers results in $n^* < N_a$. Vector r_i is the position vector relating the position of measurement location i to a user defined origin. $\left[{}^P\underline{r}_i{}^P \right]^\times$ is the skew symmetric operator equivalent of the cross product, making the matrix based computations in Equation 4.1 possible. The matrix equivalent of a vector (i.e., a coordinatized vector quantity) is denoted as ${}^{(\ )}(\_)_{(\ )}^{(\ )}$ where the right superscript and subscript identify the body and point of interest respectively, and the left superscript denotes the coordinate frame in which the vector quantity was coordinatized; e.g., ${}^P\underline{r}_i^P$ in Equation 4.1 denotes the i[th] point on body P (the platform) coordinatized in frame $\mathcal{F}_P$ - the platform's coordinate frame.

4.4.1.2 Equation 4.1 represents one equation in six unknowns, the three components of the linear acceleration of the reference point and the three components of the rigid body angular acceleration. In order to determine these quantities, at least six measurements are needed. These requirements are not as stringent as that reported in the article "On the Use of Linear Accelerometers in Six-DOF Laboratory Motion Replication"[11] because of the assumptions above (i.e., small angular velocities and rigid body).

Let's consider the most general case of N_a measurements from n^* locations. In this case, Equation 4.1 becomes:

$$\underset{(n\times 1)}{\left[a_{k_j}\right]} = \underset{(n\times 1)}{\begin{bmatrix} a_{1_j} \\ a_{2_j} \\ \vdots \\ a_{n_j} \end{bmatrix}} = \underset{(n\times 6)}{\begin{bmatrix} \underline{e}_j^T & -\underline{e}_j^T\left[{}^P\underline{r}_1^P\right]^\times \\ \underline{e}_j^T & -\underline{e}_j^T\left[{}^P\underline{r}_i^P\right]^\times \\ \vdots & \vdots \\ \underline{e}_j^T & -\underline{e}_j^T\left[{}^P\underline{r}_{n^*}^P\right]^\times \end{bmatrix}} \underset{(6\times 1)}{\begin{bmatrix} {}^P\underline{a}_o^P \\ {}^P\underline{\alpha}^P \end{bmatrix}}, \; i \in \left(1,2,\ldots,n^*\right), j \in \left(x,y,z\right)$$

which using the nomenclature defined in this TOP is of the form:

$$\underset{(n\times 1)}{\left\{\mathbf{a}\right\}_{\text{Meas}}} = \underset{(n\times 6)}{\left[\overline{\mathbf{T}}_{\mathbf{a}}\right]} \underset{(6\times 1)}{\left\{\mathbf{c}\right\}_{\text{Motion}}} \tag{4.2}$$

where $\left\{\mathbf{c}\right\}_{\text{Motion}}$ is a 6 x 1 matrix of unknown linear and angular accelerations and $\left\{\mathbf{a}\right\}_{\text{Meas}}$ is an nx1 matrix of acceleration measurements. Observe that $\left[\overline{\mathbf{T}}_{\mathbf{a}}\right]$ is entirely defined by knowledge of (i) placement, (ii) orientation, and (iii) utilized signals of the accelerometers.

Observe that if $\overline{\mathbf{T}}_{\mathbf{a}}$ is of full column rank, then $\left[\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\right]^{-1}$ exists enabling $\left\{\mathbf{c}\right\}_{\text{Motion}}$ to be solved as follows:

$$\begin{aligned}
\left\{\mathbf{a}\right\}_{\text{Meas}} &= \overline{\mathbf{T}}_{\mathbf{a}}\left\{\mathbf{c}\right\}_{\text{Motion}} \\
\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\left\{\mathbf{a}\right\}_{\text{Meas}} &= \overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\left\{\mathbf{c}\right\}_{\text{Motion}} \\
\left[\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\right]^{-1}\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\left\{\mathbf{a}\right\}_{\text{Meas}} &= \left[\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\right]^{-1}\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\left\{\mathbf{c}\right\}_{\text{Motion}} \\
\left[\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\right]^{-1}\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\left\{\mathbf{a}\right\}_{\text{Meas}} &= \left\{\mathbf{c}\right\}_{\text{Motion}}
\end{aligned}$$

Defining $\mathbf{T}_{\mathbf{a}} \equiv \left[\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}\overline{\mathbf{T}}_{\mathbf{a}}\right]^{-1}\overline{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}$, Equation 4.2 can be rewritten as:

$$\left\{\mathbf{c}\right\}_{\text{Motion}} = \left[\mathbf{T}_{\mathbf{a}}\right]\left\{\mathbf{a}\right\}_{\text{Meas}} \tag{4.3}$$

Where $\left[\mathbf{T_a}\right]$ is a 6 x *n* matrix referred to in the literature as the "Acceleration Transform Matrix" or "Input Transform Matrix". Observe that the critical requirement that $\left[\overline{\mathbf{T}}_\mathbf{a}^\mathbf{T}\overline{\mathbf{T}}_\mathbf{a}\right]^{-1}$ exists in order to derive the input transformation matrix $\left[\mathbf{T_a}\right]$, is solely a function of placement and orientation of measurement transducers.

4.4.2 Drive (Output) Transformation.

a. Although details of the Drive Transformation are not required to develop a MDOF VSD reference, a short summary of the concept is provided for general knowledge. Referring to the schematic in Figure 2, transformation matrix $\mathbf{T_s}$ transforms the N_d drive variables into N_s shaker drive signals. Reference 10 provides a formal derivation of the transformation matrix, $\mathbf{T_s}$. Note that while the "acceleration transformation" was computed based on knowledge of position and polarity of the control accelerometers, the transformation matrix, $\mathbf{T_s}$ is dependent upon the position and line of action (LOA) of the individual actuators. In this TOP and within reference 10 $\mathbf{T_s}$ is referred to as the "drive transformation" or "output transformation". The following cases summarize the computation of $\mathbf{T_s}$ and the effect on the control process.

(1) Case 1: Configurations in which the number of motion degrees-of-freedom or control signals, N_c and the number of output control variables, N_d are the same is referred to as "square" control. If the number of output control variables, N_d and the number of shakers, N_s is the same, the transformation matrix, $\mathbf{T_s}$ will simply be the Identity matrix.

(2) Case 2: Configurations in which the number of shakers N_s exceeds the number of output control variables N_d, the excitation system is said to be over-determined or over-actuated. In such cases, some of the drives will be linear combinations of other drives. Furthermore, if $\mathbf{T_s}$ is a constant which is employed in the time domain, the individual actuators must be matched (e.g. matched frequency response functions (FRFs)).

(3) Case 3: Configurations in which the number of shakers, N_s is less than the number of control signals, N_c, the excitation system is said to be under-determined or under-actuated. In such cases, exact control of the SDM is not possible.

b. In theory, $\mathbf{T_s}$ could be implemented before or after the transformation into the time domain. One advantage of placing the transformation in the frequency domain section of the control algorithm is that the matrix could then be made a function of frequency. Having the transformation matrix, $\mathbf{T_s}$, a constant assumes the shakers are matched and the desired transformation can be deduced.

4.4.2.1 Drive (Output) Transformation Derivation.

a. As previously stated, one goal of this TOP is to recommend a standard nomenclature. The following summary from reference number 10 has been restructured to the nomenclature recommended by this TOP. Figure 3 illustrates the generalized multi-axis vibration system.

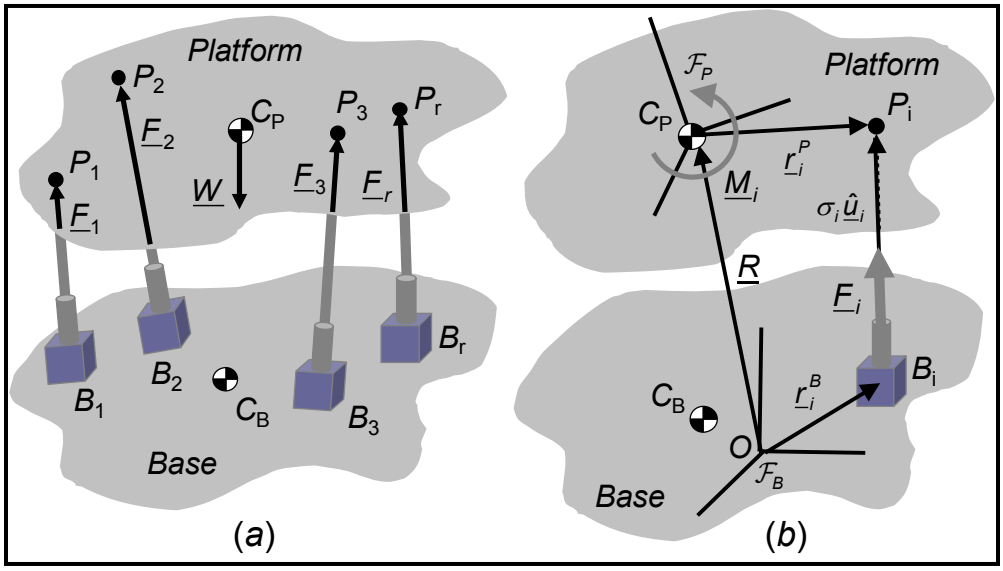

Figure 3. Generalized multi-axis vibration system.

b. Refer to reference number 10 for a detailed derivation of Equation 4.4. The following summary illustrates how the output transform, $\mathbf{T_s}$ is associated with the P-Matrix, (Plucker Matrix) discussed in the reference.

$$\underset{\mathbf{P}}{\begin{bmatrix} {}^{B}\hat{\underline{u}}_1 & {}^{B}\hat{\underline{u}}_2 & \cdots & {}^{B}\hat{\underline{u}}_{N_s} \\ {}^{P}\underline{m}_1^{P} & {}^{P}\underline{m}_2^{P} & \cdots & {}^{P}\underline{m}_{N_s}^{P} \end{bmatrix}}_{6xN_s} \underset{\mathbf{F}}{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N_s} \end{bmatrix}}_{N_s x1} = \underset{\mathbf{C}}{\begin{bmatrix} m\left({}^{B}\underline{a}_C^{P} - {}^{B}\underline{g}\right) - {}^{B}\underline{F}_E \\ {}^{P}\underline{\underline{I}}_C^{P}\,{}^{P}\dot{\underline{\omega}}^{P} + \left[\underline{\omega}^{P}\right]^{\times}\,{}^{P}\underline{\underline{I}}_C^{P}\left({}^{P}\underline{\omega}^{P}\right) - {}^{P}\underline{M}_E \end{bmatrix}}_{6x1} \tag{4.4}$$

c. In Equation 4.4, $[\mathbf{P}]$ represents the Plucker Matrix which is derived from known geometric parameters associated with the individual actuators, $[\mathbf{F}]$ represents the drive and $[\mathbf{C}]$ represents the desired motion. The variables ${}^{B}\hat{\underline{u}}_i$ represent the LOA vectors for each of the actuators and ${}^{P}\underline{m}_i^{P}$ is the moment arm associated with force f_i. Observe that the maximum dimension for the $[\mathbf{C}]$ matrix will be six, if all six traditional motion DOFs are being considered

(i.e. $N_d = 6$). As stated in paragraph 4.4.2, Case 1 scenarios will simply have an identity matrix as the output transformation matrix and Case 3 scenarios (under-actuated) will not have a unique solution. Case 2 scenarios (over-actuated) may be addressed in terms of output transformations. The objective is to determine $[\mathbf{F}]$ in Equation 4.4, yielding the N_s drive signals as follows:

(1) Define $\underset{N_s x 1}{\mathbf{F}} \equiv \underset{N_s x 6}{\mathbf{P}}^{T} \underset{6x1}{\mathbf{D}}$ and substitute into $\underset{6xN_s}{\mathbf{P}} \underset{N_s x 1}{\mathbf{F}} = \underset{6x1}{\mathbf{C}}$ yielding $\underset{6xN_s}{\mathbf{P}} \underset{N_s x 6}{\mathbf{P}}^{T} \underset{6x1}{\mathbf{D}} = \underset{6x1}{\mathbf{C}}$

(2) $\underset{6xN_s}{\mathbf{P}} \underset{N_s x 6}{\mathbf{P}}^{T}$ will be of full rank (i.e. invertible) if $[\mathbf{P}]$ is of full rank.

(a) If $[\mathbf{P}]$ is of full rank: $\underset{6x1}{\mathbf{D}} = \left[\underset{6xN_s}{\mathbf{P}} \underset{N_s x 6}{\mathbf{P}}^{T} \right]^{-1} \underset{6x1}{\mathbf{C}}$

(b) If $[\mathbf{P}]$ is not full rank, actuator placement is not sufficient to obtain the mechanical DOF's desired.

(3) Substituting results from (2) yields $\underset{N_s x 1}{\mathbf{F}} \equiv \underset{N_s x 6}{\mathbf{P}}^{T} \underset{6x1}{\mathbf{D}} = \underset{N_s x 6}{\mathbf{P}}^{T} \left[\underset{6xN_s}{\mathbf{P}} \underset{N_s x 6}{\mathbf{P}}^{T} \right]^{-1} \underset{6x1}{\mathbf{C}}$

(4) $\underset{N_s x 6}{\mathbf{T}_s} \equiv \underset{N_s x 6}{\mathbf{P}}^{T} \left[\underset{6xN_s}{\mathbf{P}} \underset{N_s x 6}{\mathbf{P}}^{T} \right]^{-1}$

d. The discussions within this paragraph and previous derivation assumed $N_d = 6$. In the event $N_d < 6$, N_d would represent the actual number of mechanical DOFs. In terms of the nomenclature of Figure 2, and assuming matched actuators are employed, voltage drives to the shakers for the over-actuated scenario would be defined as Equation 4.5:

$$\{\mathbf{s}\} = \mathbf{T_s}\{\mathbf{d}\} \tag{4.5}$$

4.5 Data Analysis.

a. Ensure transducer placements have been addressed, to guarantee the desired motion DOFs may be resolved (refer to paragraph 4.4.1.2), and that common data validity checks are performed. Then, it is recommended that appropriate combinations of the linear acceleration measurements be transformed into the 6 traditional motion DOFs through implementation of the acceleration transformation matrix. The transformed time histories will be referenced to a single point on the structure referred to as the "origin" as discussed in paragraph 4.4.1.

b. A SDM for each test configuration identified in the mission scenario should be computed. In addressing the VSD techniques for reducing an ensemble of data, in this case an ensemble of SDM's, the analyst will be required to deal with the ASD terms (the diagonal terms of the SDM) and CSD terms (the off-diagonal terms of the SDM).

4.5.1 Phase and Coherence based Representations of CSD Terms.

Although the off-diagonal terms of the SDM are computed in terms of a CSD, it is common among control system vendors to allow cross terms to be defined in terms of Phase and Coherence. This is a convenient option in that it is often easier to physically interpret SDM CSD terms in terms of Phase and Coherence. There is a direct relationship between the two techniques of defining the cross terms of the SDM that is dependent upon the definition of ordinary coherence between two signals, $\gamma_{ij}^2 = \frac{|G_{ij}|^2}{G_{ii}G_{jj}}$. Normalizing the CSD terms of the SDM by $\sqrt{G_{ii}G_{jj}}$ yields a normalized spectral density matrix (SDM_n) in which the ASD terms are not affected and the magnitude of the normalized CSD terms are defined as $\frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}}$, which is equivalent to the square root of the ordinary coherence function, while not affecting the original phase relationship of the CSD terms. Similarly, the normalized spectral density matrix, SDM_n, may be transformed back to the original CSD form of the SDM.

4.5.2 Positive Definite SDM Considerations.

a. Any specified spectral density matrix must be positive semi-definite to be physically realizable. In practice it must be positive definite. The determinate of the matrix must be ≥0. All the eigenvalues of the SDM must be ≥0. This must be true at all frequencies. It must be possible to perform a Cholesky decomposition of the specified SDM. Another property of positive semi definite matrices is from Matrix Computations[12]:

$$|\Phi_{ij}|^2 \le \Phi_{ii}\Phi_{jj} \quad or \quad 0 \le \gamma^2 = \frac{|\Phi_{ij}|^2}{\Phi_{ii}\Phi_{jj}} \le 1$$

In the terms of random vibrations the ordinary coherence, γ^2 between signals must be less than or equal to one. In practical terms, if the coherence between any pair of signals is one, the SDM will be positive semi-definite and the control system will have problems. Note that in general, if **D** is Hermitian and positive semi-definite **C** will also be Hermitian and positive semi-definite.

b. If all the eigenvalues are non-negative, the matrix is positive semi-definite. If any of the eigenvalues are zero, it implies that one or more of the rows of the spectral density matrix are a linear combination of other rows. In practice, one would typically expect to deal only with positive definite matrices. Observe that even a small amount of noise or nonlinearity will result in a positive definite matrix. If a matrix is positive definite, the matrix can always be factored using Cholesky decomposition,

$$\mathbf{\Phi = LL'}$$

where **L** is a lower triangular matrix. Which without loss of generality can be rewritten as,

$$\mathbf{\Phi} = \mathbf{LIL'}$$

where **I** is the identity matrix. In this application, **I** is not really the identity matrix. **I** is a spectral density matrix. At every frequency, **I** is a diagonal matrix of ones. The components in **I** are independent since all the off diagonal elements are zero. It is now clear why the cross spectral density matrix must be positive definite. If any of the elements in **I** are zero, it implies that there are less than N (the number of rows or columns in $\mathbf{\Phi}$) independent sources in $\mathbf{\Phi}$. Some of the rows and columns are linear combinations of other rows and columns. The identity matrix is positive definite, therefore $\mathbf{\Phi}$ must be positive definite. Using the interpretation of Random Data Analysis and Measurement Procedures[13], the diagonal elements of **I** can be interpreted as the auto-spectral densities of independent random noise sources. The maximum number of independent noise sources is N. If some of the elements in **I** are zero, the problem can still be solved by making the corresponding rows and columns of **L** zero. This is the positive semi-definite case. This case corresponds to the case where there exists less than N independent sources. Some of the N sources are linear combinations of other sources. This case will be very difficult for the control system. In general one may make some of the sources small but not zero. Part of this document will discuss the generation of a desired control SDM to make the control problem achievable and hopefully relatively easy for the control system to implement.

c. In general the control problem is an inverse problem. The desired control SDM (the output of the system under test) is known, and the drive (input to the system under test) SDM must be computed. There is a potential point of confusion here. The control system manufacturers think the drive SDM is the output of the control system, which is the input to the shaker system. Similarly, the control system input is the output of the shaker system. Paragraphs 4.4.1 and 4.4.2 provide nomenclature employed for input and output transformations as they are applied within this document.

d. Inverse problems can be very difficult as multiplication by a matrix inverse is required. If the matrix is ill-conditioned, the result will be similar to dividing by zero for the scalar case.

For the case in which the number of inputs and outputs are the same; **H** is a square matrix of FRF's. The solution is to invert **H**. The solution for the drive matrix is then given by:

$$\mathbf{Z} = \mathbf{H}^{-1}$$
$$\mathbf{D} = \mathbf{ZRZ'}$$

This of course assumes **H** is well conditioned and the inverse exists. Part of this document will discuss issues to help the process of achieving a well conditioned **H** matrix.

The **H** matrix is typically estimated from:

$$\hat{\mathbf{H}} = \hat{\mathbf{S}}_{\mathbf{CD}}\hat{\mathbf{D}}^{-1}$$

The inverse of $\hat{\mathbf{D}}$ must exist. This implies that $\hat{\mathbf{D}}$ must be positive definite. The initial estimate of **H** is determined by exciting the system with a set of independent white inputs in a pretest environment. If **H** is to be corrected during the test, $\hat{\mathbf{D}}$ must be positive definite during the test or special provisions must be used to avoid the inversion of $\hat{\mathbf{D}}$ at frequencies where $\hat{\mathbf{D}}$ is not positive definite. This is one of the reasons the reference **R** rarely has any of the coherences equal to unity.

4.5.3 Data Compression.

a. Use of time compression techniques such as Miner-Palmgren may be employed to modify the ASD terms. References numbers 1 and 3 provide discussions on time compression. In the simplest terms, the Miner-Palmgren Hypothesis (Miner's rule) is a set of mathematical equations used to scale vibration spectra levels and their associated test times. It provides a convenient means to analyze fatigue damage resulting from cyclical stressing. The mathematical expression and variable descriptions for this technique are illustrated below in Equation 4.6:

$$\frac{t_2}{t_1} = \left[\frac{S_1}{S_2}\right]^M \tag{4.6}$$

where:

t_1 = equivalent test time

t_2 = in-service time for specified condition

S_1 = severity (root mean square ((rms)) at test condition

S_2 = severity (rms) at in-service condition

(The ratio S_1/S_2 is commonly known as the exaggeration factor.)

M = a value based on (but not equal to) the slope of the S-N curve for the appropriate material where S represents the stress amplitude and N represents the mean number of constant amplitude load applications expected to cause failure. For the MDOF VSD work at hand, the default of $M = 7$ was selected per reference number 1.

b. It is recommended that the final vibration specification ASD terms are no greater than 3 decibel (dB) higher than maximum values measured in the field. Miner-Palmgren will be employed to the ASD portion of the SDM in the same manner as one would employ for a traditional 1-DOF scenario. Details such as maintain common test durations between mechanical DOFs are address in Paragraph 6.

4.5.4 Limiting Strategies.

Traditional notching techniques may also be employed if impedance mismatches lead to unrealistically high test item response. Notching techniques may be employed across all actuators with equal weighting or by weighting notching at each actuator as a function of coherence between the actuators and the location of interest. In addition to traditional notching based on acceleration spectra, it is also possible to consider limiting based on other parameters

(e.g. von Mises Stress or Force limiting). As with any notching scheme, it is critical that any resulting deviations to the test or test tolerances must be approved by the appropriate test authority and must be clearly documented in the test plan and final report.

4.5.5 Minimum Drive Considerations.

A number of challenges have been identified in addressing the objective of establishing a reference SDM for multiple exciter test (MET) scenarios. One major area of concern is related to the fact that it is highly likely that there will be mechanical impedance differences between the field and laboratory conditions. Given these impendence mismatch issues, it is undesirable to force the test item into what could potentially be an unnatural state as fixtured in the laboratory. Optimally, achieving the specified autospectra without excessively taxing the excitation system is desired. Smallwood made a general approach to establishing minimum drive criteria in the article "MIMO Linear Systems Extreme Inputs/Outputs"[14]. Unfortunately, the technique does not always guarantee the resulting SDM to be positive semi-definite.

4.5.5.1 Independent Drives.

a. Although an active area of research, general techniques to address minimum drive criteria have not been formally established at the time of this publication. A proposed approach for trending drive voltages towards minimums while maintaining a positive- definite SDM, is discussed in the article "A Proposed Method to Generate a Spectral Density Matrix for a MIMO Vibration Test"[15], and is summarized below:

(1) Taking a clue from the modal test community, assume the drive signals to the excitation system will be uncorrelated. Typically for a vibration test, the drives are the voltage inputs to the shakers. For a simulation, the inputs into a model are often forces. It is always possible to excite the system with uncorrelated inputs. This is standard practice in the modal community, and is standard practice when performing the system identification for MIMO test systems. This leads to the logical question: Is it possible to generate a set of uncorrelated inputs that will produce a desired set of response autospectra (the diagonal of the output SDM)?

(2) The general equation relating the control point accelerations to the drive voltages is given in Random Vibrations, Theory and Practice[16]:

$$\mathbf{S_Y} = \mathbf{H}\mathbf{S_X}\mathbf{H'}$$

where $\mathbf{H'}$ is the conjugate transpose of $\mathbf{H}$, and $\mathbf{S_X}$ and $\mathbf{S_Y}$ are SDM's. $\mathbf{H}$ is a matrix of frequency response functions relating the output to the input of the excitation system. In our case, ideally, $\mathbf{S_X}$ will be a diagonal matrix. Let $\overline{\mathbf{X}}$ be a column vector of the diagonal of $\mathbf{S_X}$ or, $\overline{\mathbf{X}} = \text{diag}(\mathbf{S_X})$, and $\overline{\mathbf{Y}} = \text{diag}(\mathbf{S_Y})$. The relationship between the autospectra, as shown in Annex D proof 1, is given by:

$$\overline{\mathbf{Y}} = \overline{\mathbf{H}\mathbf{X}}$$

where:

$$\overline{\mathbf{H}} = \mathbf{H}.*\text{conj}(\mathbf{H})$$

where: * indicates an element by element multiplication. $\overline{H}_{ij} = |H_{ij}|^2$.
The solution is given by:

$$\overline{\mathbf{X}} = \overline{\mathbf{H}}^{-1}\overline{\mathbf{Y}}$$

b. In some cases the result will include negative elements in $\overline{\mathbf{X}}$. This is not physically possible. It indicates that the desired ASD's cannot be achieved with independent drives. In this case the negative values are set to zero, and the output SDM is recomputed from $\mathbf{S_Y} = \mathbf{HS_X H'}$ using the modified input spectral density matrix (the negative values set to zero). The resulting control point acceleration autospectra, will not be at the desired levels. To correct this problem, the control point acceleration autospectra are rescaled to the desired levels, keeping the phase and ordinary coherence the same. This is accomplished by pre and post multiplying the SDM by a diagonal matrix whose elements are the square root of the ratio of the desired ASD to the computed ASD:

$$\mathbf{S_{Ynew}} = \mathbf{S}_s\mathbf{S_{Yold}}\mathbf{S}_s$$

where $\mathbf{S_s}$ is a diagonal matrix and

$$S_{s,ii} = \sqrt{\frac{Y_{ii,new}}{Y_{ii,old}}}$$

Note: Setting $\boldsymbol{S_{Yii,new}}$=1, provides an efficient way to compute the normalized SDM where the diagonals are one and the magnitude of the off diagonals squared are the ordinary coherence and the phase of the off diagonal elements is the phase of the cross spectra.

The drive SDM can then be computed as:

$$\mathbf{S_{Xnew}} = \mathbf{ZS_{Ynew}Z'}$$

where $\mathbf{Z} = \text{pinv}(\mathbf{H})$, the Moore-Penrose pseudo inverse. If **H** is square and full rank, the solution typically ends here. If **H** is not square or not full ranked:

$$\mathbf{S_{Y}}_{new} = \mathbf{HS_{X}}_{new}\mathbf{H'}$$

The $\text{diag}(\mathbf{S_{Y}}_{new})$ may not yield the desired ASD's. In this case, an iterative approach will often improve the result.

4.6 Independent References.

a. It is sometimes desirable to define the reference spectrum in terms of a diagonal matrix of autospectra. Several reasons drive us in this direction. One common case is that only knowledge of the autospectra from the field environments is available. Several factors can result in this situation. First the field data may have been acquired without phase information. Second, the resulting cross spectra can have a very complicated structure which is impractical to implement in a specification. Enveloping amplitudes is possible, but enveloping the phase is much more difficult. Third, the specification may be a composite of several environments, making the definition of cross spectra very difficult. Fourth, the vehicle on which the field data were taken may not be identical to the test vehicle. Fifth, the boundary conditions in the field may be different from the boundary conditions in the laboratory.

b. Small changes in the modal frequencies caused by any of the above factors can change the phase at any frequency near a mode by a large amount. All these factors make the specification of the cross spectra difficult. An option is to ignore the cross spectra and set them all to zero. This has the theoretical advantage of providing an excitation that in some sense covers the control variable response space.

c. The drive signals can readily be computed yielding uncorrelated motion (in this case the SDM of the uncorrelated reference spectra **Y** is diagonal) from:

$$\mathbf{S}_{X0} = \mathbf{Z}\mathbf{S}_{Y0}\mathbf{Z}'$$

This approach is currently available in commercial control systems. You simply specify the reference SDM as a diagonal matrix with the cross spectra (or equivalently the coherences) zero or near zero. This is typically a conservative approach.

d. In contrast to the independent drive discussion in paragraph 4.5.5.1, the danger with the independent reference concept is that this specification of control variables may be overly conservative near frequencies dominated by a single mode. An important clue that the result may be overly conservative is the trace of the drive voltages. This trace should be monitored and if overly large in some band of frequencies, limits can be negotiated and implemented.

4.7 Recommended Practices Summary.

The following list provides recommendations and general guidance to be considered when addressing the multi-axis VSD.

a. If possible, specify the test in terms of the rigid body motion.

b. Over specification of the control accelerometers is desirable. Use more control accelerometers than degrees of freedom in the test.

c. If possible, the entire SDM should be specified. A method to automate the generation of envelopes may be desired. This will permit the generation of the envelopes to be less developer specific.

d. If the entire SDM is specified, it is suggested that the coherence be set to near zero if the desired coherence is below 0.2. It should be recognized that the estimation of coherence is a biased result (the result will always be positive). It is recognized that the estimated coherence will never be zero; however, the control software can attempt to make the coherence as low as possible. The tolerance on the coherence must recognize the bias. If the coherence is small the phase is not important. For convenience, establishing a zero phase is a reasonable specification when the coherence is low.

e. If step c becomes too complicated, it is recommended that the test be run with near zero coherence.

f. If step e results in unrealistic responses, try using the independent drive option.

g. Consider a compromise position between independent reference criteria of step e and independent drive criteria as recommended in step f.

h. If the drive requirements are excessive at some frequencies, allow the test to be modified to reduce the drive requirements as discussed in paragraphs 4.5.4 and 4.5.5.

i. It is understood that MIMO testing is more complicated than single-input single output (SISO) testing. The specifications must reflect the desires of a knowledgeable environmental test engineer. Good communication between the project team, the environmental test engineer and the test lab must be maintained to achieve the desired test results.

5. DATA REQUIRED.

Just as discussed in NATO AECTP Leaflet 2410[3], field data must be acquired based upon the anticipated mission scenario of the unit under test (UUT). As detailed in paragraph 4.4.1.1 and reference number 1, transducer placement and orientation are critical and must be thoroughly documented.

5.1 Reference SDM Development.

As stated in paragraph 4.5, a SDM in terms of the six traditional rigid body modes should be computed for each test configuration identified in the mission scenario.

5.1.1 SDM Ensemble CSD Characteristics.

Based on the characteristics of the CSD terms of the ensemble of SDMs, the VSD process will yield a vibration specification consistent with one of the three cases that follow:

a. Case 1. Coherence Terms Approaching Zero (Independent Motion DOFs) – This is the easiest situation to deal with in that each motion DOF ASD may be addressed individually via the same techniques employed in 1-DOF VSD as discussed in reference number 3. When programming the vibration control system, it is recommended that coherence be set to a low non-zero level (i.e. $\gamma^2 = 0.1$) over the test bandwidth of interest. For such a small coherence, the phase parameter is essentially a random variable and establishing a phase specification is not required.

A special situation that may lead an analyst to develop a MDOF vibration specification with independent motion DOFs, would be a composite specification that encompasses multiple vehicles (i.e. a composite wheeled vehicle specifications comparable to those in MIL-STD-810G, Method 514). As each vehicle will tend to have its own CSD characteristics, it is not possible to define CSD terms such that a single coherence and phase relationship addresses each vehicle. Enveloping techniques that work well in addressing magnitude based ASD terms are simply not applicable in addressing phase relationships between mechanical DOFs.

b. Case 2. Non-Zero Coherence across a Portion of the Test Bandwidth – When developing a MDOF vibration specification based on a single platform, one would expect the CSD terms measured across the range of scenarios addressed in the mission scenario to be similar in nature. The dynamic characteristics of the structure and often the proximity of the measurement transducers will greatly influence the CSD characteristics. There are often situations in which coherence between motion DOFs are high and phase is well defined, but only over a portion of the test spectrum. This is a common observation on many wheeled vehicles where coherence is high at lower frequencies (i.e. frequencies below 50 Hertz (Hz) and near zero at higher frequencies. In such scenarios, one would only establish coherence and phase specifications for the portion of the spectrum with high coherence. The remainder of the spectrum would be treated as in Case 1. Also, in establishing CSD reference criteria, the analyst must ensure the resulting criteria is physically realizable (refer to paragraph 4.5.3 for additional detail).

c. Case 3. Non-Zero Coherence across the Full Test Bandwidth – This scenario is comparable to Case 2 with coherence being defined across the entire test bandwidth. It is anticipated that this would be the least likely scenario in a MDOF VSD effort. However, it is also the configuration that will be the most difficult to deal with from both a VSD development aspect and from an implementation perspective. In addition to the issue of ensuring the resulting SDM reference is physically realizable, the classic problem of mechanical impedance mismatch between field and laboratory are often major concerns in implementing a fully defined SDM reference criterion for a laboratory test. Specifically, if the mechanical impedance between field and laboratory are not very well matched (and they usually are not), there may be portions of the spectrum in which coherence may be significantly different than specified and simply not controllable. While this situation is also possible in Case 2, it is almost certain to be an issue in a scenario such as Case 3, in which the entire test bandwidth has a CSD reference criteria. This topic of uncontrollable coherence associated with mechanical impedance mismatches is a control issue for all three Cases and is discussed further in the minimum drive consideration of paragraph 4.5.6.

d. Regardless of which of the three cases the SDM is characterized by, it is highly likely that there will be mechanical impedance differences between the field and laboratory conditions. In some cases these impedance differences may result in excessive drive signals. Although the various control system vendors address this situation in varying degrees, it may still be necessary to address this issue through test operator intervention via techniques such as those identified in paragraphs 4.5.4 and 4.5.5.

5.2 Test Tolerance Recommendations.

Setting tolerances for a MIMO test is challenging given the large amount of information encompassed by the reference autospectra and cross spectra involved. Additionally, the overall energy is not necessarily distributed evenly about each mechanical DOF and dominant DOFs often tend to dominate the control. The objective here is to establish a reasonable starting point in establishing test tolerances. Experience with specific test configurations may be employed to refine the basic requirements defined below. As usual, any test specific test tolerances should be clearly documented within the test plan.

a. Autospectra[(1)]: ±3 dB for $f \le 500 Hz$ and $\pm$6dB for $f > 500 Hz$.
[(1)]The portion of the spectrum that actually reaches the maximum tolerance limits is anticipated in narrow bandwidths. The tolerance on the overall Grms level of each controlled DOF shall be within $\pm$15% of the corresponding reference.

b. Cross spectra: Define tolerances in terms of Phase and Coherence

(1) Coherence: For ordinary coherence in the range $0.5 \le \gamma^2 < 1.0$, set the tolerance to be ±0.1 (avoid establishing a coherence reference or tolerance of 1.0).

(2) Phase: If $\gamma^2 < 0.5$, any phase is acceptable. If $0.5 \le \gamma^2 < 1.0$ and $f_h + 3\Delta f < f < f_h - 3\Delta f$, where f_h is a frequency where the reference rate of phase change is more than $10° / Hz$ and Δf is the line spacing of the reference spectra, then the default tolerance on phase will be $\pm 40°$. Otherwise the default tolerance on phase will be $\pm 10°$.

c. Limiting: See paragraph 4.5.4.

5.3 Laboratory Data.

In the case the reference SDM is directly employed as the reference in a MET test (i.e. input/output (I/O)) Transformation Control as discussed in reference number 9), it is not a mathematical based hard requirement that the control accelerometers be placed in the exact same location in the laboratory as they were used in the original data acquisition phase. The critical parameter is that all control locations employed in the laboratory test are referenced to the same "origin" as selected in the original VSD development. However, it is often desirable, based on making position specific comparisons between field and laboratory data, to match the laboratory control locations to the original measurement points.

6. MDOF VSD METHODS.

6.1 Options Considered.

Having reviewed the data acquisition and analysis requirements, this section is dedicated to defining the steps for two MDOF VSD methodologies[17]. Method I is processed in the SDM domain and Method II conducts averaging steps in the Cholesky Domain. An example follows in paragraph 6.3.

6.1.1 Method I.

The following is a 10 step outline of Method I (SDM Domain) MDOF VSD:

Step 1: Prepare to convert field measurements into motion DOFs.
- Identify position vectors $r_1 - r_n$ and row selection vectors e_j as defined in paragraph 4.4.1.1, corresponding to the field measurements.
- Identify the mission scenario.
- Identify the frequency bandwidth of interest.
- Identify the sampling frequency of the field measurements.

Step 2: Transform the field measurements into motion DOF's per equation (4.3) for each "Run" identified in the mission scenario.

Step 3: Compute the SDM for each run identified in Step 2. The dimension of the resulting SDM's will be [6x6xd], where d is the number of spectral lines being considered, addresses the frequency bandwidth of interest.
- Since the SDM is computed from measured field data, it should be positive definite; however, a check should be performed to verify that each individual SDM is positive definite. This serves as an excellent data quality check.
- Refer to the guidance in Step 7 if minor corrections are required to force an individual SDM to be positive definite.

Step 4: Convert the CSD terms (the off-diagonal terms of the SDM) into a normalized form in which the magnitude squared of the cross terms correlates to the ordinary coherence while leaving the phase unchanged.
- This is accomplished by normalizing (dividing) the CSD terms by $\sqrt{G_{xx}G_{yy}}$.
- While it is not absolutely necessary to conduct this step, it is often easier to understand the physical meaning of the CSD terms when viewed in terms of phase and coherence.

Step 5: Either organize all of the SDM's for the Runs of interest into a logical structure or merge them into one file of known matrix structure such as [SDM_Run1,SDM_Run2….SDM_RunN] to optimize the conduct basic statistics.

Step 6: Compute a weighted average SDM of the N SDM's of Step 5.

- It is critical that the weighted average be performed on the true complex CSD terms (**not** the normalized SDM).
- The weighting factor on the average will be directly correlated to the mission scenario times identified in Step 1. If the individual Runs are positive definite, the resulting average should also be positive definite. However, numerical issues may yield non-positive definite results. To minimize numerical issues, average only the lower triangular portion of the SDM and fill in the upper triangular portion of the SDM by taking advantage of the Hermitian structure of the matrix [16].
- Any type of enveloping operation should be avoided as it is highly likely to yield a non-positive definite result.

Step 7: As SDM data are manipulated through activities such as averaging, it is advisable to verify the results remain positive definite. As discussed above, occasional numerical issues may be of concern in some instances. If required, force the SDM computed in Step 6 to be positive definite.

- This is done by systematically reducing the magnitude of the cross spectral density terms until the Cholesky decomposition is possible at each depth (spectral line) of the SDM. (If required, this process may be somewhat conservative in its reduction of the coherence between DOFs in that the systematic reduction of cross term magnitudes is applied to each cross term equally).

Step 8: Scale the diagonal terms of the autospectra (the diagonal terms of the SDM) resulting from Step 7 to the maximum rms level of each of the N SDM's in Step 5 on an individual DOF basis using Miner-Palmgren.

- Observe that a new total test time will be computed for each DOF and that it highly probably that the resulting test times for each DOF will not be the same.
- Since the magnitude of the autospectra are being increased while not modifying the cross-spectral density terms, the resulting scaled SDM should still be positive definite. However, as discussed in Step 7, it is highly recommended that anytime a SDM is manipulated, it should be verified that the resulting SDM remains positive definite.

Step 9: Review the test time associated with each DOF resulting from Step 8 and select a reasonable test time to which the entire SDM may be referenced to.

- Just as in the case of a 1-DOF VSD development, one should consider the general guidance to keep the final test amplitudes resulting from time compression to be no more than 3 dB above the maximum measured field data. Once a test time is selected, reapply Miner-Palmgren as required per DOF. Again make sure the resulting SDM is positive definite and modify as required per Step 7.

Step 10: Scale the results from Step 9 up by 3 dB to account for uncontrolled variables such as fleet variations and scenario conditions not considered in the mission scenario. The resulting SDM and the test time association per Step 9 define the final specification.

- This is accomplished by pre and post multiplying the SDM by the square root of the ratio of the desired scaling factor as:

$\mathbf{S}_{\mathbf{Ynew}} = \mathbf{S}_s \mathbf{S}_{\mathbf{Yold}} \mathbf{S}_s$ (e.g. to scale the SDM ASD terms by 3 dB while keeping the phase and ordinary coherence the same, the diagonal terms of $\mathbf{S}_s$ would be defined as $S_{s,ii} = \sqrt{2}$).

6.1.2 Method II.

The following is a 10 step outline of Method II (Cholesky Domain) MDOF VSD:

Steps 1-4: Correlate directly to Method I Outline.

Step 5: Perform a Cholesky decomposition on the individual SDM associated with each Run in the mission scenario.

- Since each individual Run was based on a physical event, the individual SDM's should be positive definite, thereby making the Cholesky decomposition possible. (Recall all Runs would have been tested to verify each was positive definite or corrected as required per Step 3).
- Either organize all of the lower triangular matrices resulting from the Cholesky decomposition for the Runs of interest into a logical structure or merge them into one file of known matrix structure such as [CHOL_Run1,CHOL_Run2….CHOL_RunN] to optimize the conduct basic statistics.

Step 6: Compute a weighted average Lower Triangular Matrix of the N Cholesky decompositions of Step 5.

- The weighting factor on the average will be directly correlated to the mission scenario identified in Step 1. Note that the resulting average will still consist of positive eigenvalues implying that when converted back into the SDM format that the result will be positive definite.
- Once converted back into the SDM domain, the resulting CSD terms will generally be highly comparable to the average CSD values computed in Step 6 of Method I. However, the rms levels of the ASD terms will not be the same. In addition, the spectral shape of the ASD terms will generally have been slightly modified.

Step 7: Rescale the ASD terms of the SDM resulting from Step 6 to match the rms levels of those in Method 1 Step 6.

- Convert the CSD terms (the off-diagonal terms of the SDM) into a normalized form in which the magnitude squared of the cross terms correlates to the ordinary coherence while leaving the phase unchanged. (Again, while it is not absolutely necessary to conduct this step, it is often easier to understand the physical meaning of the CSD terms when viewed in terms of phase and coherence).
- The resulting SDM phase and coherence are expected to be very similar to those of the averaged field data produced in Method I. The ASD terms spectral shapes are expected to be slightly different (i.e. < 3 dB per spectral line).

Steps 8-10: Correlate directly to Method I Outline.

6.2 Example.

a. To illustrate the process discussed above, a simple example was derived (Method I is addressed first). Using an available wheeled vehicle, the input to an onboard missile storage rack was instrumented as shown in Figure 4. The transducer at the center of Figure 4 was placed at the user defined origin, position [0,0,0], in terms of a Cartesian coordinate system. In a traditional right hand orientation, the forward direction of the vehicle was defined as the positive x-axis, towards the vehicle driver's side was considered positive y-axis, and through the vehicle roof was considered the positive z-axis. All transducers are referenced in terms of their relative placement to the origin as discussed previously in the acceleration transformation section of this TOP.

Figure 4. Transducer placement (input to missile rack).

b. Method I Example.

(1) Having established a clear coordinate system definition, the key parameters discussed in Step 1 are identified. In distance units of inches, the positions of the four corner accelerometer locations used in this example are defined as:

$r1=[-17,-6,0]'$, $r2=[-17,6,0]'$, $r3=[17,-6,0]'$, $r4=[17,6,0]'$, which in skew symmetric form are:

$$\left[{}^{P}r_1^{P}\right]^{x}=\begin{bmatrix}0 & 0 & -6\\ 0 & 0 & 17\\ 6 & -17 & 0\end{bmatrix},\left[{}^{P}r_2^{P}\right]^{x}=\begin{bmatrix}0 & 0 & 6\\ 0 & 0 & 17\\ -6 & -17 & 0\end{bmatrix},\left[{}^{P}r_3^{P}\right]^{x}=\begin{bmatrix}0 & 0 & -6\\ 0 & 0 & -17\\ 6 & 17 & 0\end{bmatrix},\left[{}^{P}r_4^{P}\right]^{x}=\begin{bmatrix}0 & 0 & 6\\ 0 & 0 & -17\\ -6 & 17 & 0\end{bmatrix}$$

For convenience, the instrumentation team placed the tri-axial transducers such that the channel used to measure the y-axis motion was actually 180 degrees out of phase with respect to the referenced coordinate system. This issue is addressed by simply defining row selection vectors as $e_x^T=[1,0,0], e_y^T=[0,-1,0], e_z^T=[0,0,1]$. Matrix $\overline{T}_a$ and matrix T_a may now be computed as per the discussion in paragraph 4.4.1.1 as:

$$\overline{T}_a=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 6\\ 0 & -1 & 0 & 0 & 0 & 17\\ 0 & 0 & 1 & -6 & 17 & 0\\ 1 & 0 & 0 & 0 & 0 & -6\\ 0 & -1 & 0 & 0 & 0 & 17\\ 0 & 0 & 1 & 6 & 17 & 0\\ 1 & 0 & 0 & 0 & 0 & 6\\ 0 & -1 & 0 & 0 & 0 & -17\\ 0 & 0 & 1 & -6 & -17 & 0\\ 1 & 0 & 0 & 0 & 0 & -6\\ 0 & -1 & 0 & 0 & 0 & -17\\ 0 & 0 & 1 & 6 & -17 & 0\end{bmatrix}$$

$$T_a=\begin{bmatrix}0.2500 & 0 & 0 & 0.2500 & 0 & 0 & 0.2500 & 0 & 0 & 0.2500 & 0 & 0\\ 0 & -0.2500 & 0 & 0 & -0.2500 & 0 & 0 & -0.2500 & 0 & 0 & -0.2500 & 0\\ 0 & 0 & 0.2500 & 0 & 0 & 0.2500 & 0 & 0 & 0.2500 & 0 & 0 & 0.2500\\ 0 & 0 & -0.0417 & 0 & 0 & 0.0417 & 0 & 0 & -0.0417 & 0 & 0 & 0.0417\\ 0 & 0 & 0.0147 & 0 & 0 & 0.0147 & 0 & 0 & -0.0147 & 0 & 0 & -0.0147\\ 0.0046 & 0.0131 & 0 & -0.0046 & 0.0131 & 0 & 0.0046 & -0.0131 & 0 & -0.0046 & -0.0131 & 0\end{bmatrix}$$

The field data were sampled at 4096 Hz and the bandwidth of interest is 500 Hz. For the example at hand, a mission scenario was established using a Beta distribution as discussed in reference number 3, and is illustrated in Table 1. Allowing for the time associated with speeds below 5 miles per hour (mph), the total mileage represented is approximately 300.

TABLE 1. MISSION SCENARIO

ROAD CLASSIFICATION	SPEED (mph)	TIME (hrs)	DISTANCE (miles)
Embedded Rock	5	.690	3.45
	10	1.545	15.45
	15	.737	11.05
Cross Country	10	5.18	51.80
	20	6.332	126.64
	30	2.002	60.06
Radial Washboard	5	.811	4.055
	7	1.841	12.88
	10	1.183	11.83

(2) The field data were then converted into motion DOFs, $\{\mathbf{c}\}_{\text{Motion}}$, using Equation 4.3 per Step 2.

(3) The time histories, $\{\mathbf{c}\}_{\text{Motion}}$ were then transformed into the frequency domain in the form of a SDM per run as described in Step 3. Each SDM was tested per the Cholesky decomposition property and verified to be positive definite.

(4) Each SDM was then normalized as suggested in Step 4 to allow the analyst to review the degree of coherence between DOFs.

(5) Per Step 5, the SDMs were configured into a convenient structure to allow statistical analysis. MATLAB was employed in this example and the data were configured as SDM_all=[SDM_Run1,SDM_Run2….SDM_Run8] . Observe only 8 of the 9 runs identified in the scenario are being considered. In reviewing the field data, the 5 mph radial washboard data were significantly lower than the rest of the Runs, determined to have no effect on fatigue, and were not considered in computing the basic statistics of the ensemble.

(6) Next, per Step 6, a weighted average in terms of the time per road condition as defined in Table 1 was computed. This average should be computed in terms of complex CSD terms, not the normalized SDM. The resulting weighted average SDM was then tested at each spectral line to establish whether or not the positive definite criterion was met. Figure 5 illustrates the weighted average SDM. Taking advantage of the Hermitian property of a SDM, Figure 5 is laid out such that the lower triangular section represents the phase between DOFs, the upper triangular portion represents the square root of the ordinary coherence, and the diagonal terms are the ASDs of the 6 rigid body DOFs. Although too small to review in detail on a single page as shown, the coherence plots are all scaled between 0.1 and 1.0. This is to illustrate there is some level of coherence, particularly below 100 Hz in the example at hand, between DOFs. Using the VSD process proposed, the analyst will try to keep as much coherence in the final specification as possible while still ensuring the final result is positive definite.

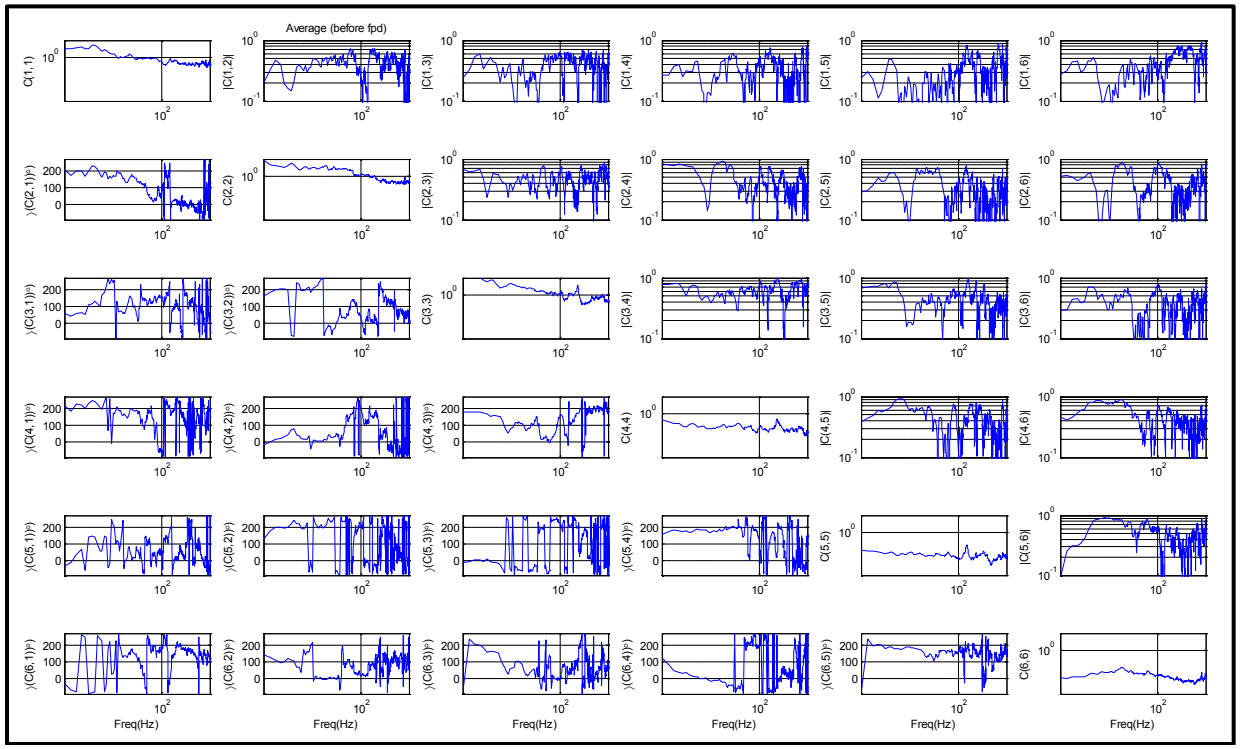

Figure 5. Normalized weighted average SDM.

(7) In order to address the possibility of having to deal with non-positive definite results, a utility was written which gradually and equally reduces the magnitudes of the cross spectral density terms until the positive definite criterion is met per Step 7. This technique actually reduces the cross term magnitudes of some CSDs more that what is required. Addressing this potential shortcoming is one of the motivations for the development of Method II.

(8) At this point, per Step 8, the rms level was computed for each ASD (diagonal SDM Entry) over the bandwidth of interest (3-500 Hz in this example). Each ASD was then scaled to the level of the maximum rms level via Equation 4.6.

(9) Per Step 9, the new test times associated with each ASD were also documented. As expected, the new times associated with each DOF were no longer the same. Since the VSD effort is designed to yield a simultaneous 6-DOF reference, it will be necessary to choose a common test time and rescale all ASD entries to the selected test duration. For the example at hand, a test duration of 15 minutes was selected. As is always the case with selection of compressed test durations, one should adhere to the guidance of not exaggerating the ASD power levels by more than 2:1. Of course when dealing with 6 ASD terms, this is not always possible. In such cases, the analyst should avoid increasing the dominant DOFs or DOFs with known structural shortcomings by more than 3 dB above maximum measured ASD levels.

(10) The terms comprising the SDM were based on average ASD and CSD estimates, which is in contrast to the guidance provided in reference number 3, in which the ASD levels carried through the calculations of a 1-DOF VSD were actually based on an ASD computed as the sum of a Mean ASD and standard deviation computed on a per spectral line basis. Working directly with the mean ASD levels is intended to avoid excessive conservatism in the VSD process. Conservatism intended to address uncontrolled variables such as fleet variations and conditions not considered in the mission scenario are addressed by a single scalar (+3 dB in this example) in Step 10. Clearly the analyst has the ability to modify the final conservatism level based on knowledge of the specific VSD effort.

The final reference SDM produced by Method I is shown in Figure 6. Observe that the phase and coherence terms are essentially unchanged from that of the average SDM of Figure 5.

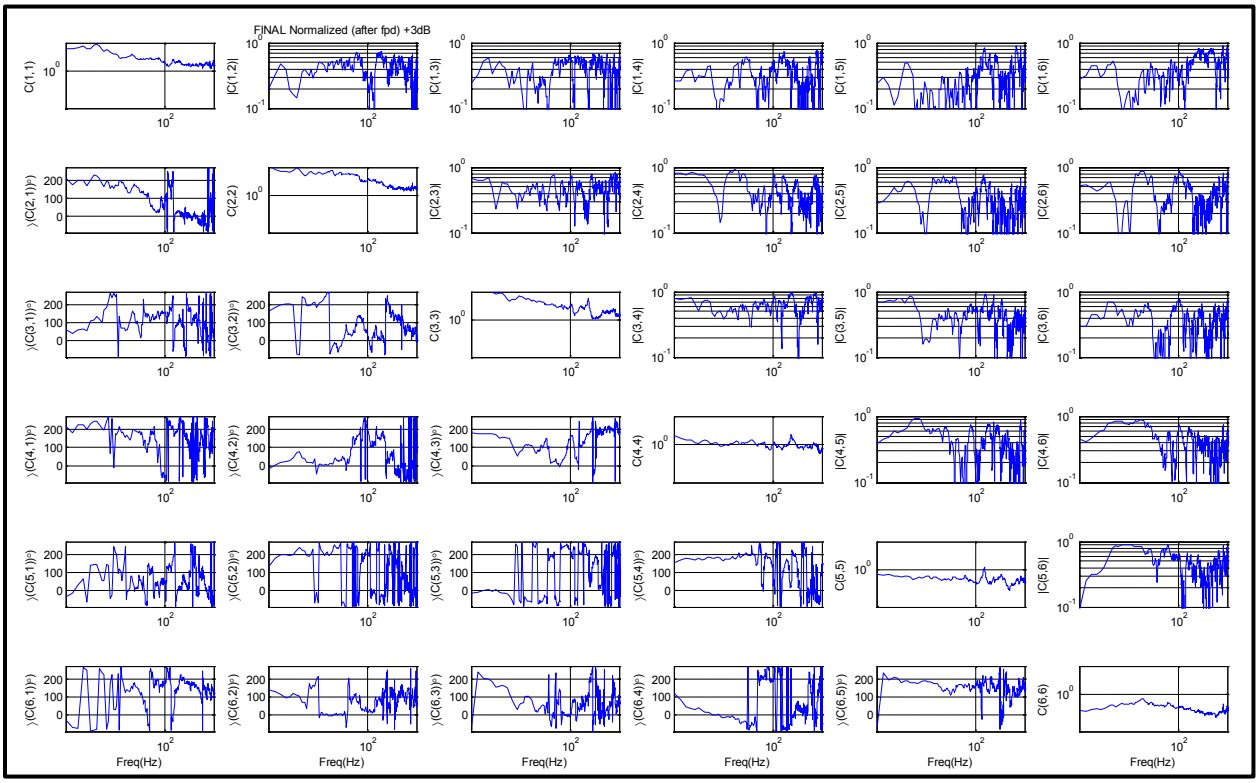

Figure 6. Method I normalized reference SDM.

c. Method II Example. The first four steps of Method II correlate directly to that of Method I. The major deviation in Method II is that all averaging will be computed in the Cholesky domain. In Step 5, Cholesky decompositions are carried out on the individual SDM's associated with each Run in the mission scenario. Since each individual Run was based on a measured physical event, the individual SDMs were positive definite as expected, thereby making the Cholesky decomposition possible. In the event that a given Run had failed the Cholesky decomposition and all measurement locations and relative polarities were verified;

investigate the spectral lines at which the decomposition fails. If the decomposition is failing at only a few spectral lines, it may be possible to salvage the measurement employing the CSD magnitude reduction techniques proposed in Method I. The Cholesky domain data were then organized into a convenient structure for statistical analysis. As in Method I, Matlab was used to compute the weighted averages and the Cholesky domain data were organized as: CHOL_all=[CHOL_Run1, CHOL_Run2…..,CHOL_RunN] . In Step 6, a weighted average in terms of the time per road condition as defined in Table 1 was computed over the lower triangular matrix of the eight Cholesky decompositions of Step 5. The weighted average was then converted back into the SDM domain. As expected, the coherence characteristics of the resulting SDM were comparable with that of Figure 5 and the rms levels of the ASD terms required rescaling per Step 7. Steps 8-10 were carried out directly as stated in the Method I outline.

d. The reference SDM resulting from Method II (Figure 7) yielded similar phase and coherence characteristics to that of the reference SDM resulting from Method I (Figure 6). Note that the Method I example took advantage of averaging only the lower triangular CSD terms, avoiding potential numerical issues, thereby not requiring the SDM to be forced positive definite in a manner that would result in lowering the coherence in a more conservative manner than required.

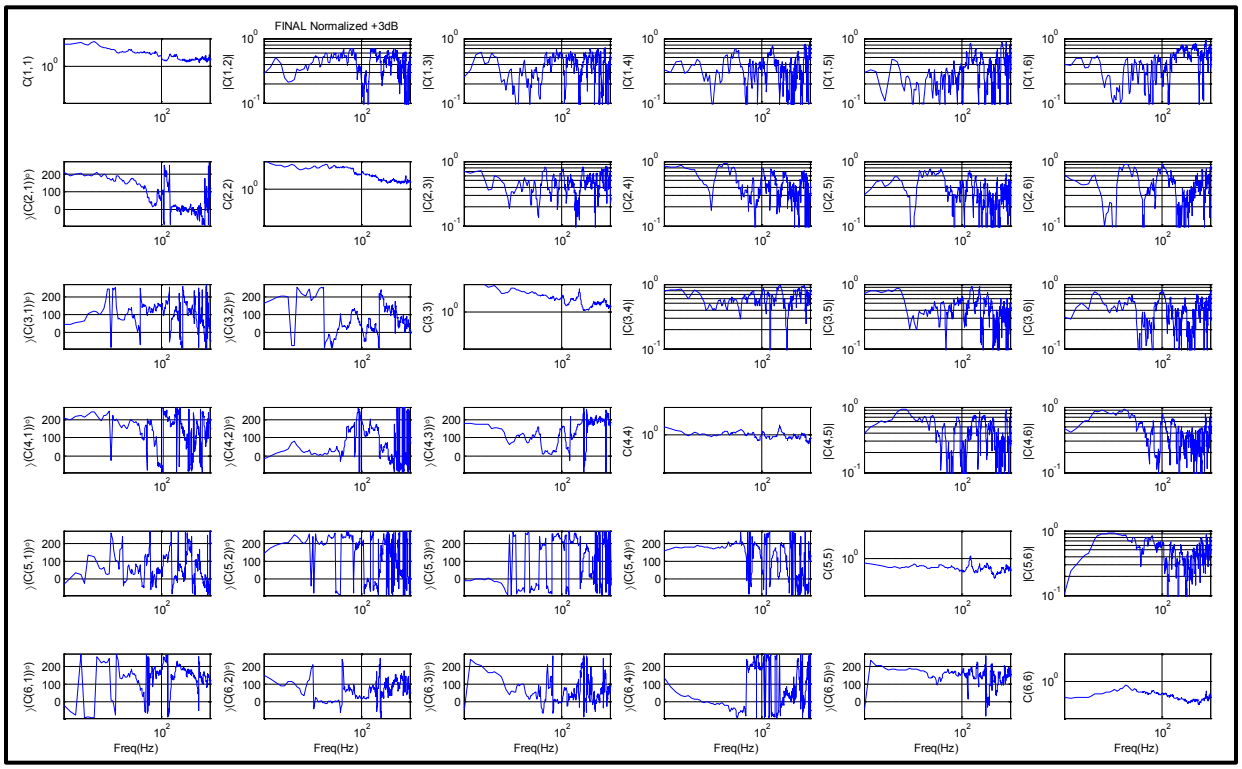

Figure 7. Method II reference SDM.

e. ASD Comparisons. Next, the minor spectral shape deviations between the ASD resulting from the two VSD methods discussed will be illustrated. Figures 8 and 9 show the ASD references for the Z axis (vertical) and rotation about Z axis (Rz) respectively, as produced from both VSD methods. The ASD references are superimposed with the raw (unexaggerated) reference data from which the specifications were created. Observe that the ASD shapes envelope the field data without excessive conservatism.

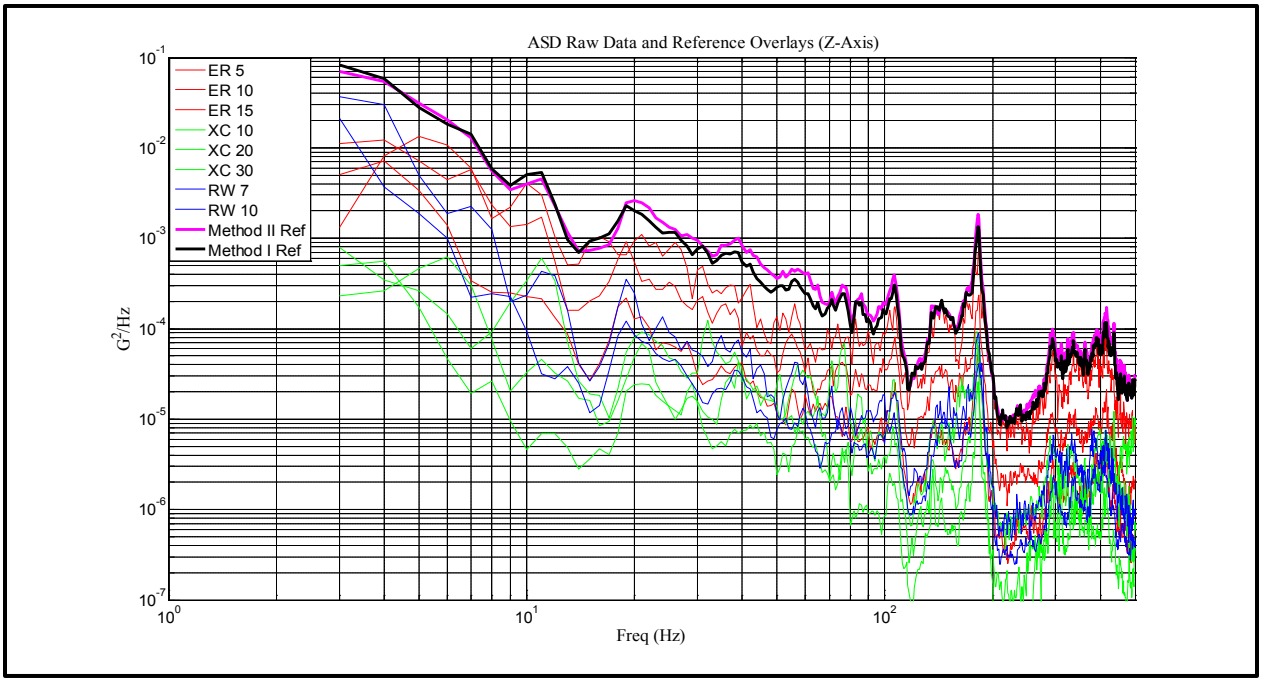

Figure 8. ASD references for the Z axis.

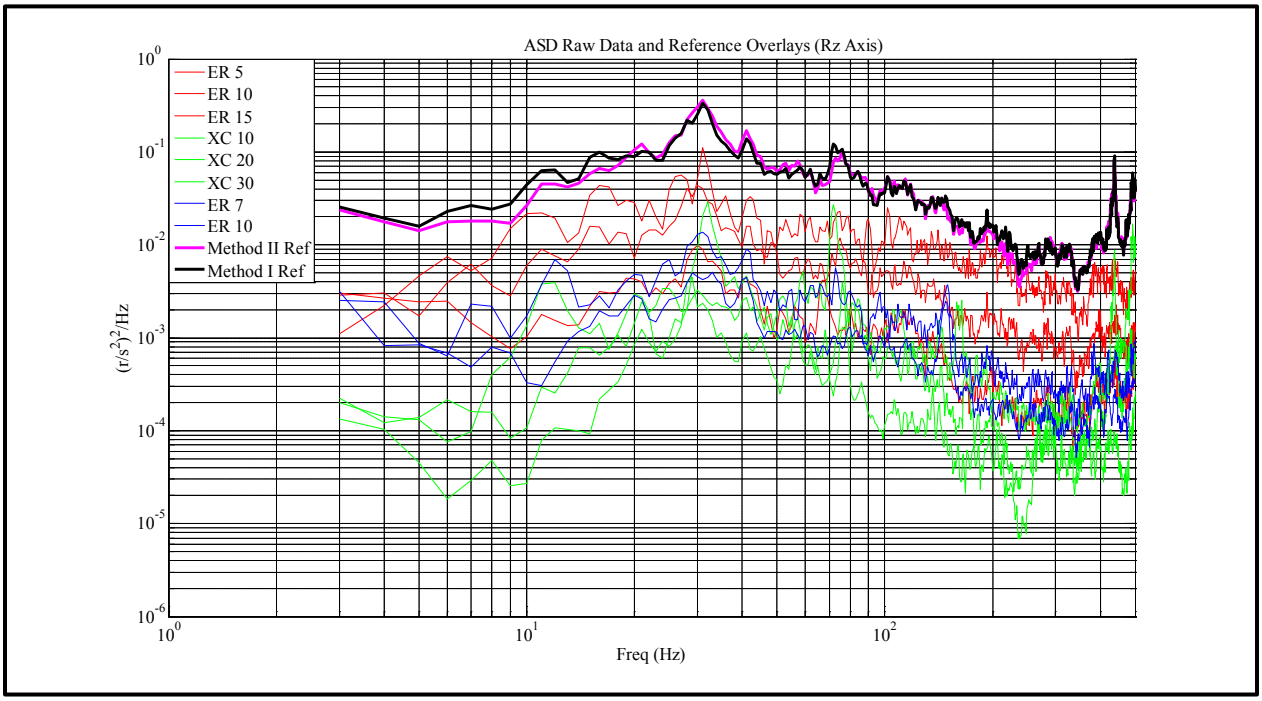

Figure 9. ASD references for rotation about Z axis (Rz).

f. As stated previously, the test duration for the reference SDM yielded by both Methods in this example was established to be 15 minutes. Clearly, as illustrated in Figures 8 and 9 the associated ASD references are highly correlated.

6.3 Concluding Remarks.

a. Two techniques were defined for establishing an input specification for a MDOF system. It was shown that simple enveloping techniques are not appropriate when considering CSD terms due to the sensitivity of such operations associated with maintaining a physically realizable reference. The resulting SDM references yielded through the process outlined are fully populated SDM's. Importing the fully populated SDM into the MDOF control system in an efficient manner is essential due to the volume of information involved.

b. While synthesizing a drive signal with CSD characteristics of the field data is desired, it is recognized that the mechanical impedance of the laboratory configuration is highly unlikely to match that of the field data. Therefore, it will be difficult to maintain CSD characteristics across the spectral bandwidth of interest and thus, the control hierarchy will generally place emphasis on the ASD terms. Also, it is not uncommon in MDOF tests for a specific mechanical degree-of-freedom to consist of a very small percentage of the composite energy across all mechanical degrees-of-freedom. In such cases, the associated error for the low DOF will often be higher than the desired test tolerances and considering global test tolerances may need to be considered.

c. Care was taken in the examples provided to limit the amount of conservatism in the VSD process. One quickly realizes that the amount of conservatism is cumulative across degrees of freedom and if not managed carefully will yield test levels significantly higher than the measured environment. Unlike, the common technique of essentially adding 3 dB to all measurements prior to conducting averaging or enveloping techniques in the 1-DOF arena per reference number 3, all weighted averages in the 6-DOF examples shown were based on raw averaged data. Conservatism to account for variables such as fleet variability and mission scenario omissions were added in the final step. Magnitude amplification associated with time compression techniques was limited to no more than maximum measured levels. Also, on the subject of tolerances, one may find it reasonable to define phase and coherence tolerances over only a portion of the test bandwidth. In the example provided, the coherence dropped off considerably at frequencies above 50 Hz. Since the phase term is essentially a random variable for low coherence, setting tolerances for frequencies greater than 50 Hz would not be recommended for the example shown.

APPENDIX A. GLOSSARY.

Term	Definition
Laboratory Vibration Test Schedule (LVTS)	All information required to perform a vibration test on a vibration exciter. Information typically includes: a broadband spectra (or profile), sine or narrowband information (if used), test run time, control accelerometer locations, control methods and tolerances, and any test specific information required.
Multi-Axis (MA)	Excitation or response measurement that requires more than one unique vector for description.
Multi-Degree-of-Freedom (MDOF)	Motion defined by test item movement along or about more than one axis whose description requires two or more coordinates to completely define the position of the item at any instant.
Multiple Exciter/Multiple-Axis (MEMA)	Application of multiple exciters providing dynamic input to the test item in a way that requires more than a single vector for complete description of excitation and measurement. Note that many multi-axis test platform configurations have been built in recent years. Many are optimally configured such that the number of mechanical DOFs is equivalent to the number of exciters. There are also over-actuated systems consisting of more exciters than mechanical DOFs. In each case, the dynamic properties vary between designs and must be considered in the design of a MET.
Multiple-Exciter/ Single-Axis (MESA)	Application of multiple exciters providing dynamic input to the test item in a single vector direction. For example, extended materiel might require excitation at the forward and aft end in a single vector axis. For the case in which the two exciters are driven to a common specification with respect to both phase and amplitude, the output may be described basically in the one axis of excitation. For the case in which the two exciters are driven to independent magnitude and/or phase specifications, the output may need to be described in terms of a forward axis and aft axis and, perhaps, a rotational axis about the test item's center-of-gravity (CG).
Multiple-Input/Multiple-Output (MIMO)	Refers to input of a multiple drive signals to an exciter system configuration in a MDOF configuration, and multiple measured outputs from the fixture or test item in a MDOF configuration. It is important to note that generally there is no one-to-one correspondence between inputs and outputs, and the number of inputs and number of outputs may be different.

APPENDIX A. GLOSSARY.

Term	Definition
Multiple-Input/Single-Output (MISO)	Refers to input of a multiple drive signals to an exciter system configuration in a MDOF configuration and a single measured output from the fixture or test item in a SDOF configuration. This terminology is most used in measurement data processing where the single output is a composite of measurements from multiple inputs.
Scenario	A tabulation of expected exposure events and the corresponding durations.
Single-Axis (SA)	Excitation or response measurement in a unique single vector direction (linear or rotational). For rotational axis, the vector direction is perpendicular to the plane of rotation of the exciter or test item.
Single-Degree-of-Freedom (SDOF)	Motion defined by materiel movement along or about a single axis whose description requires only one coordinate to completely define the position of the item at any instant.
Single-Exciter/Single-Axis (SESA)	Application of a single exciter providing dynamic input to the test item in a single vector direction.
Single-Input/Multiple-Output (SIMO)	Refers to input of a single drive signal to an exciter system in a SDOF configuration, and multiple measured outputs from the fixture or test item in a MDOF configuration. In general, for specification purposes the dynamic behavior of the test item will not be assumed to contribute to the output DOF, i.e., measured rotation of an extended test item that is being excited in a cantilever mode will still basically be considered as a SET with linear acceleration characterizing the output.
Single-Input/Single-Output (SISO)	Refers to input of a single drive signal to an exciter system in a SDOF configuration and a single measured output from the fixture or test item in a SDOF configuration.

APPENDIX B. ABBREVIATIONS.

AECTP	Allied Environmental Conditions Test Publication
ASD	auto spectral density (also referred to as the power spectral density (PSD)
CG	center of gravity
CSD	cross spectral density
dB	decibel
DFT	discrete Fourier transform
DOF	degree of freedom
DTC	US Army Developmental Test Command
FRF	frequency response function
g/V	gravitational units/volts of drive
Hz	hertz
I/O	input/output
IEEE	Institute of Electrical and Electronics Engineers
IES	Institute of Environmental Sciences
IEST	Institute of Environmental Sciences and Technology
LCEP	Life Cycle Environment Profile
LOA	line of action
LVTS	Laboratory Vibration Test Schedule
MA	multi-axis
MDOF	multiple degree-of-freedom
MEMA	multiple-exciter multiple-axis
MESA	multiple-exciter single-axis
MET	multiple exciter test
MIL-STD	Military Standard
MIMO	multiple-input multiple-output
MISO	multiple-input single-output
NATO	North Atlantic Treaty Organization
NBROR	narrowband random on random
pinv	Moore Penrose pseudo inverse
PSD	power spectral density
rms	root mean square
RTC	US Army Redstone Test Center

APPENDIX B. ABBREVIATIONS.

SA	single-axis
SDM	spectral density matrix
SDOF	single degree-of-freedom
SESA	single-exciter/single-axis
SIMO	single-input multiple-output
SISO	single-input single-output
SOR	sine-on-random
TOP	Test Operations Procedure
TWR	Time Waveform Replication
UUT	unit under test
VSD	Vibration Schedule Development

APPENDIX C. NOMENCLATURE.

Term	Definition
{ }	A vector where each element is a discrete time history or function of frequency, the discrete Fourier transform (DFT) of a time history. In general lower case letters will be used for functions of time and upper case letters will be used for functions of frequency. Sometimes lower case letters are used to designate an element in a matrix.
[ ]	Will denote a matrix. Usually a third dimension will denote time samples or samples as a function of frequency.
[ ]'	The complex conjugate transpose of a matrix.
$[\,]^{\dagger}$	The Moore Penrose pseudo inverse of a matrix.
$[\,]^{T}$	The transpose of a matrix.
^	Over a variable will denote an estimated value.
{a}	The vector of return acceleration signals.
A	The spectral density matrix of the return signals, typically in units of g^2/Hz.
$\{c\}$	A vector of the control signals from a MIMO system. Each element in the vector is a function of time. It can be thought of as a 2 dimensional matrix. First dimension is the input number. The second dimension is the time index.
$\{C\}$	The DFT of $\{c\}$.
C	The spectral density matrix of the control signals. The diagonal elements are the real auto-spectral densities of the control signals. The off diagonal elements are complex functions of frequency giving the cross spectral density between pairs of control signals.
{d}	A vector of drive signals into a MIMO system. Each element in the vector is a function of time. It can be thought of as a 2 dimensional matrix. First dimension is the input number. The second dimension is the time index.

APPENDIX C. NOMENCLATURE.

Term	Definition
[**D**]	The drive signals in the frequency domain. $\{d\}$ is formed from [D] using a method called time domain randomization. Initially **D** = **ZRZ'**.
E[ ]	The expected value.
g	The acceleration of gravity.
[**H**]	A matrix of frequency response functions (FRF's) relating the control system response to the drive signals. Typically the elements will have units of g/V. Each element is a frequency response function. A third dimension typically is the amplitude as a function of a set of frequencies relating to the DFT of the input and response signals.
N_s	The number of drive signals, the number of shakers.
N_c	The number of control signals.
N_a	The number of acceleration return signals.
N_d	The number of output control variables.
R	The reference control spectral density matrix; the desired spectral density matrix.
{s}	The vector of shaker drive voltages.
S	The spectral density matrix of the drives in shaker space.
$\mathbf{S}_{CD}$	The spectral density matrix between the control signal and the drives to the shakers.
$\mathbf{T_a}$	The acceleration to control space transformation matrix.
$\mathbf{T_s}$	The drive in the control space to voltages {s} to the shakers transformation matrix.
$\mathbf{Z= H^{\dagger}}$	The system impedance matrix, typically units of volts/g.

APPENDIX D. MATRIX ALGEBRA REVIEW.

A matrix is an array of numbers arraigned in rows and columns. The size of the matrix is typically stated an [n,m] or n x m, where n is the number of rows and m is the number of columns. In this document 3 dimensional matrices are also used where the third dimension is typically samples in either the time or frequency domain. This Appendix will discuss only two dimensional matrices. It is assumed that if the matrix has 3 dimensions, that the operations can be performed on each 2 dimensional matrix along the third dimension. For example if the matrix is a matrix of frequency response functions, matrix operations will be performed at each frequency line. The definitions provided in this appendix are based on information provided primarily in reference numbers 12 and 13.

a. SDM: A spectral density matrix is a 3 dimensional matrix. At each frequency line (the 3rd index) the matrix is a square complex matrix. Each diagonal element is the autospectrum of the corresponding element. Loosely an element in the SDM is defined as:

$$G_{ji}(k) = 2\lim_{T\to\infty}\frac{1}{T}\mathrm{E}[X_j(k,T)X_i^*(k,T)]$$

where: $G_{ji}(k)$ is the cross spectral density between the j'th and i'th random processes. $X_j(k,T)$ and $X_i(k,T)$ are the discrete Fourier transforms of the time histories, and k is the frequency index. If i = j, the spectrum is called the autospectrum or the power spectrum. In reality, the true spectral density is generally not known and an estimate is employed. Some authors define the elements as:

$$G_{ij}(k) = 2\lim_{T\to\infty}\frac{1}{T}\mathrm{E}[X_x^*(k,T)X_j(k,T)]$$

The SDM matrix is Hermitian positive definite.

b. Hermitian Matrix: A matrix, A, is Hermitian if the diagonal elements are real positive numbers and the corresponding off diagonal elements are complex conjugate pairs:

$$a_{ii} = \text{positive real number}$$
$$a_{ji} = a_{ij}^* = conj(a_{ij})$$

where: a_{ji} is the element form j'th row, i'th column of A.

Note: All valid spectral density matrices (SDM) are Hermitian.

APPENDIX D. MATRIX ALGEBRA REVIEW.

c. Positive Definite Matrix and Positive Semi-Definite Matrix: If a square Hermitian matrix, A, has all positive eigenvalues, the matrix is positive definite. If the matrix has zero eigenvalues the matrix is positive semi-definite. A Cholesky decomposition is possible for all positive definite matrices.

$$\mathbf{A} = \mathbf{LL'}$$

where: **L** is a lower triangular matrix with real positive values on the diagonal. **L'** is the complex conjugate transpose of **L**. If the matrix, **A**, is positive semi-definite, special care must be taken in computing **L**. If a zero element is found on the diagonal of **L**, the entire column must be set to zero. Computing the Cholesky decomposition is actually the easiest way to check for positive definite. If the algorithm fails the matrix, **A** is not positive definite.

d. Transformation of a Positive Definite Matrix:

$$\textbf{Let } \mathbf{B} = \mathbf{HAH'}$$

If the matrix A is positive definite, B is positive definite.

Note: All valid SDMs are positive semi-definite or positive definite. Because some noise is always present in measured data, a measured SDM will always be positive definite.

e. Ordinary Coherence, γ^2: The ordinary coherence between two signals is defined as:

$$\gamma_{12}^2 = \frac{|G_{12}|^2}{G_{11}G_{22}}$$

G_{12} is the cross spectral density between the signals and G_{11} and G_{22} are the two autospectra.

The ordinary coherence is bounded by $0 \le \gamma_{12}^2 \le 1$.

Coherence is a measure of the linear relationship between the signals. If the coherence is unity, a perfect linear relationship exists between the signals. If the coherence is zero, the signals are said to be independent, and there is no linear relationship between the signals.

If one or more of the ordinary coherences in a SDM are in unity at any frequency, the matrix is positive semi-definite at that frequency.

APPENDIX D. MATRIX ALGEBRA REVIEW.

f. Singular Value Decomposition: Singular value decomposition has several applications in MIMO testing. Singular value decomposition is defined as:

$$\mathbf{M=USV'}$$

M is any matrix. **U** and **V'** are orthonormal. This implies that:

$$\mathbf{UU'=I} \text{ and } \mathbf{VV'=I}$$

S is a diagonal matrix of non-negative real numbers. A common convention is to order the diagonal elements of **S** in a non-increasing fashion.

g. Pseudo inverse: The Moore Penrose pseudo inverse is used often in MIMO control. Some of the properties are discussed below. The Moore Penrose pseudo inverse can be derived as follows:

$$\mathbf{M} = \mathbf{USV'}$$
$$\mathbf{U'M} = \mathbf{U'USV'} = \mathbf{SV'}$$
$$\mathbf{S^{-1}U'M} = \mathbf{S^{-1}SV'} = \mathbf{V'}$$
$$\mathbf{VS^{-1}U'M} = \mathbf{VV'} = \mathbf{I}$$

$\mathbf{M}^{\dagger} = \mathbf{VS}^{-1}\mathbf{U'}$ is known as the pseudo inverse of **M**.

The inverse of the reduced **S** is easy since the matrix is diagonal. To compute $\mathbf{S}^{-1}$ the elements greater than a tolerance are inverted and kept, the elements less than a tolerance are replaced by zero.

$$\mathbf{MM}^{\dagger}\mathbf{M} = \mathbf{M} \text{ and } \mathbf{M}^{\dagger}\mathbf{M}\,\mathbf{M}^{\dagger} = \mathbf{M}^{\dagger}$$
$\mathbf{MM}^{\dagger}$ and $\mathbf{M}^{\dagger}\mathbf{M}$ are Hermitian

If the number of columns in **M** exceed the number of rows and the rows are independent $\mathbf{MM}^{\dagger}=\mathbf{I}$. If the number of rows in **M** exceeds the number of columns and the columns are independent $\mathbf{M}^{\dagger}\mathbf{M}=\mathbf{I}$. For a more complete discussion see the help file for pinv in MATLAB.

h. Matrix Rank: The rank of a matrix, M, equals the number of non-zero singular values in M. In numerical linear algebra, the singular values can be used to determine the effective rank of a matrix. Define a measure of singular values as the ratio of the singular values and the largest singular value. Let r be the number values greater than a threshold. Where the measure is less than the threshold, set the singular values to zero. The number of non-zero singular values in the resulting matrix is the effective rank of the matrix. The effective rank of the matrix is r. For a square matrix, if r is less than the number of rows and columns in the matrix, the matrix is said to be ill conditioned.

APPENDIX D. MATRIX ALGEBRA REVIEW.

i. Matrix Approximation: Let $\tilde{M} = usv'$

where: **s** = a diagonal matrix of the singular values greater than a threshold defined as the ratio of the singular values divided by the largest singular value. Let n = the number of kept singular values. **s** has n rows and columns. **u** is the first n columns of **U**. **v'** is the first n rows of **V'**.

$$\tilde{M} \text{ minimizes } \left\| S - usv' \right\|_F$$

Hence, $\tilde{M}$ is a very useful approximation of **M**.

j. Frobenius Norm: The Frobenius Norm of matrix M is defined as:

$$\left\| M \right\|_f = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \left| m_{ij} \right|^2} = \sqrt{trace(A'A)} = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i^2}$$

where: σ_i are the singular values of M.

k. Trace: The trace of a positive definite matrix is defined as the sum of the diagonal elements. An important property of the trace often of use is:

$$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$$

l. Rescaling the Autospectra: When generating a SDM it might sometimes be useful to rescale the autospectra and be assured that the result remains positive definite. This can be accomplished by pre and post multiplying by a diagonal matrix of scaling factors. The triple product will rescale the autospectra while keeping the coherence and phase between pairs of channels unchanged.

$$\mathbf{G}_{\mathbf{new}} = \mathbf{S}\mathbf{G}_{\mathbf{old}}\mathbf{S}'$$

where: $\mathbf{G}_{\mathbf{new}}$ is the new positive definite SDM, $\mathbf{G}_{\mathbf{old}}$ is the original positive definite SDM, and $\mathbf{S}$ is a diagonal matrix of scaling factors. Each autospectra will be scaled by the corresponding element in $\mathbf{S}^2$.

This is a convenient way to generate the normalized SDM (the diagonal elements are the autospectra and the magnitude squared of the off diagonal terms are the ordinary coherence and the phase is the phase of the cross spectra). The normalized form is computed by rescaling the SDM to unity autospectra by pre and post multiplying the SDM by a diagonal matrix whose terms are the inverse square root of the autospectra. The resulting unity autospectra are then replaced by the original autospectra.

APPENDIX D. MATRIX ALGEBRA REVIEW.

The inverse is computed by replacing the diagonal autospectra by ones and then rescaling by pre and post multiplying by a diagonal matrix whose terms are the square root of the original autospectra.

m. Proof 1:

An element in $\boldsymbol{S_Y}$ is given by, where n = number of inputs, and m = number of outputs

$$Y_{ij} = \sum_{r=1}^{n}\sum_{k=1}^{n} X_{rk} H_{ir} H_{jr}^{*} \qquad i = 1:m \quad j = 1:m$$

A diagonal element is given by:

$$Y_{ii} = \sum_{r=1}^{n}\sum_{k=1}^{n} X_{rk} H_{ir} H_{ir}^{*} = \sum_{r=1}^{n}\sum_{k=1}^{n} X_{rk} \mid H_{ir} \mid^{2} \qquad i = 1:m$$

If $\boldsymbol{S_X}$ is diagonal, $X_{rk} = 0$, if $r \neq k$, (a-3) reduces to:

$$Y_{ii} = \sum_{r=1}^{n} X_{rr} \mid H_{ir} \mid^{2} \qquad i = 1:m$$

This can be written as a set of linear equations:

$$\overline{\mathbf{Y}} = \overline{\mathbf{HX}}$$

Which can be solved for $\overline{\mathbf{X}}$ as: $\overline{\mathbf{X}} = \overline{\mathbf{H}}^{-1}\overline{\mathbf{Y}}$

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