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ENGINEERING DESIGN HANDBOOK.

DEVELOPMENT GUIDE FOR RELIABILITY.

PART SIX.

MATHEMATICAL APPENDIX AND GLOSSARY.

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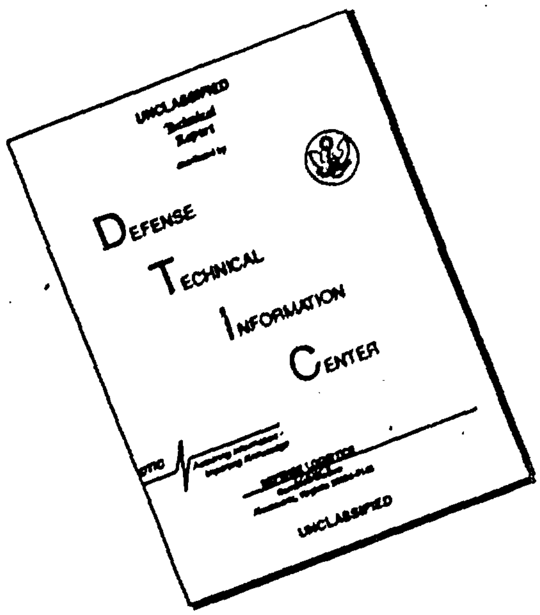

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DEPARTMENT OF THE ARMY
HEADQUARTERS UNITED STATES ARMY MATERIEL COMMAND
5001 Eisenhower Ave., Alexandria, VA 22333

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NO. 706-200

8 January 1976

ENGINEERING DESIGN HANDBOOK
DEVELOPMENT GUIDE FOR RELIABILITY, PART SIX
MATHEMATICAL APPENDIX AND GLOSSARY

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PREFACE

This handbook, *Mathematical Appendix and Glossary*, is the last in a series of five on reliability. The series is directed largely toward the working engineers who have the responsibility for creating and producing equipment and systems which can be relied upon by the users in the field.

The five handbooks are:

1. *Design for Reliability*, AMCP 706-196
2. *Reliability Prediction*, AMCP 706-197
3. *Reliability Measurement*, AMCP 706-198
4. *Contracting for Reliability*, AMCP 706-199
5. *Mathematical Appendix and Glossary*, AMCP 706-200.

This handbook is directed toward reliability engineers and managers who need to be familiar with or need to have access to statistical tables, curves, and techniques, or to special terms used in the reliability discipline. References are given to the literature for further information.

Much of the handbook content was obtained from many individuals, reports, journals, books, and other literature. It is impractical here to acknowledge the assistance of everyone who made a contribution.

The original volume was prepared by Tracor Jitco, Inc. The revision was prepared by Dr. Ralph A. Evans of Evans Associates, Durham, N.C., for the Engineering Handbook Office of the Research Triangle Institute, prime contractor to the US Army Materiel Command. Technical guidance and coordination on the original draft were provided by a committee under the direction of Mr. O. P. Bruno, Army Materiel System Analysis Agency, US Army Materiel Command.

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CHAPTER 1

GLOSSARY

LIST OF SYMBOLS

AOQ	=	average outgoing quality
AOQL	=	average outgoing quality limit
AQL	=	acceptable quality level
ASN	=	average sample number
ATE	=	automatic test equipment
Cdf	=	cumulative distribution factor
$E\{x\}$	=	expected value of x
FMECA	=	failure mode, effects, and criticality analysis
$g(t)$	=	state of system under usual conditions
$G(t)$	=	state of system under unusual conditions
LTPD	=	lot tolerance percent defective
mse	=	mean square error
MTBF	=	mean time-between-failures
MTF	=	mean time-to-failure
MTFF	=	mean time-to-first-failure
MTTR	=	mean time-to-repair
MTX	=	arithmetic or *s*-expected value for xxxxtime
N	=	population size
OC	=	operating characteristic
pdf	=	probability density function
pmf	=	probability mass function
QC	=	quality control
QPL	=	qualified products list
R	=	reliability
rms	=	square root of arithmetic mean of the squares
RQL	=	rejectable quality level
Sf	=	survivor function
s-	=	denotes statistical definition
t	=	time
T	=	time interval
x	=	value of random variable X
$\bar{x}$	=	population mean
X	=	name of random variable
α	=	producers risk
β	=	consumers risk
θ	=	$1/\lambda$

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λ = failure rate

μ = mean value

σ = standard deviation

$\tau(t)$ = function of time

Some words (phrases) have more than one definition. No relative importance is implied by the order in which they appear. When there is more than one definition of a word (phrase), they are numbered with an initial superscript.

A definition indicated by a * has more complete explanations of the term and fewer ambiguities than other definitions. The definitions in this Glossary try to impart knowledge. The accompanying notes help to provide understanding. Knowledge without understanding can be costly. Do not apply any of these concepts blindly.

See Refs. 1-3 for the definitions of many concepts not listed here.

When the precise statistical definition of a word is intended, the word has "*s*-" as a prefix; e.g., *s*-normal, *s*-independence, *s*-reliability.

A

accelerated life test. A life test under test conditions that are more severe than usual operating conditions. It is helpful, but not necessary, that a relationship between test severity and the probability distribution of life be ascertainable.

Note 1. The phrase "more severe" is actually defined by the fact that the *Cdf* of life is everywhere greater than the *Cdf* of life under usual conditions.

Note 2. Where there is more than one **failure mode**, the concept of acceleration is not simple. Conceivably, a set of test conditions which accelerates some failure modes could be more benign for other failure modes.

Note 3. **Accelerated life tests** can be qualitatively useful in finding potential **failure modes** even when they are not quantitatively useful.

See also: **acceleration, true**

acceleration factor. Notation:

$\tau(t)$ ≡ the time transformation from more-severe test conditions to the usual test conditions.

The **acceleration factor** is $\tau(t)/t$.

The differential **acceleration factor** is $d\tau(t)/dt$.

Note 1. **acceleration factor** is defined only for **true acceleration.** If the acceleration is not true, the concept is meaningless (*see:* [2] **acceleration, true** (Note 3).

Note 2. It helps, but is not necessary, if the acceleration factor is independent of time. In practical situations, it usually is assumed to be independent of time. A good reason for so doing is that there is rarely enough statistical evidence to dispute this simple, convenient hypothesis.

See also: **acceleration, true.**

[1] **acceleration, true.** Acceleration is true if and only if the system, under the more-severe test conditions, passes reasonably through equivalent states and in the same order it did at usual conditions. (Adapted from Ref. 4.)

Note 1. Acceleration need not be true to be useful.

Note 2. The word "reasonably" is used because the needs and desires of the people involved change from time to time. Things

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need only be close enough for the purposes at hand.

Note 3. "System state" describes only those characteristics of the system which are important for the purposes at hand (just as is true in thermodynamics).

Note 4. Two states of a system are "equivalent" if and only if one can be reversibly transformed into the other by changing the test conditions.

Note 5. Mathematical definition.

$g(t) \equiv$ state of system under usual conditions.

$G(t) \equiv$ equivalent state of system under more-severe test conditions. It is not the state at the more-severe test conditions, but is the state after being reversibly transformed to the usual conditions.

$\tau(t) \equiv$ a function of time.

There is true acceleration if and only if:

(a) $G(t) \equiv g(\tau[t])$

(b) $\tau(t)$ is strictly monotonically increasing

(c) $G(0) = g(0)$

(d) $\tau(0) = 0$ (this is a logical consequence of (a) and (c)).

The acceleration factor is defined as $\tau(t)/t$. Incremental acceleration factor is defined as $d\tau(t)/dt$.

[2] **acceleration, true.** Acceleration is true if and only if the probability distribution of life for each important **failure mode**, under the more-severe test conditions, can be changed (by a time transformation) to the probability distribution of life for that **failure mode**, under the usual test conditions, and:

(a) The time transformation is the same for each such **failure mode**.

(b) The time transformation is strictly monotonically increasing.

Note 1. Acceleration need not be true to be useful.

Note 2. Let the time transformation be $\tau(t)$, then **acceleration factors** are defined as in[1] **acceleration, true** (Note 5).

Note 3. **True acceleration** could be defined singly for each important **failure mode.**

See also: **acceleration factor.**

accept/reject test. A test, the result of which will be the action to accept or to reject something, e.g., an hypothesis or a batch of incoming material.

The test will have a set of constants which are selected before the test, and it will have an **operating characteristic**. For example, a common fixed-sample-size attribute test has the constants: sample-size and acceptance-number; a set of procedures to select a random-sample, to test every item for good/bad (and evaluation criteria therefor), and to stop the test where all items are tested; and an **operating characteristic** that shows the probability of acceptance (or rejection) as a function of the true fraction-bad of the population from which the sample was a random one.

Note 1. The data also can be used for estimating parameters of the probability distribution of the population. For many kinds of tests, this may be intractable because the test procedures were chosen to minimize resources consumed in the test rather than to make parameter estimation easy.

Note 2. The accept/reject criterion must have only 1-dimension. That is, even if several characteristics are measured (for example, major and minor defects) the numbers so obtained must be combined in some way to get a single number that is then compared against the accept/reject criterion. The accept/reject criterion can be complicated, e.g., accept if the average sample length is between 4.0 and 5.0 in., reject otherwise.

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Note 3. This kind of test is used largely for theoretical hypothesis testing and for quality-control acceptance-sampling.

See also: **operating characteristic, random sample.**

*[1] **acceptable quality level** (AQL). A point on the quality coordinate of the **operating characteristic** of an **attribute acceptance-sampling plan** which is in the region of good quality and reasonably low rejection probability.

Note 1. The rejection probability at the AQL is often called the **producer risk** α.

Note 2. The conventional definitions (*see:* defs. 2 and 3) tend to endow this point with very special properties which it does not really have. Conventionally this point (AQL, α) is one of two that define the **acceptance sampling plan** and its **operating characteristic**. But any 2 points on that **operating characteristic** will generate exactly the same **acceptance sampling plan**. That is why this modified, more usable definition is given.

Note 3. An example of an AQL is 1.5% defective at a rejection probability (**producer risk**) of 10%.

Note 4. The term itself can be very misleading, especially to non-specialists in Quality Control. Its use ought to be avoided in material written for such people.

See also: **operating characteristic.**

[2] **acceptable quality level** (AQL). The maximum percent defective (or the maximum number of defects per hundred units) that, for purposes of acceptance sampling, can be considered satisfactory as a process average.

Note. When a consumer designates some specific value of AQL for a certain characteristic or group of characteristics, he indicates to the supplier that his (the consumer's) **acceptance sampling plan** will accept the great majority of the lots that the supplier submits, provided that the process average level of percent defective in these lots is no greater than the designated value of AQL. Thus the AQL is a designated value of percent defective (or of defects per hundred units) that the consumer indicates will be accepted a great majority of the time by the acceptance sampling procedure to be used. The AQL alone does not describe the protection to the consumer for individual lots but more directly relates to what might be expected from a series of lots, provided that the steps called for in the reference AQL system of procedures are taken. It is necessary to refer to the OC curve of the sampling plan that the consumer will use, or to the AOQL of the plan, to determine what protection the consumer will have. (Ref. 3)

[3] **acceptable quality level** (AQL). The maximum percent defective (or the maximum number of defects per hundred units) that, for the purposes of sampling inspection, can be considered satisfactory as a process average. (Refs. 1 and 7)

*[1] **acceptance number.** The largest number of **defects** that can occur in an **acceptance sampling plan** and still have the lot accepted.

Note 1. In a 1-sample plan, this is a straightforward concept. In an m-sample plan ($m > 1$) the concept usually is applied to each of the samples; so there are m acceptance numbers. In a sequential test, the acceptance number is the boundary of the plan which separates "continue testing" from "accept"; it is a function of the number tested, total test time, or whatever variable represents the amount of testing done so far.

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Note 2. The concept is limited to those plans which have a discrete dependent variable that can be interpreted as **defects**.

See also: **defect**.

[2]**acceptance number**. The largest number of defectives (or **defects**) in the sample or samples under consideration that will permit the acceptance of the inspection lot. (Ref. 3.)

[3]**acceptance number**. The maximum number of **defects** or defective units in the sample that will permit acceptance of the inspection lot or batch. (Ref. 1.)

*[1] **acceptance sampling plan**. An **accept/reject test** whose purpose is to accept or reject a lot of items or material.

Note 1. Rejection may involve 100% inspection or some other scheme rather than outright rejection.

Note 2. These plans often come in sets, so that the user can pick the best one of the set for his purposes.

See also: **accept/reject test**.

Note 3. Each acceptance sampling plan has an accept/reject (decision) boundary in the "number of failures (defects)" vs "amount of sampling" plans. If the "reject line" has m values it is an m-sample plan. "$m = 1$" is most common and is referred to as a single-sample plan. "$m = 2$" is referred to as a double-sample plan. "$m > 2$" is referred to as a multiple-sample plan. "$m >> 2$" often is referred to as a truncated sequential-sample plan.

Note 4. The data can be used to estimate a parameter of the probability distribution, but often the sampling characteristics of such an estimator are not easy to calculate.

[2]**acceptance sampling plan**. A specific plan that states the sample size or sizes to be used and the associated acceptance and rejection criteria. (Ref. 3.)

Note: A specific **acceptance sampling plan** may be developed for any acceptance situation, but inspection systems usually include sets of **acceptance sampling plans** in which lot sizes, sample sizes, and acceptance criteria are related.

[3]**acceptance sampling plan**. A sampling plan indicates the number of units of product from each lot or batch which are to be inspected (sample size or series of sample sizes) and the criteria for determining the acceptability of the lot or batch (acceptance and rejection numbers). (Definition of sampling-plan from Ref. 7.)

*[1] **acceptance test**. Test to determine conformance to specifications/requirements and which is used to determine if the item can be accepted at that point in the life-cycle.

Note 1. If the item is accepted, the life-cycle continues. If the item is not accepted, continuing with development of the item is done according to contract and/or agreement of all parties concerned.

Note 2. *See also:* Acceptance in Ref. 1.

[2]**acceptance test**. (1) A test to demonstrate the degree of compliance of a device with purchaser's requirements. (2) A *conformance test* (in contrast, is)... without implication of contractual relations ... . (Ref. 5.)

active element. A part that converts or controls energy; e.g., transistor, diode, electron tube, relay, valve, motor, hydraulic pump. (Ref. 6.)

active element group. An active element and

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its associated supporting (passive) parts; e.g., an amplifier circuit, a relay circuit, a pump and its plumbing and fittings. (Ref. 6.)

ambient. Used to denote surrounding, encompassing, or local conditions. Usually applied to environments (e.g., ambient temperature, ambient pressure).

arithmetic mean. The arithmetic mean of n numbers is the sum of the n numbers, divided by n.

Note. This is the conventional average. The term is used to distinguish it from other kinds of mean; e.g., geometric, harmonic.

assembly. A number of parts or subassemblies joined together to perform a specific function. (Ref. 6.)

assurance. A qualitative term relating to degree of belief. It often is applied to the achievement of program objectives.

*[1]**attribute**. A characteristic or property of an item such that the item is presumed either to have it or not to have it; there is no middle ground.

Note. The term is used most often in testing where the attribute is equivalent to good/bad.

[2]**attribute**. A characteristic or property which is appraised in terms of whether it does or does not exist (e.g., go or not-go) with respect to a given requirement. (Adapted from Ref. 3.)

[3]**attribute**. A term used to designate a method of measurement whereby units are examined by noting the presence (or absence) of some characteristic or **attribute** in each of the units in the group under consideration and by counting how many units do (or do not) possess it. Inspection by **attributes** can be of two kinds—either the unit of product is classified simply as defective or nondefective or the number of **defects** in the unit of product is counted, with respect to a given requirement or set of requirements. (Adapted from Ref. 3.)

attribute testing. Testing to evaluate whether or not an item possesses a specified attribute. *See:* **go/no-go**.

automatic test equipment (ATE). Test equipment that contains provisions for automatically performing a series of preprogrammed tests.

Note. It usually is presumed that the ATE evaluates the test results in some way.

[1]**availability**. The fraction of time that the system is actually capable of performing its **mission**. (Ref. 5.)

[2]**availability**. A measure of the degree to which an item is in the operable and committable state at the start of the **mission**, when the **mission** is called for at an unknown (random) point in time. (Ref. 2.)

[3]**availability (operational readiness)**. The probability that at any point in time the system is either operating satisfactorily or ready to be placed in operation on demand when used under stated conditions.

[4]***s*-availability**. The fraction of time, in the long run, that an item is up.

Note 1. The item is presumed to have only 2 states (up and down) and to cycle between them.

Note 2. The definition of being up can be important in a redundant system.

availability, intrinsic. The **availability**, except that the times considered are **operating**

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time and **active repair time**. (Adapted from Ref. 6.)

Added Note:
Note. This definition does not have widespread use and the term can be misleading. It would be wise to define it wherever it is used.

average. A general term. It often means arithmetic mean, but can refer to *s*-expected value, median, mode, or some other measure of the general location of the data values.

*[1] **average outgoing quality** (AOQ). The expected value (for a given **acceptance sampling plan**) of the outgoing quality of a lot, for a fixed incoming quality, when all rejected lots have been replaced by equal lots of perfect quality and all accepted lots are unchanged.

Note 1. Quality is measured by fraction defective. The terms AOQ and AOQL are not applicable otherwise.

Note 2. The inspection/sorting/replacement process usually is assumed to be perfect.

Note 3. It often is assumed that all bad parts found during inspection are replaced by good parts. Slight discrepancies in calculated AOQ's can occur if this fact is ignored when it is true.

Note 4. As implied in the definition, the AOQ is a function of incoming quality.

[2] **average outgoing quality** (AOQ). The *s*-expected average quality of outgoing product for a given value of incoming product quality. The AOQ is computed over all accepted lots plus all rejected lots after the latter have been inspected 100% and the defective units replaced by good units. (Ref. 3.)

Note. In practical cases, different numerical values of AOQ may be obtained, depending on whether or not the defectives found in samples or in 100% inspection of rejected lots are replaced by good units.

[3] **average outgoing quality** (AOQ). The average quality of outgoing product including all accepted lots, plus all rejected lots after the rejected lots have been effectively 100 percent inspected and all defectives replaced by nondefectives. (Refs. 1 and 7.)

[1] **average outgoing quality limit** (AOQL). The maximum AOQ over all possible values of incoming product quality, for a given **acceptance sampling plan**. (Ref. 3.)

[2] **average outgoing quality limit** (AOQL). The maximum AOQ for all possible incoming qualities for a given sampling plan. (Adapted from Ref. 1.)

average sample number (ASN). The average number of sample units inspected per lot in reaching decisions to accept or reject. (Ref. 3.)

Added Notes:
Note 1. The ASN usually is applied only where the sample number (size) is a random variable.

Note 2. It is usually a function of incoming quality.

B

bad-as-old. A term which describes repair. The repaired item is indistinguishable from a nonfailed item with the same operating history. Its internal clock stays the same as it was just before failure.

Note. If the **failure rate** is constant, **good-as-new** and **bad-as-old** are the same.

basic failure rate. The **basic failure rate** of an

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item derived from the catastrophic failure rate of its parts, before the application of use and tolerance factors. The failure rates contained in MIL-HDBK-217 are "base" failure rates. (Adapted from Ref. 6.)

bathtub curve. A plot of failure rate of an item (whether repairable or not) vs time. The failure rate initially decreases, then stays reasonably constant, then begins to rise rather rapidly. It has the shape of a bathtub.

Note. Not all items have this behavior.

bias. The difference between the ***s*-expected value** of an estimator and the value of the true parameter.

breadboard model. A preliminary assembly of parts to test the feasibility of an item or principle without regard to eventual design or form.

Note It usually refers to a small collection of electronic parts.

*[1] **burn-in**. The initial operation of an item for the purpose of rejecting or repairing it if it performs unsatisfactorily during the **burn-in** period.

Note 1. The **burn-in** conditions need not be the same as operating conditions.

Note 2. The purpose is to get rid of those items that are more likely to fail in use.

Note 3. The method of **burn-in** and description of desired results need careful attention. **Burn-in** can do more harm than good.

[2] **burn-in**. The operation of an item to stabilize its characteristics. (Ref. 2.)

C

[1] **capability**. A measure of the ability of an item to achieve **mission** objectives given the conditions during the **mission**. (Ref. 2.)

[2] **capability**. A measure of the ability of an item to achieve **mission** objectives, given that the item is working properly during the **mission**.

censored. A set of data from a fixed sample is **censored** if the data from some of the items are missing.

Note 1. In a **censored** life test, it is known only (for censored items) that they survived up to a certain time

Note 2. The reason for the censoring in a life test must have nothing to do with the apparent remaining life of the item.

Note 3. Statisticians sometimes give special names to censoring, depending on which order statistics are censored.

checkout. Tests or observations on an item to determine its condition or status. (Adapted from Ref. 2.)

Added notes:

Note 1. **Checkout** is often assumed to be perfect, i.e., to judge properly the condition of each part and to do no damage to anything. **Checkouts** are rarely perfect.

Note 2. It sometimes is implied that any nonsatisfactory condition is remedied (perfectly or otherwise).

coefficient of variation. The **standard deviation** divided by the **mean**.

Note 1. The term is rarely useful except for positive random variables. It is not defined if the mean is zero, or if the data have been coded by anything other than a scale factor.

Note 2. It is a relative measure of the dispersion of a random variable.

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complexity level. A measure of the number of **active elements** required to perform a specific system function. (Ref. 6.)

***s*-confidence**. A specialized statistical term. It refers to the truth of an assertion about the value of a parameter of a probability distribution.

Note 1. ***s*-confidence** ought always to be distinguished from engineering confidence; they are not at all the same thing. One can have either without the other.

Note 2. Incorrect definitions of this and related terms often are encountered in the engineering literature.

Note 3. For more details, consult a competent statistician or competent statistics book.

***s*-confidence interval**. The interval within which it is asserted that the parameter of a probability distribution lies.

Note. The interval is a measure of the statistical uncertainty in the parameter estimate, given that the model is true. There might be more important sources of uncertainty involved with the model not being true.

See also: ***s*-confidence**, ***s*-confidence limits**.

***s*-confidence level**. The fraction of times an ***s*-confidence** statement is true.

Note 1. The larger the ***s*-confidence** level, the wider the ***s*-confidence** interval, for a given method of generating that interval.

Note 2. Sometimes the asserted level is a lower bound, all that is known is that the actual level is above the stated level. This is especially common where the random variable is discrete.

Note 3. This refers to the totality of times the procedure of calculating an ***s*-confidence** statement from a new set of data is effected.

See also: ***s*-confidence**, ***s*-confidence interval**.

***s*-confidence limits**. The extremes of an ***s*-confidence** interval.

Note. When only 1 limit is given (along with the modifier "upper" or "lower") the interval includes the rest of the domain of the random variable on the appropriate side of the limit.

***s*-consistency**. A statistical term relating to the behavior of an estimator as the sample size becomes very large. An estimator is ***s*-consistent** if it stochastically converges to the ***s*-population** value as the sample size becomes "infinite". It is one of the important characteristics of an estimator as far as reliability engineers are concerned.

continuous sampling plan. In **acceptance sampling**, a plan, intended for application to a continuous flow of individual units of product, that (1) involves acceptance and rejection on a unit-by-unit basis and (2) uses alternate periods of 100% inspection and sampling, the relative amount of 100% inspection depending on the quality of submitted product. **Continuous sampling** plans usually are characterized by requiring that each period of 100% inspection be continued until a specified number of consecutively inspected units are found clear of defects.

Note. For single-level **continuous sampling plans**, a single sampling rate (e.g., inspect 1 unit in say 5 or 1 unit in 10) is used during sampling. For multilevel **continuous sampling plans**, two or more sampling rates may be used, the rate at any time depending on the quality of submitted product. (Adapted from Ref. 3.)

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controlled part. An item which requires the application of specialized manufacturing, management, and procurement techniques.

controlled process. A process which requires the application of specialized manufacturing, management, and procurement techniques.

***s*-correlation**. A form of statistical dependence between 2 variables. Unless otherwise stated, linear ***s*-correlation** is implied.

Note. In writing for engineers, it is better to write the full phrase "**linear *s*-correlation**" to avoid ambiguity.

See also: ***s*-correlation coefficient**.

[1]***s*-correlation coefficient**. A number between − 1 and + 1 which provides a normalized measure of **linear *s*-correlation**.

Note 1. See *Part Three* for mathematical expressions (for both discrete and continuous random variables).

Note 2. Values of + 1 and − 1 represent a deterministic linear relationship. Value of 0 implies no linear relationship.

[2]***s*-correlation coefficient**. A number between − 1 and + 1 that indicates the degree of linear relationship between two sets of numbers. Correlations of − 1 and + 1 represent perfect linear agreement between two variables; $r = 0$ implies no linear relationship at all. (Adapted from Ref. 3.)

cost-effectiveness. A measure of the value received (effectiveness) for the resources expended (cost).

criticality. A measure of the indispensability of an item or of the function performed by an item.

Note. **Criticality** is often only coarsely quantified.

criticality ranking. A list of items in the order of their decreasing **criticality**.

cumulative distribution function *Cdf*. The probability that the random variable whose name is X takes on any value less than or equal to a value x, e.g.,

$$F(x) = Cdf\ \{X\} \equiv Pr\ \{X \leqslant x\}$$

Note 1. The *Cdf* need not be continuous or have a derivative. Its value is 0 below the lowest algebraic value of the random variable and is 1 above the largest algebraic value of the random variable. The *Cdf* is a nondecreasing function of its argument.

Note 2. It is possible to have a joint *Cdf* of several random variables.

Note 3. The concept applies equally well to discrete and continuous random variables.

See also: ***pdf*, *pmf*, *Sf***

D

[1]**debugging**. A process of "shakedown operation" of a finished equipment performed prior to placing it in use. During this period, defective parts and workmanship errors are cleaned up under test conditions that closely simulate field operation.

Note. The debugging process is not intended to detect gross weaknesses in system design. These should have been eliminated in the preproduction stages. (Adapted from Ref. 6.)

[2]**debugging**. A process to detect and remedy inadequacies, preferably prior to operational use. (Ref. 2.)

*[1]**defect**. A deviation of an item from some ideal state. The ideal state usually is given in a formal specification.

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Note 1. The **defect** need not be harmful to the item in any way, even when it is readily detectable.

Note 2. This unmodified word is often misunderstood, because an ordinary meaning of the word implies "harmful". Thus it is always wise to be explicit about the kind of **defect** to which reference is being made.

Note 3. Improved nondestructive evaluation techniques often can detect deviations that are completely unimportant, even from a cosmetic viewpoint. Specifications ought to avoid the phrase "detectable defect".

[2]**defect.** An instance of failure to meet a requirement imposed on a unit of product with respect to a single quality characteristic.

Note. The term "defect",as used in quality control, signifies a deviation from some standard–a condition "in defect of" strict conformance to a requirement. The term thus covers a wide range of possible severity; on the one hand, it may be merely a flaw or a detectable deviation from some minimum or maximum limiting value or, on the other, a fault sufficiently severe to cause an untimely product failure. (Ref. 3.)

[3]**defect.** Any nonconformance of a characteristic with specified requirements. (Ref. 1.)

[1]**defect, critical.** A. A **defect** that could result in hazardous or unsafe conditions for individuals using, maintaining, or depending upon the item.

B. For a major system–such as aircraft, radar, or tank–a **defect** that could prevent performance of its tactical function. (Adapted from Ref. 6.)

[2]**defect, critical.** A **defect** that judgment and experience indicate is likely to result in hazardous or unsafe conditions for individuals using, maintaining, or depending upon the product; or a defect that judgment and experience indicate is likely to prevent performance of the tactical function of major end item such as an aircraft, communication system, land vehicle, missile, ship, space vehicle, surveillance system, or major part thereof. (Ref. 1.)

[1]**defective.** A unit of product which contains one or more **defects.** (Ref. 1.)

[2]**defective.** A **defective** unit; a unit of product that contains one or more **defects** with respect to the quality characteristics under consideration. (Adapted from Ref. 3.)

See also: [2]**defect.**

dependability. A measure of the item operating condition at one or more points during the mission, including the effects of reliability, maintainability, and survivability, given the item condition(s) at the start of the **mission.** It may be stated as the probability that an item will (1) enter or occupy any one of its required operational modes during a specified **mission,** (2) perform the functions associated with those operational modes. (Adapted from Ref. 2.)

*[1]**derating.** The technique of using an item at severity levels below rated values to achieve higher reliability.

Note 1. This is the opposite of **accelerated testing.**

Note 2. It is not always obvious how to derate an item. Considerable knowledge about the structure and behavior of the item often is required.

See also: **accelerated testing.**

[2]**derating.** (1) Using an item in such a way

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that applied stresses are below rated values, or (2) the lowering of the rating of an item in one stress field to allow an increase in rating in another stress field. (Ref. 2.)

design adequacy. The probability that the system will satisfy effectiveness requirements, given that the system design satisfies the design specification. (Ref. 6.)

discrimination ratio. A measure of the "distance" between the two points on the **operating characteristic** which are used to define the **acceptance sampling plan**.

Note 1. It is not an absolute measure of the discriminating ability of an **acceptance sampling plan**.

Note 2. It often is used in place of one of the measures of quality to define the **acceptance sampling plan**.

Note 3. It ought always to be defined when used; although since it is ambiguous and not necessary, its use is wisely avoided.

Note 4. A given **acceptance sampling plan** can have many discrimination ratios depending on which 2 points are used to define it.

distribution. General short name for **probability distribution**.

Note. It is general in that it does not imply a particular descriptive format such as *pdf* or *Cdf*.

[1]**downtime**. The total time during which the system is not in condition to perform its intended function.

Note. **Downtime** is subdivided conveniently into active repair time, logistic downtime, and administrative downtime. (Adapted from Ref. 6.)

[2]**downtime**. That element of time during which the item is not in condition to perform its intended function.

downtime, administrative. That portion of **downtime** not included under **active repair time** and logistic **downtime**. (Adapted from Ref. 6.)

downtime, logistic. That portion of **downtime** during which repair is delayed solely because of waiting for a replacement part or other subdivision of the system. (Adapted from Ref. 6.)

duty cycle. A specified operating time of an item, followed by a specified time of nonoperation.

Note. This often is expressed as the fraction of operating time for the cycle, e.g., the duty cycle is 15%.

E

[1]**early failure period**. That period of life, after final assembly, in which failures occur at an initially high rate because of the presence of **defective** parts and workmanship. (Ref. 6.)

[2]**early failure period**. The early period, beginning at some stated time and during which the failure rate of some items is decreasing rapidly.

Note. This definition applies to the first part of the **bathtub curve** for failure rate. (Adapted from Ref. 5.)

effectiveness. The probability that the product will accomplish an assigned **mission** successfully whenever required. (Ref. 6.)

s-efficiency. A statistical term relating to the dispersion in values of an estimator. It is between 0 and 1; and the closer to 1, the better. It is one of the important characteristics of an estimator as far as reliability engineers are concerned.

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element. One of the constituent parts of anything. An **element** may be a **part**, a **subassembly**, an **assembly**, a unit, a set, etc. (Adapted from Ref. 6.)

environment. The aggregate of all the external conditions and influences affecting the life and development of the product. (Ref. 6.)

equipment. A product consisting of one or more units and capable of performing at least one specified function. (Ref. 6.)

***s*-expected value**. A statistical term. If x is a random variable, and $F(x)$ is its *Cdf*, then $E\{x\} \equiv \int x dF(x)$, where the integration is over all x. For continuous random variables with a *pdf*, this reduces to $E\{x\} = \int x \; pdf\{x\} \, dx$.

For discrete random variables with a *pmf*, this reduces to $E\{x\} = \Sigma x_n pmf\{x_n\}$ where the sum is over all n.

exponential distribution. A 1-parameter distribution ($\lambda > 0, t \geqslant 0$) with:

$pdf\{t\} = \lambda \exp(-\lambda t)$

$Cdf\{t\} = 1 - \exp(-\lambda t)$

$Sf\{t\} = \exp(-\lambda t)$

failure rate $= \lambda$, mean time-to-failure $= 1/\lambda$.

Note 1. This is the constant failure-rate distribution.

Note 2. This has many convenient properties, and so is widely used--even when not strictly applicable.

Note 3. This distribution often is chosen, because of its tractability, when there are not enough data to reject it.

Note 4. Often parameterized with $\theta \equiv 1/\lambda$.

Note 5. It can be converted to a 2-parameter distribution by **substituting** $(t - t_o)$ for t everywhere.

F

[1]**failure**. The termination of the ability of an item to perform its required function. (Refs. 3 and 5.)

Added notes:

Note 1. It is presumed that the item either is or is not able to perform its required function. Partial ability is not considered in this definition.

Note 2. Virtually all failures discussed in these Handbooks are **random failures**.

[2]**failure**. The inability of an item to perform within previously specified limits.

[1]**failure, catastrophic**. A **failure** that is both sudden and complete. (Ref. 5.)

[2]**failure, catastrophic**. A sudden change in the operating characteristics of an item resulting in a complete loss of useful performance of the item. (Ref. 6.)

failure, chance. This is a term that is misused so often that it ought to be avoided. *See:* **failure, random.**

failure, critical. A **failure** of a component in a system such that a large portion of the **mission** will be aborted or such that the crew safety is endangered.

Note. **Criticality** is often assumed to have degrees, as in Failure Modes, Effects, and Criticality Analysis.

failure, degradation. A **failure** that occurs as a result of a gradual or partial change in the operating characteristics of an item. (Adapted from Ref. 6.)

failure, *s*-dependent. Any **failure** whose occurrence is ***s*-dependent** on other **failures.**

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failure, s-independent. Any **failure** whose occurrence is **s-independent** of other **failures.**

failure, infant. A **failure** that occurs during the very early life of an item.

Note 1. The **failure-rate** is usually decreasing.

Note 2. It is usually a **random failure.**

Note 3. It often is ascribed to grossly bad conditions of manufacture, although that need not be true.

[1]**failure mechanism.** The cause in the item of the observed failure mode of the item. It is one level down from the failure mode.

Note. *See.* note on [1]**failure mode.**

See also: **failure mode.**

[2]**failure mechanism.** The physical, chemical, or other process that results in a failure. (Adapted from Ref. 5.)

[1]**failure mode.** The observable behavior of an item when it fails; e.g., **failure modes** of electric motor might be classified as bearing seizure, winding short, winding open, overheating.

Note. The distinction between **failure mode** and **failure mechanism** is arbitrary and depends on the level at which observations are made. For example, a **failure mode** of a radar is antenna failure, the mechanism might be motor failure. If the motor is observed, the **failure mode** might be bearing seizure and the **failure mechanism** might be loss of lubrication. If the bearing is observed, its **failure mode** might be loss of lubrication, and the **failure mechanism** might be seal failure. If the seal is observed, ...

See also: **failure mechanism.**

[2]**failure mode.** The effect by which a **failure** is observed; e.g., an open- or short-circuit condition, or a gain change. (Adapted from Ref. 5.)

failure mode, effects, and criticality analysis (FMECA). An analysis of possible modes of failure, their cause, effects, criticalities, s-expected frequency of occurrence, and means of elimination.

Note 1. It often is called FMEA (without criticality).

Note 2. The analysis can include more such as (1) estimated cost to eliminate or mitigate the failure, (2) listing the items in ranked order of cost-benefit ratio to fix them.

failure, primary. A **failure** whose occurrence is not caused by other **failures.**

Note. This is sometimes ambiguously called an **independent failure.**

[1]**failure, random.** Any **failure** whose occurrence is unpredictable in an absolute sense but which is predictable in a probabilistic sense. (Adapted from Ref. 2.)

Added note.
Note 1. This term is often improperly used to imply "a constant failure rate process" or "some state of maturity of a design".

Note 2. Virtually all failures discussed in these Handbooks are **random failures.**

[2]**failure, random.** Any **failure** whose cause and/or mechanism make its time of occurrence unpredictable. (Ref. 5.) (*See:* added notes in definition 1.)

*[1]**failure rate λ, A.** The conditional probability density that the item will fail just after time t, given that the item has not failed up to time t.

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$\lambda(t) \equiv pdf\{t\}/Sf\{t\}$.

The *pdf* is normalized by the fraction still alive at the time.

Note 1A. This definition is only for continuous random variables whose *Sf* is well-behaved enough for the *pdf* to be well defined.

Note 2A. In this case (Note 1),

$$\lambda(t) = \frac{d}{dt}[-\ln R(t)]$$

where $R(t) \equiv Sf\{t\}$ is the **s-reliability**.

Note 3A. The variable need not be time, it can be any continuous measure of life such as operating time, calendar time, or distance.

Note 4A. It has many names such as hazard rate, force of mortality (especially for people's lives), instantaneous failure rate (a poor choice), and conditional failure rate

Note 5A. It must be distinguished from the *pdf* with which it is occasionally mistaken in the engineering literature.

Note 6A. Its most popular use is where the failure rate λ is constant. Then $Sf\{t\} = \exp(-\lambda t)$.

B The conditional probability that the item will fail at the next time point t_n given that the item has not failed before that time point t_n.

$$\lambda_n \equiv pmf\{t_n\}/Sf\{t_n\}.$$

The *pmf* is normalized by the fraction still alive just before t_n.

Note 1B. This definition is only for discrete random variables.

Note 2B. The variable need not be discrete time points, it can be any time-like discrete measure of life such as cycles of operation or events.

Note 3B. This is not a common use of the concept. The random variable is virtually always continuous.

General note.
Note This concept is directly applicable only to either:
(a) Nonrepairable items, or

(b) Repairable items where repair time is ignored and repair is to good-as-new. Each such item is considered to be brand new. For other repairable items, this concept must be defined further before it can be useful.

See also: ***pdf*, *pmf*, *Sf*.**

[2]**failure rate.** The number of failures of an item per unit measure of life (cycles, time, miles, events, etc., as applicable for the item). (Ref. 2.)

Added note:
Note. This may be ambiguous because it could refer to the *pdf*; *see*: [1]**failure rate,** Note 5A. Its use is best avoided unless the ambiguity can be avoided.

[3]**failure rate.** The incremental change in the number of failures per associated incremental change in time. (Adapted from Ref. 5.)

Added note:

Note. This may be ambiguous because it could refer to the *pdf*; *see*: [1]**failure rate,** Note 5A. Its use is best avoided unless the ambiguity can be avoided.

[4]**failure rate.** The rate of change of the number of items that have failed, divided by the number of items surviving. (Adapted from Ref. 5—definition of instantaneous **failure** rate.)

[5]**failure rate.** The *s*-expected number of

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failures in a given time interval. (Adapted from Ref. 6.)

Added note:

Note. This definition is ambiguous and ought not to be used.

failure, secondary. A **failure** caused either directly or indirectly by the **failure** of another item. (Adapted from Ref. 5.)

Note. This is sometimes ambiguously called a **dependent failure.**

*[1] **failure, wearout.** Any **failure** whose time of occurrence is governed by a rapidly increasing failure rate.

Note 1. The failure rate must "become infinite" as time "becomes infinite".

Note 2. The conditional mean remaining-time to failure must go to zero as the consumed life of the item "becomes infinite".

Note 3. An **s-normal distribution** of life satisfies those requirements and often is used as a typical wearout distribution.

Note 4. It may not be possible to tell, by looking at a failed item what classification of failure is involved. Some of the classifications are for mathematical convenience only.

Note 5. It is usually a **random failure.**

[2] **failure, wearout.** Any of the usual failures that occur due to mechanical wear of a part.

Note. This is the prototype for definition 1.

G

Gaussian distribution. A 2-parameter distribution ($\sigma > 0$) with

$$pdf\{x\} = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$Cdf\{x\} = gauf(x)$$

$$Sf\{x\} = gaufc\,(x).$$

"mean value of x" = μ, "standard deviation of x" = σ.

Note 1. This has several convenient properties, and so is widely used—even when not strictly applicable.

Note 2. This distribution is sometimes implied by the phrase "pure random", but it may refer to other distributions as well.

Note 3. More commonly this is called the **s-normal distribution.**

geometric mean. The **geometric mean** of n numbers is the nth root of their product.

Note. The term is applicable only to positive numbers.

go/no-go. This expression implies that only 2 states will be considered: either it "goes" or it "doesn't go", i.e., is either good or bad. It is the same as **attribute.**

good-as-new. A term which describes repair. The repaired item is indistinguishable from a brand new item. Its internal clock has been turned back to zero.

Note 1. It does not always imply perfection, especially if the item contains redundancy.

Note 2. If the **failure rate** is constant, **good-as-new** and **bad-as-old** are the same.

goodness of fit. A statistical term that quantifies how likely a sample was to have come from a given probability distribution.

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H

hazard rate. Same as **failure rate.**

homogeneous. A. The state of being reasonably close together with respect to one or more important properties.

B. Describable by one of the simple, common, tractable probability distributions.

Note. This is a qualitative term and suggests that its user is satisfied with his description of the events. It usually is ambiguous and ought to be replaced by a more accurately descriptive phrase.

human engineering. The area of **human factors**, which applies scientific knowledge to the design of items to achieve effective man-machine integration and utilization. (Ref. 2.)

human factors. A body of scientific facts about human characteristics. The term covers all biomedical and psychosocial considerations: it includes, but is not limited to, principles and applications in the areas of human engineering, personnel selection, training, life support, job performance aids, and human performance evaluation. (Ref. 2.)

human performance. A measure of man-functions and actions in a specified environment. (Ref. 2.)

hypothesis. An assertion that is to be tested by means of sampling and statistical analysis.

Note. This restricted definition is the way the term generally is used in statistical reliability procedures. Other, more general, definitions are valid also.

See also: **hypothesis, null.**

hypothesis, null. An **hypothesis** that there is no difference between some characteristics of the parent populations of several different samples, i.e., that the samples came from similar populations.

Note 1. This is usually tested by:

(a) Being specific about the characteristics of the **population**

(b) Pooling the sample data in some way

(c) Seeing how often one would get samples that differ as much as the samples at hand.

Note 2. An alternate **hypothesis** is often specified or implied. The more narrowly and specifically the alternate **hypothesis** is framed, the easier it is to distinguish between the **null-** and **alternate-hypotheses.**

Note 3. It is easy to reject the **null hypothesis** when the occurrence of the observed sample differences (or worse) would be very unlikely. One should, however, be very suspicious when the samples are very alike—someone may have taken liberties (perhaps unintentional or well intentioned) with the data.

Note 4. In some cases, such as goodness-of-fit tests, there is only one sample, and the null hypothesis is that it came from a particular family of distributions.

Note 5. Example. It is hypothesized that 2 samples came from 2 ***s*-normal** populations that have the same standard deviation and the same means. (The **null hypothesis** here refers to assuming there is no difference in the means.) The alternate **hypothesis** is exactly the same, except that the means are different. This is a quite restrictive alternate **hypothesis** and can be tested quite sharply.

Note 6. The discriminating ability depends not only on the form of the alternate **hypothesis** but on the amount of the data.

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It is always possible to have so few data that one can never reject the **null hypothesis** or so many data that one will always reject the **null hypothesis**. The engineering interpretation of the results of hypothesis testing are often quite different from the statistical interpretation.

I

infant mortality. Premature catastrophic failures occurring at a much greater rate than during the period of useful life prior to the onset of substantial wearout.

Note 1. This term is used in analogy to the human situation, where the **bathtub curve** holds for the death rate. Infants have a higher death rate than do older children. Many infant deaths are due to subnormal characteristics of the infant. Likewise, early failures in many equipments are due to substandard characteristics.

Note 2. **Infant mortality** often can be reduced by stringent quality control and design efforts.

[1]**inspection**. The examination and testing of supplies and services (including, when appropriate, raw materials, components, and intermediate assemblies) to determine whether they conform to specified requirements. (Ref. 1; Source:ASPR 14-001.3.)

[2]**inspection**. The process of measuring, examining, testing, gaging, or otherwise comparing the unit with the applicable requirements. The unit of product may be a single article, a pair, a set; or a specimen, a length, an area, a volume; or an operation, a service, a performance. (Ref. 3.)

inspection by attributes. Inspection whereby either the unit of product or characteristics thereof, is classified simply as **defective** or nondefective, or the number of **defects** in the unit of product is counted, with respect to a given requirement. (Ref. 1.)

See also: **attribute**.

inspection by variables. Inspection wherein certain quality characteristics of a sample are evaluated with respect to a continuous numerical scale and expressed as precise points along this scale. Variable inspections record the degree of conformance or nonconformance of the unit with specified requirements for the quality characteristics involved. (Ref. 1.)

inspection level. An indication of the relative sample size for a given amount of product. (Ref. 1.)

Added note:

Note. When the **inspection level** is changed, the new **operating characteristic** will generally cross the old one near the region of "indifference". This means that **consumer-** and **producer-risks** will both generally rise or fall when the inspection is reduced or tightened, respectively.

[1]**inspection lot**. A collection of units of product bearing identification and treated as a unique entity from which a sample is to be drawn and inspected to determine conformance with the acceptability criteria. (Ref. 1.)

[2]**inspection lot**. A collection of similar units or a specific quantity of similar material offered for **inspection** and acceptance at one time. (Ref. 3.)

inspection, normal. **Inspection** in accordance with a sampling plan that is used under ordinary circumstance. (Ref. 3.)

See also: **inspection level**.

inspection, reduced. **Inspection** in accordance with a sampling plan requiring smaller

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sample sizes than those used in **normal inspection**. **Reduced inspection** is used in some inspection systems as an economy measure when the level of submitted quality is sufficiently good and other stated conditions apply. (Ref. 3.)

Note. The criteria for determining when quality is "sufficiently good" must be defined in objective terms for any given inspection system.

See also: **inspection level.**

[1]**inspection, tightened.** Inspection under a sampling plan using the same quality level as for **normal inspection**, but requiring more stringent acceptance criteria. (Ref. 1.) (Source: MIL-STD-109A.)

See also: **inspection level.**

[2]**inspection, tightened.** Inspection in accordance with a sampling plan that has more strict acceptance criteria than those used in **normal inspection. Tightened inspection** is used in some inspection systems as a protective measure when the level of submitted quality is sufficiently poor. It is expected that the higher rate of rejections will lead the supplier to improve the quality of submitted product. (Ref. 3.)

Note. The criteria for determining when quality is "sufficiently poor" must be defined in objective terms for any given inspection system.

See also: **inspection level.**

item. A very general term. It can refer to anything, from very small parts to very large systems.

Note. This term often is used to avoid being specific about the size or complexity of the thing to which reference is made.

L

life test. A test, usually of several items, made for the purpose of estimating some characteristic(s) of the probability distribution of life.

longevity. Length of useful life of a product, to its ultimate wearout requiring complete rehabilitation. This is a term generally applied in the definition of a safe, useful life for an equipment or system under the conditions of storage and use to which it will be exposed during its lifetime.

lot. *See:* **inspection lot.**

lot quality. The true fraction defective in a **lot.**

Note. This applies to **attributes.** Other definitions would be needed for variables.

*[1]**lot tolerance percent defective** (LTPD). A point on the quality coordinate of the **operating characteristic** of an **attribute acceptance-sampling-plan** which is in the region of bad quality and reasonably low acceptance probability.

Note 1. The rejection probability at the LTPD is often called the **consumer risk** β.

Note 2. The conventional definitions (*see:* defs. 2 and 3) tend to endow this point with very special properties which it does not really have. Conventionally this point (LTPD, β) is one of two that define the **acceptance sampling plan** and its **operating characteristic.** But any 2 points on that **operating characteristic** will generate exactly the same **acceptance sampling plan.** That is why this modified, more usable definition is also given.

Note 3. An example of an LTPD is 20% defective at an acceptance probability (consumer risk) of 10%.

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Note 4. The term itself can be very misleading, especially to non-specialists in Quality Control. Its use ought to be avoided in material written for such people.

[2] **lot tolerance percent defective** (LTPD). Expressed in percent defective, the poorest quality in an individual lot that should be accepted. Also referred to as rejectable quality level (RQL). (Ref. 3.)

Note. The LTPD is used as a basis for some inspection systems and commonly is associated with a small consumer's risk.

M

s-maintainability. A characteristic of design and installation which is expressed as the probability that an item will be retained in or restored to a specified condition within a given period of time, when the maintenance is performed in accordance with prescribed procedures and resources. (Refs. 1 and 2.)

maintenance. All actions necessary for retaining an item in or restoring it to a specified condition. (Ref. 2.)

Added note:

Note. **Maintenance** usually is assumed to be perfect, i.e., to restore all parts to good-as-new and to do no damage to anything. The assumption is rarely true.

maintenance, corrective. This is the same as **repair**.

See also: **maintenance**.

maintenance, preventive. The **maintenance** performed in an attempt to retain an item in a specified condition by providing systematic inspection, detection and prevention of incipient failure. (Adapted from Ref. 2.)

See also: **maintenance** (and added note).

maintenance ratio. The man-hours of **maintenance** required to support each hour of operation.

Note. This figure reflects the frequency of failure of the system, the amount of time required to locate and replace the faulty part, and to some extent the overall efficiency of the maintenance organization. This method of measurement is valuable primarily to operating agencies, since, under a given set of operating conditions, it provides a figure of merit for use in estimating maintenance manpower requirements. The numerical value for maintenance ratio may vary from a very poor rating of 5 or 10, down to a very good rating of 0.25 or less. (Adapted from Ref. 6.)

maintenance time, corrective. *See:* **repair time**.

malfunction. Anything that requires **repair**. It is purposely a general word.

Note. It can be anything from a minor degradation to a complete system breakdown.

marginal testing. A test in which item environments such as line voltage or temperature are changed to worsen (reversibly) the performance. Its purpose is to find out how much margin is left in the item for degradation.

mean. A. The arithmetic mean; the **s-expected value**.

B. As specifically modified and defined, e.g., harmonic mean (reciprocals), geometric mean (a product), logarithmic mean (logs).

Note. Definition A is implied unless otherwise modified. It is wise to be explicit if there is any possibility of misunderstanding.

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mean cycles-between-failures. *See:* mean life-between-failures.

mean cycles-to-failure. *See:* **mean life.**

mean distance-between-failures. *See:* **mean life-between-failures.**

mean distance-to-failure. *See:* **mean life.**

mean life. $\int_o^T R(t)dt$

where

$R(t)$ ≡ the **s-reliability** of the item

T ≡ the interval over which the **mean life** is desired, usually the useful life(**longevity**).

Note 1. The concept is defined only for items which are either

(a) Not repaired, or

(b) Repaired to a good-as-new condition, and returned to stock, i.e., after repair they are treated as brand new items. The repair process itself is irrelevant to the concept.

Note 2. T is "infinity" in most definitions. Suppose $R(t) = \exp(-\lambda t)$, the often treated case. Then

$$MTF = [1 - \exp(-\lambda T)]/\lambda.$$

(a) Suppose T is short compared to $1/\lambda$, i.e., $\lambda T << 1$. The $MTF \approx T$.

(b) Suppose T is long compared to $1/\lambda$, i.e., $\lambda T >> 1$. The $MTF \approx 1/\lambda$.

This example helps to clear the confusion between $1/\lambda$ (which is often called the mean-life) and the longevity T. If the longevity is "infinite", then the mean life (for constant failure rate) is $1/\lambda$. The mean lives in the literature are virtually always $1/\lambda$, the distinction in this note is very rarely made elsewhere.

Note 3. The concept is applicable to any measure of life, such as calendar time, operating time, cycles of something, distance, or events. The phrase is ambiguous unless the measure of life is clearly and explicitly defined.

Note 4. When $T \to \infty$, the $MTF \to \infty$ for some **s-reliability** functions. In that case, it is important that T not be allowed to "go to infinity".

Note 5. For a sample of N, mean life is just the usual average life—add the lives of N units, and divide by N.

Note 6. There are many definitions of this concept in the literature, some of which are misleading and/or ambiguous. Be extremely wary of any definition that is not equivalent to the one given here.

Note 7. The **s-reliability** of an item is a function of many things, e.g., all the mission conditions.

Note 8. This concept may be modified by such terms as estimated, extrapolated, or observed. See Ref. 5, pp. 340-341.

See also: **s-reliability.**

mean life-between-failures. This concept is the same as **mean life** except that it is for repaired items, and is the mean up-duration of the item. The formula is the same as for **mean life** except that $R(t)$ is interpreted as the distribution of up-durations.

Note 1. The concept is applied, virtually always, only to items where the up-durations are exponentially distributed, i.e., $R(t) = \exp(-\lambda t)$. If it is applied to any other up-duration distribution, there are severe conceptual difficulties and the whole repair philosophy must be carefully and explicitly detailed. With exponentially distributed up-durations (usual case) the repair process itself is irrelevant to the concept.

Note 2. When up-durations are exponen-

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tially distributed (rate parameter λ), the **bad-as-old** and **good-as-new** repair philosophies are exactly the same, because the item at any point in its up-duration has the same $R(t)$ as at any other point, or as any other item with the same rate parameter. If the down-durations are ignored (compressed to zero), then the failure events form a Poisson process with rate parameter λ.

Note 3. The concept is only applied when $\lambda T >> 1$ so that mean life between failures is $1/\lambda$. (*See:* **mean life**, Note 2.) If one tries to apply it in other situations, the definition must be extended to include the entire maintenance philosophy.

Note 4. The concept is applicable to any measure of life, such as calendar time, operating time, cycles of something, distance, or events.

Note 5. For a sample of N, mean up-duration is just the usual average up-duration—add the up-durations of N units, and divide by N.

Note 6. There are many definitions of this concept in the literature, some of which are misleading and/or ambiguous. Be extremely wary of any definition that is not equivalent to the one given here.

Note 7. The up-duration of an item is a function of many things, e.g., all the mission conditions.

Note 8. This concept may be modified by such terms as estimated, extrapolated, or observed. *See:* Ref. 5, pp. 340-341.

See also: **mean life, s-reliability.**

mean square error (mse). A property of a statistical estimator. It is similar to variance except that it is referred to the true **population** mean instead of its own mean.

$$mse = (\text{bias})^2 + \text{variance}$$

Note. The *mse* is often a very useful concept, more so than variance. But the *mse* is much less tractable than variance and so is less often used.

*[1] **mean time-between-failures** (MTBF). *See:* **mean life-between-failures.**

[2] **mean time-between-failures** (MTBF). For a particular interval, the total functioning life of a **population** of an item divided by the total number of failures within the **population** during the measurement interval. The definition holds for time, cycles, miles, events, or other measure of life units. (Ref. 2.)

mean time-to-failure (MTF). *See:* **mean life.**

mean time-to-first-failure (MTFF). Same as **mean life**, but can apply to repairable equipment (although behavior subsequent to the first failure is irrelevant unless the item is restored to **good-as-new** and is treated as any other brand new item).

*[1] **mean time-to-repair** (MTTR). Similar to **mean life** except that repair time is used instead of life

$$MTTR = \int_0^T \bar{G}(t)\, dt$$

where

$G(t) \equiv$ *Cdf* of repair time

$\bar{G}(t) \equiv 1 - G(t)$

$T \equiv$ maximum allowed repair time, i.e., item is treated as nonrepairable at this echelon and is discarded or sent to a higher echelon for repair.

Note 1. The value of T can be important for distributions with very long tails, e.g., lognormal.

Note 2. Suppose the repair rate,

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$\mu \equiv \frac{d}{dt}$ $[-\ell n\, \bar{G}(t)]$, is constant, then

$\bar{G}(t) = \exp(-\mu t)$ and
$MTTR = [1 - \exp(-\mu T)]/\mu$

If T is long compared to $1/\mu$, the usual case, then $MTTR \approx 1/\mu$. This supposition of constant repair rate is not considered to be realistic although it is often made. Conventional wisdom suggests a lognormal distribution.

Note 3. *See:* notes on **mean life.**

[2] **mean time-to-repair** (MTTR). The total corrective maintenance time divided by the total number of corrective maintenance actions during a given period of time. (Ref. 2.)

mean time-to-xxxx (MTX). This is simply the arithmetic mean (for a sample) or the **s-expected** value (for a population) of the xxxxtime.

$$MTX = \int_o^T \bar{\phi}(t)\, dt$$

where
xxxx = any event

T ≡ the maximum considered xxxxtime

$\bar{\phi}(t)$ ≡ Sf of xxxxtime

See also: **mean life.**

mission. The objective or task, together with purpose, which clearly indicates the action to be taken (Ref. 2.)

Added notes:
Note 1. In reliability it is presumed that the mission description includes conditions under which the performance is to be obtained, the time duration (where appropriate), and the definition of failure/success.

Note 2. The mission can consist of sub-missions (phases) each of which is defined as a mission in itself. The sub-missions can be time sequential or occur at the same time (e.g., multiple missions).

module. An item which is packaged for ease of maintenance of the next higher level of assembly. (Adapted from Ref. 6.)

N

s-normal distribution. *See:* **Gaussian distribution.**

O

operating characteristic (OC). A. For an **accept/reject test**: the relationship between probability of accepting an hypothesis and the true value of a parameter in that hypothesis.

B. For **acceptance sampling**: the relationship between probability of accepting a lot and the true quality (usually measured by fraction defective) of the lot.

Note 1. Probability of acceptance is the same as longrun fraction of lots accepted.

Note 2. The OC is most usually presented as a curve and referred to as the OC curve.

*[1] **operating characteristic curve** (OC curve). The curve which shows the relationships of the **operating characteristic.**

See also: **operating characteristic.**

[2] **operating characteristic curve.** The curve of a sampling plan which shows the percentage of lots or batches which may be expected to be accepted under the specified sampling plan for a given process quality. (Ref. 1.)

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[3] **operating characteristic curve.** A. A curve showing, for a given sampling plan, the probability of accepting a lot, as a function of the lot quality.

B. A curve showing, for a given sampling plan, the probability of accepting a lot, as a function of the quality of the process from which the lots come. Also, as used for some types of plans—such as chain sampling plans and continuous sampling plans—a curve showing the percentage of lots, or product units, that may be expected to be accepted as a function of the process quality. (Ref. 3.)

Note. For sampling plans, the terms OC curve, consumer's risk, producer's risk, and the like, are used in two senses, referred to as type A and type B, depending on whether interest centers on (A) probabilities associated with sampling from a lot of stated quality or on (B) probabilities associated with sampling the output (series of lots, units, etc.) from a process of stated quality. For sampling from a lot, the values of probabilities, risks, and the like, are based on sampling from a finite population, and for sampling from a process, they are based on sampling from an infinite population.

operational. Of, or pertaining to, the state of actual usage (being up, being in operation). (Adapted from Ref. 2.)

overstress. A condition wherein the severity levels of operation (use, etc.) are more than usual or more than the specification.

Note. Often the term is applied where the stress is increased slowly (perhaps in steps) until failure occurs or until an adequate ability to resist the stress is demonstrated.

P

parallel. Items that are connected so that the total flow is through all, and what flows through one item does not flow through another.

Note. The term is often ambiguous because it can refer to a logic diagram as well as a physical diagram, and the two do not always agree. It is wise to modify the term explicitly to be clear.

part. An item that will not be disassembled for maintenance.

Note. It is a loose term, and applies to the purposes at hand.

passive element. An element that is not active.

See also: **active element.**

population. The totality of the set of items, units, elements, measurements, and the like, real or conceptual, that is under consideration. (Adapted from Ref. 3.)

Added notes:

Note 1. A synonym is universe.

Note 2. In practice, where the sampling is actual, rather than hypothetical, the **population** is likely to be defined (by working backwards) as that group from which the sample was actually a random sample. This working backwards may arrive at a rather different population than originally was intended. The actual vs "hoped-for" population has been at the root of many statistical errors.

precision. Degree of mutual agreement among individual measurements. Relative to a method of test, precision is the degree of mutual agreement among individual measurements made under prescribed like conditions. (Ref. 3.)

predicted. That which is expected at some future date, postulated on analysis of past experience. (Adapted from Ref. 2.)

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probability density function *pdf*. The derivative of the *Cdf* with respect to the random variable.

Note 1. For continuous random variables only.

Note 2. The *Cdf* must be well behaved enough for the operation to be performed. Otherwise the *pdf* will not be defined at the ill behaved places.

See also: ***Cdf, pmf.***

probability distribution. A general term that refers to the way a random variable is distributed. It is often used in association with a name such as Gamma, Gaussian, exponential, or Weibull. The probability distribution has quantitative properties such as a *Cdf* and *Sf*. If the random variable is continuous and well behaved enough, there will be a *pdf*. If the random variable is discrete, there will be a *pmf*.

probability mass function *pmf*. The amount of probability assigned to each value of the random variable.

Note. For discrete random variables only.

See also: ***Cdf, pdf.***

Q

qualification. The entire process by which products are obtained from manufacturers or distributors, examined and tested, and then identified on a Qualified Products List. (Source: DSM 4120.3-M.) (Ref. 1.)

qualified product. A product that has been examined and tested and listed on or qualified for inclusion on the applicable Qualified Products List. (Source: DSM 4120.3-M.) (Ref. 1.)

qualified product list (QPL). A list of products, qualified under the requirements stated in the applicable specification, including appropriate product identification and test reference with the name and plant address of the manufacturer or distributor, as applicable. (Source: DSM 4120.3-M.) (Ref. 1.)

[1]**quality**. The totality of features and characteristics of a product or service that bear on its ability to satisfy a given need. (Ref. 3.)

[2]**quality**. The composite of all the attributes or characteristics, including performance, of an item or product. (Source: DOD-D-4155.11.) (Ref. 1.)

[1]**quality assurance**. A system of activities whose purpose is to provide assurance that the overall **quality-control** job is in fact being done effectively. The system involves a continuing evaluation of the adequacy and effectiveness of the overall **quality-control** program with a view to having corrective measures initiated where necessary. For a specific product or service, this involves verifications, audits, and the evaluation of the quality factors that affect the specification, production, inspection, and use of the product or service. (Adapted from Ref. 3.)

See also: [1]**quality control.**

[2]**quality assurance**. A planned and systematic pattern of all actions necessary to provide adequate confidence that the item or product conforms to established technical requirements. (Source: DOD-D-4155.11.) (Ref. 1.)

quality characteristics. Those properties of an item or process which can be measured, reviewed, or observed, and which are identified in the drawings, specifications, or contractual requirements. Reliability becomes a **quality characteristic** when so defined. (Ref. 6.)

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[1]**quality control** (QC). The overall system of activities whose purpose is to provide a quality of product or service which meets the needs of users; also, the use of such a system.

The aim of **quality control** is to provide quality that is satisfactory, adequate, dependable, and economic. The overall system involves integrating the quality aspects of several related steps, including the proper specification of what is wanted; production to meet the full intent of the specification: inspection to determine whether the resulting product or service is in accordance with the specification; and review of usage to provide for revision of specification.

The term "quality control" often is applied to specific phases in the overall system of activities, as, for example "process quality control".

Note. Broadly, **quality control** has to do with making quality what it should be, and **quality assurance** has to do with making sure that **quality control** is what it should be. In some industries, quality assurance is used as an all-inclusive term combining both functions. (Ref. 3.)

[2]**quality control.** A management function whereby control of quality of raw or produced material is exercised for the purpose of preventing production of defective material. (Ref. 1.)

R

[1]**random sample.** As commonly used in acceptance sampling theory, the process of selecting sample units in such a manner that all units under consideration have the same probability of being selected.

Note: Actually, equal probabilities are not necessary for random sampling; what is necessary is that the probability of selection be ascertainable. The stated properties of published sampling tables, however, are based on the assumption of random sampling with equal probabilities. An acceptable method of random selection with equal probabilities is the use of a table of pseudorandom numbers in a standard manner.

(The definition of "sampling at random" adapted from Ref. 3.)

See also: **population.**

[2]**random sample.** A sample selected in such a way that each unit of the **population** has an equal chance of being selected. (Ref. 1.)

See also: **population.**

*[1]**redundancy.** The existence of more than one means for accomplishing a given function.

Note 1. Each means of accomplishing the function need not be identical. (Adapted from Ref. 2.)

Further notes:

Note 2. In the qualified definitions of **redundancy** in the Glossary, the collection of all means for accomplishing the given function is called a group.

Note 3. The changeover (switching) often is presumed to be perfect, i.e., no information or product is lost, the changeover takes negligible time, the system performance never "knows" that the failure occurred. Perfection rarely is observed in practice. Loss of information in computer systems is especially important.

Note 4. Some action is often necessary to disconnect a failed item and possibly to connect a good item. If much action is necessary, it is often called maintenance.

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The distinction between **maintenance** and **redundancy** is one of degree of effort to effect the changeover.

[2]**redundancy**. The introduction of auxiliary elements and components to a system to perform the same functions as other elements in the system for the purpose of improving reliability and safety. (Ref. 5.)

*[1]**redundancy, active**. A type of redundancy wherein all items in the group are operating simultaneously.

Note 1. A failed item might need to be disconnected from the system; e.g., centrifugal pumps physically in parallel, might have a check valve physically in series with each pump.

Note 2. The failure behavior of each operating item in the group usually is presumed to be the same, although that behavior might be a function of the number of operating units.

Note 3. This often is presumed to be the same, mathematically, as hot standby.

Note 4. This often is presumed to be the opposite of **passive redundancy** and **standby redundancy**.

See also: **redundancy**.

[2]**redundancy, active**. That redundancy wherein all redundant items are operating simultaneously rather than being switched on when needed. (Refs. 2 and 5.)

redundancy, passive. This usually is **standby redundancy**.

*[1]**redundancy, standby**. A type of redundancy wherein some items in the group are not operating, i.e., are on standby.

Note 1. A failed item might need to be disconnected from the system.

Note 2. Some action is usually necessary to connect the new item into the system.

Note 3. The failure behavior of the standby items is not always clear when this term is used. Often **cold standby** is implied, but **warm- or hot-standby** might actually be occurring. It is wise always to be explicit about the failure behavior of standbys—it may even be worse than for operating items.

See also: **redundancy**.

*[2]**redundancy, standby**. That **redundancy** wherein the alternate means of performing the function is inoperative until needed and is switched on upon failure of the primary means of performing the function. (Adapted from Refs. 2 and 5.)

*[1]**reliability**. The ability of an item to complete its mission successfully.

[2]**reliability**. The ability of an item to perform a required function under stated conditions for a stated period of time. (Adapted from Ref. 5.)

[3]**reliability**. A general term denoting some measure of the failure characteristics of an item.

*[1]***s*-reliability**. The probability that the item successfully completes its mission, given that the item was in proper condition at the mission beginning.

Note 1. The characteristics of the mission, such as length, environments, and the definition of failure must be defined clearly.

Note 2. The method for assuring "proper condition at the beginning of the mission" must be defined clearly. This is important when the item contains any nominal **redundancy**.

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Note 3. The mission can be either 1-shot (such as an explosive bolt) or over a length of time, such as a radar.

Note 4. The mission must be reasonably simple, otherwise other concepts will be more appropriate, e.g., system effectiveness.

Note 5. The concept can be modified by such words as assessed, estimated, predicted, extrapolated, or operational.

Note 6. Sometimes a long range reliability implicitly is being considered, and mission **reliability** is to be calculated for a short mission during that time. Such a concept requires careful delineation of the conceptual model and its implications. See Ref. 5, pp. 488-489.

Note 7. If repair is to be allowed, the assumptions concerning repair must be stated clearly and explicitly. See notes under **mean life** and **mean life-between-failures.**

[2]***s*-reliability**. The probability that an item will perform its intended function for a specified interval under stated conditions. (Refs. 1 and 2.)

Added note:

Note. This is the conventional definition. It lacks some of the important features of [1]***s*-reliability**, e.g., "1-shot missions", and "condition at mission beginning".

[3]***s*-reliability**. The probability that a device will function without failure over a specified time period or amount of usage.

Note 1. This is used most commonly in engineering applications. In any case where confusion may arise, specify the definition being used.

Note 2. The probability that the system will perform its function over the specified time should be equal to or greater than the reliability.
(Adapted from Ref. 5.)

reliability, achieved. The **reliability** actually demonstrated (with appropriate statistical considerations) by hardware tests, at a given calendar time.

reliability apportionment. The assignment of reliability goals to subitems (e.g., from system to its subsystems) in such a way that:

(a) The item will have the required **reliability**.

(b) The resources consumed in meeting the goals will be minimized.

reliability growth. Any design is incomplete, inadequate, and wrong in places. The failure rate of initially produced items often will be 10 times the hoped-for value. Reliability growth is the effort, the resource commitment, in improving design, purchasing, production, inspection procedures to improve the **reliability**.

Note. Reliability growth is one of the main reasons that **inherent reliability** is a poor phrase to use.

***s*-reliability, inherent**. This is a poor term to use; it is very ambiguous and subject to gross misuse. It can cause much misunderstanding. Very often it means ***s*-reliability** calculated using only those failures that an imaginative, aggressive, intelligent designer cannot blame on someone else. This concept violates the very foundation of reliability growth.

reliability measure. A general term denoting the ***s*-reliability**, ***s*-unreliability** or some function thereof.

Note. This term is used most often when the constant failure rate assumption is made. The measures usually being consid-

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ered are then *s*-**reliability**, *s*-**unreliability**, **failure rate**, **mean life**, **mean life-between-failures**.

See also: [3]**reliability**.

s*-reliability, mission.** *See:* [1]s*-reliability**, Note 6.

reliability, operational. This is a vague term. It usually refers to a method of calculating reliability using handbook failure rates and severity factors. Its use is best avoided unless its meaning is clearly explained.

reliability, predicted. The **reliability** of an equipment computed from its design considerations and from the **reliability** of its parts in the intended conditions of use. (Ref. 5.)

Added note:

Note. The prediction does not say what the **reliability** *will* be, but what the reliability can be if there is a reasonable **reliability growth** program.

See also: **reliability**.

reliability-with-repair. The **reliability** that can be achieved when maintenance is allowed under circumstances such that the system is never officially down (i.e., any downtime is not charged against reliability).

Note. When using this concept, the circumstances of allowable maintenance and definition of system states must be defined carefully and explicitly.

repair. The **maintenance** performed, as a result of failure, to restore an item to a specified condition. (Adapted from Ref. 2.)

risk. The probability of making a poor decision.

See also: **risk, consumer**; **risk, producer**.

*[1]**risk, consumer** β. A point on the acceptance-probability axis of the **operating characteristic** of an **attribute acceptance-sampling-plan** which is in the region of bad quality and reasonably low acceptance probability.

Note 1. The bad quality corresponding to β is often called the **lot tolerance percent defective** (LTPD).

Note 2. The conventional definition (*see:* def. 2) tends to endow this point with very special properties which it does not really have. Conventionally this point (LTPD, β) is one of two that define the **acceptance sampling plan** and its **operating characteristic**. But any 2 points on that **operating characteristic** will generate exactly the same **acceptance sampling plan**. That is why this modified, more usable definition is also given.

See also: **lot tolerance percent defective**, **operating characteristic**.

[2]**risk, consumer** β. For a given sampling plan, the probability of acceptance for a designated numerical value of relatively poor submitted quality.

Note. The exact risk depends on whether "submitted quality" relates to lot quality or process quality.

(Adapted from Ref. 3.)

*[1]**risk, producer** α. A point on the rejection-probability curve of the **operating characteristic** of an **attribute acceptance-sampling plan** which is in the region of good quality and reasonably low rejection-probability.

Note 1. The good quality corresponding to α is often called the **acceptable quality level** (AQL).

Note 2. The conventional definition (*see:*

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def. 2) tends to endow this point with very special properties which it does not really have. Conventionally this point (AQL, α) is one of two that define the **acceptance sampling plan** and its **operating characteristic**. But any 2 points on that **operating characteristic** will generate exactly the same **acceptance sampling plan**. That is why this modified, more usable definition is also given.

See also: **acceptable quality level, operating characteristic.**

[2]**risk, producer** α. For a given sampling plan, the probability of rejection for a designated numerical value of relatively good submitted quality.

Note. The exact risk depends on whether "submitted quality" relates to lot quality or process quality.

(Adapted from Ref. 3.)

root mean square (rms). The square root of the arithmetic mean of the squares.

S

s statistic. *See:* **standard deviation.**

safety. The conservation of human life and its effectiveness, and the prevention of damage to items, consistent with mission requirements. (Ref. 2.)

safety factor. A general term relating to the ability of the item to withstand more than the nominal "stresses".

Note. Whenever this is used in a specific sense, it must be clearly defined.

sampling plan. *See:* **acceptance sampling plan.**

[1]**sampling plan, double.** A specific type of attribute sampling plan in which the inspection of the first sample leads to a decision to accept, to reject, or to take a second sample. The inspection of a second sample, when required, then leads to a decision to accept or reject. (Source: MIL-STD-105) (Adapted from Ref. 1.)

[2]**sampling plan, double.** Sampling inspection in which the inspection of the first sample leads to a decision to accept a lot, to reject it, or to take a second sample; and the inspection of a second sample, when required, then leads to a decision to accept or to reject the lot. (Ref. 3.)

[1]**sampling plan, multiple.** A specific type of attribute sampling plan in which a decision to accept or reject an inspection lot may be reached after one or more samples from that inspection lot have been inspected, and always will be reached after not more than a designated number of samples have been inspected. (Source: MIL-STD-105) (Adapted from Ref. 1.)

[2]**sampling plan, multiple.** Sampling inspection in which, after each sample is inspected, the decision is made to accept a lot, to reject it, or to take another sample; but in which there is a prescribed maximum number of samples, after which a decision to accept or to reject the lot must be reached.

Note. Multiple sampling as defined here sometimes has been called "sequential sampling" or "truncated sequential sampling". The term "multiple sampling" is recommended.

(The definition of "multiple sampling" from Ref. 3.)

sampling plan, sequential. A specific type of sampling plan in which the sample units are selected one at a time. After each unit is inspected, the decision is made to accept, reject, or continue inspection until the acceptance or rejection criteria are met. Sampling terminates when the inspection

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results of the sample units determine that the acceptance or rejection decision can be made. The sample size is not fixed in advance, but depends on actual inspection results. (Source: Handbook H53.) (Ref. 1.)

Added note:

Note. In practice most such plans are truncated and then become like **multiple sampling** plans. The term multiple is used most often when there are only a few decision points, say up to 5, whereas the term sequential is used most often where there are many, say more than 5, decision points.

See also: **multiple sampling plan.**

series. Items which are connected so that what flows through one item flows through another.

Note. The term is often ambiguous because it can refer to a logic diagram as well as a physical diagram, and the two do not always agree. It is wise to modify the term explicitly to be clear.

severity level. A general term implying the degree to which an environment will cause damage and/or shorten life.

*[1] **s-significance.** A statistical term that relates to the probability that an observed test statistic would be as bad (or worse) than it was, if the hypothesis under test were true.

Note 1. One must distinguish between **s-significance** and engineering significance; there can be one without the other.

Note 2. It would be wise to obtain the services of a competent statistician if **s-significance** tests are to be used.

[2] **s-significance.** Results that show deviations between an **hypothesis** and the observations used as a test of the **hypothesis**, greater than can be explained by random variation or chance alone, are called statistically significant. (The definition of "statistical significance, statistically significant" from Ref. 3.)

*[1] **s-significance level.** The probability that, if the **hypothesis** under test were true, a sample test statistic would be as bad or worse than the observed test statistic.

Note 1. The **operating characteristic** (probability of rejection) gives the **s-significance** level for any given test.

Note 2. In many situations, there is a numerical relationship between **s-confidence** and **s-significance.**

[2] **s-significance level.** The probability (**risk**) of rejecting an hypothesis that is true. This is also referred to as **producer risk** in sampling inspection (**acceptance sampling**). (Adapted from Ref. 3.)

standard deviation. The root mean square deviation from the mean. It is a measure of dispersion of a random variable or of data. Four cases are important:

(1) For a continuous random variable x,

$$\sigma^2 \equiv \int (x-\mu)^2 \, pdf\{x\} dx$$

where

σ = population standard deviation

μ $\equiv$ population mean $\equiv \int x \, pdf\{x\} \, dx$

and the probability distribution is well behaved enough for the expressions to have meaning

and the integrations are over all values of x (the domain of x)

(2) For a discrete random variable x_n,

$$\sigma^2 \equiv \sum (x_n - \mu)^2 \, pmf\{x_n\}$$

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where

σ = population standard deviation
μ = population mean $\equiv \Sigma \; x_n pmf \; x_n$
the probability distribution is well behaved enough for the expressions to have meaning
the sum is over all values of x_n (the domain of x)

(3) For a finite population of size N with random variable x_n,

$$\sigma^2 \equiv \frac{1}{N} \sum_{n=1}^{N} (x_n - \bar{x})^2$$

where
σ = population standard deviation

$\bar{x}$ = population mean $\equiv \frac{1}{N} \sum_{n=1}^{N} x_n$

(4) For a sample of size N (from an "infinite" population) with data x_n,

$$\sigma^2_{\text{sample}} \equiv \frac{1}{N} \sum_{n=1}^{N} (x_n - \bar{x})^2$$

$$s^2 \equiv \frac{1}{N-1} \sum_{n=1}^{N} (x_n - \bar{x})^2$$

where
σ_{sample} = sample standard deviation

s = s-statistic

$\bar{x}$ = sample mean $\equiv \frac{1}{N} \sum_{n=1}^{N} x_n$

There is considerable controversy, confusion, and misunderstanding in the literature concerning whether σ_{sample} or s is the sample standard deviation. The simple answer is that it all depends on what you want to get. There is no question that σ_{sample} is the rms deviation from the mean, nor that it is the maximum likelihood estimator for the population standard deviation of an **s-normal** distribution. There is likewise no question that s^2 is the unbiased estimate of the population variance (although s is a biased estimate of the population standard deviation) and that, for **s-normal** distributions, the s- and s^2-statistics are extremely useful. But the utility of s and s^2 for **s-normal** distributions does not make s the rms deviation from the mean, nor is unbiasedness very important for s^2 as an estimator for the population variance (rarely, if ever, does one wish to estimate the population variance for its own sake). When the probability distribution is **s-normal**, one is virtually always interested in the s-statistic, not the σ_{sample} because s is used in calculating a Student's t statistic, a χ^2/ν statistic, an F-statistic, and for **s-confidence** or **s-significance statements**.

standby. A reserve item, often considered to be part of **redundancy**.

Note. Nothing is implied about its failure behavior, either absolutely or relative to operating equipment. Often **cold standby** is implied, but the term is ambiguous.

See also: **redundancy**.

standby, cold. A **standby** which is not degrading in any way and which cannot fail. Its **failure rate** is zero and is **good-as-new** when put in service.

standby, hot. A **standby** whose failure and degradation behavior is exactly that of a like operating item.

Note. **Hot standbys** are usually indistinguishable from **active redundancy**.

standby, warm. A **standby** whose failure and degradation behavior is not specified. It

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often is presumed to be between **hot-** and **cold-standby**, but (for mathematical convenience) often is presumed to include both.

Note. It is implied that the failure and degradation behavior is never worse than **hot standby**.

stress. A general and ambiguous term used as an extension of its meaning in mechanics as that which could cause failure. It does not distinguish between those things which cause permanent damage (deterioration) and those things which do not (in the absence of failure).

See also: **severity level**.

subassembly. A. A general term implying a lower level than an assembly, i.e., an **assembly** is made up of subassemblies.

B. Two or more parts which form a portion of an assembly, or form a unit replaceable as a whole, but having a part or parts which are replaceable as individuals. (Ref. 6.)

See also: **assembly**.

subsystem. A major subdivision of a **system** which performs a specified function in the overall operation of a **system**. (Ref. 6.)

Survivor function *Sf*. The probability that the random variable whose name is X takes on any value greater than or equal to a value x, e.g.,

$$\bar{F}(x) = Sf\{X\} \equiv Pr\{X \geq x\}.$$

Note 1. The *Sf* need not be continuous nor have a derivative. Its value is 1 below the lowest algebraic value of the random variable and is 0 above the highest algebraic value of the random variable. The *Sf* is a nonincreasing function of its argument.

Note 2. It is permissible to have a joint *Sf* of several random variables.

Note 3. The concept applies equally well to discrete and continuous random variables. For continuous random variables with continuous *Sf* (and thus continuous *Cdf*), $Sf + Cdf \equiv 1$; otherwise the identity does not hold.

Note 4. Since the identity in Note 3 holds so often, sometimes the *Sf* is defined that way. (Where there is no chance of misunderstanding, it may appear that way in some Parts of this Handbook series.)

system. A combination of complete operating equipments, assemblies, components, parts, or accessories interconnected to perform a specific operational function.

T

test category. Category I: A test in which US Army Test and Evaluation Command (TECOM) is responsible for establishing the test objective, preparation, and approval of the plan of test, and the processing and distribution of the report of test. The results of this category of tests may lead to type classification of the materiel undergoing tests.

Category II: A test in which TECOM is performing a service for the requesting agency and in which the test objectives, plan of test, and the processing and distribution of the report of test are the responsibilities of the requestor.

test severity. The **severity level** at which a test is run. If there is more than one failure mode, the concept might be ambiguous unless only overall failure rate is considered.

tolerance failure. A **drift-** or **degradation failure**.

See also: **failure, degradation**.

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tractable. Easy to work with mathematically and statistically.

truncation. A. Deletion of portions of the domain of a random variable greater-than and/or less-than specified value(s).

B. (For a sequential test) closing the decision boundary so that a decision always is made within a reasonable amount of testing.

U

use factor. A factor for adjusting base failure rate, as determined from MIL-HDBK-217, to specific use environments and packaging configurations other than those applicable to ground based systems. (Adapted from Ref. 6.)

V

variable. (in testing) The opposite of attribute; i.e., the characteristic under examination can have many (or a continuum of) values.

variance. The square of the standard deviation. The term often is used in theoretical statistics because it avoids taking the square root of a calculation. Variance is the second central moment.

REFERENCES

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2. MIL-STD-721, *Definitions of Effectiveness Terms for Reliability, Maintainability, Human Factors, and Safety.* (All reproduced material was taken from 721B, 25 August 1966, revised 10 March 1970.)

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CHAPTER 2

PROBABILITY DISTRIBUTIONS, SOME CAUTIONS AND NAMES

2-1 CAUTIONS

The common tractable *PrD's* (probability distributions) have no magic power to transform sample data into absolute knowledge, but many people act as if they did. Some important cautions are listed:

(1) Avoid assuming that the selected *PrD* represents the physical data outside the range of the sample data, merely because the sample data might reasonably (statistically) have come from it. Gross extrapolation beyond the range of the data is very misleading.

(2) Do not use point estimates of the parameters of the *PrD* without calculating some measure of their uncertainty such as *s-confidence** limits or a standard deviation.

(3) Avoid fitting sample data too closely by brute force, possibly by using a multiparameter *PrD* for each of several segments of the random variable. If one wishes a very close fit, there are several old fashioned methods such as power series which do not clothe brute force in a comely cloak. In samples of less than 10 or so, there can be tremendous scatter in the shape of a sample *pdf*, all from the same *PrD*.

(4) Avoid fitting a *PrD* to the data merely because it can be done.

(5) Avoid extensive calculations that select the family of *PrD's* which gives the best fit (in some sense) to the sample data. If that is the only reason for choosing a family of *PrD's*, it is not a good enough reason. It is especially bad practice when the desired results depend heavily on the shape of the *PrD* outside the region of the data.

The reason for all the cautions to the amateur analyst (and even some professional analysts) is not that he will violate some purist theory, but that he will outsmart himself. After having outsmarted and fooled himself, he will proceed to mislead others. One of the main functions of statistics in reliability engineering is to tell the engineer what he does NOT know from the data.

The main purpose of fitting a *PrD* to the data is for a summary. Once the data are presumed to be a random sample from a *PrD*, there is no need to save the data.

It is always possible to have so few data that they could reasonably have come from almost any family of *PrD's*. It is also always possible to have so much data that they could not have come from any given family of *PrD's*.

When the purpose of fitting a *PrD* to the data is to estimate some characteristic of the *PrD*—e.g., mean, standard deviation, or median—then using the corresponding sample characteristic directly always ought to be considered. That way no delusion of increased accuracy is generated by the extra mathematical manipulations. If this can't be done because extrapolation is necessary, then the uncertainties ought to be faced directly, without the delusion of mathematical precision.

*The prefix "*s*-" indicates the word is being used in the statistical sense.

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Always ask yourself why you want to do a particular statistical calculation, and will it really help you, or will it just let you fool yourself into thinking you know more about your data than you really do.

2-2 NAMING PROBABILITY DISTRIBUTIONS

Engineers and statisticians generally approach statistics from different points of view. It is very convenient for an engineer to have a name for each function he uses; statisticians seem not to mind the lack of names for many *PrD's*.

This handbook has adopted the convention of giving a base name to each *PrD*, and then adding a suffix to imply a particular function. The base name consists of 3 letters which are reasonably mnemonic.

(1) gau = gaussian

(2) csq = chi-square (χ^2)

(3) csn = chi-square/nu (χ^2/ν)

(4) fis = fisher-snedecor (F)

(5) exp = exponential

(6) wei = weibull

(7) lgn = lognormal

(8) gam = gamma

(9) bet = beta

(10) poi = poisson

The suffix *f* implies the *Cdf*, the suffix *fc* implies the *Sf*. For continuous *Cdf*'s, the *Sf* is the complement of the *Cdf*, from which name (complement) the *c* is derived for the suffix *fc*. The suffix *hr* implies the failure rate (hazard rate). The hazard rate for a *PrD* generally is defined for a location parameter of zero and a scale parameter of one.

When each *Cdf* and *Sf* have a short name, it is much easier to write equations.

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CHAPTER 3

BINOMIAL DISTRIBUTION

3-0 LIST OF SYMBOLS

bin = base name for binomial distribution

binf = *Cdf* for binomial distribution

binfc = *Sf* for binomial distribution

C = *s*-Confidence

Cdf = Cumulative distribution function

C, L, U = subscripts that imply a *s*-confidence; C is general, L is lower, U is upper.

$CM_i\{\ \}$ = ith central moment

Conf$\{\ \}$ = *s*-Confidence level

csqfc = *Sf* for the chi-square distribution

CV$\{\ \}$ = coefficient of variation: StDv$\{\ \}/E\{\ \}$

$E\{\ \}$ = *s*-Expected value

f, x = notation used in linear interpolation (often with subscripts)

I_p = incomplete beta function

$M_i\{\ \}$ = ith moment about the origin

$NCM_i\{\ \}$ = normalized ith central moment; $CM_i\{\ \}/[\text{StDv}\{\ \}]^i$

p, N = parameters

pdf = probability density function

pmf = probability mass function

$Pr\{\ \}$ = Probability

PrD = probability distribution

r = discrete random variable

R = *s*-Reliability

s- = denotes statistical definition

Sf = Survivor function

StDv$\{\ \}$ = standard deviation

Var$\{\ \}$ = variance

η = a uniformly distributed random variable

$\bar{\ }$ = the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

$\{\cdot;\cdot\}, (\cdot;\cdot)$ = the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon

$\hat{\ }$ = estimate

3-1 INTRODUCTION

The binomial distribution arises when repeated trials have only 2 outcomes. Each trial is under the same conditions as all the repeated trials. One of the outcomes is

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labeled, and the number of times it occurs is counted. The probability parameter refers to the labeled outcome. The other outcome is not considered further.

The base name *bin* is given to the binomial distribution (for binomial). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf*. The *Cdf* and *Sf* are not complementary because the random variable is discrete.

3-2 FORMULAS

N = number of trials, fixed in advance. N is a parameter of the distribution but is always known—never estimated from the data.

p = probability parameter. It turns out to be the long run relative frequency of the labeled outcome.

r = random variable, $r = 0, 1, 2, \ldots, N$

$\bar{p} = 1 - p$

$$pmf\{r; p, N\} = \binom{N}{r} p^r \bar{p}^{N-r} = pmf\{N-r; \bar{p}, N\} \tag{3-1}$$

$$Cdf\{r; p, N\} = binf(r; p, N) = \sum_{i=0}^{r} \binom{N}{i} p^i \bar{p}^{N-i} = binfc(N-r; \bar{p}, N) \tag{3-2}$$

$$Sf\{r; p, N\} = binfc(r; p, N) = \sum_{i=r}^{N} \binom{N}{i} p^i \bar{p}^{N-i} = binf(N-r; \bar{p}, N) \tag{3-3}$$

Table 3-1 shows a few examples of the binomial *pmf*. Some of the symmetries in the binomial distribution are shown in Eqs. 3-1 through 3-3.

It is easier to remember the *pmf* in the form of Eq. 3-4.

$$pmf\{r_1, r_2; p_1, p_2, N\} = \frac{N!}{r_1! r_2!} p_1^{r_1} p_2^{r_2}$$

$$p_1 + p_2 = 1$$

$$r_1 + r_2 = N \tag{3-4}$$

Eq. 3-4 is also easy to extend to the multinomial form, e.g., for 4 possible outcomes:

$$pmf\{r_1, r_2, r_3, r_4; p_1, p_2, p_3, p_4, N\} = \left(\frac{N!}{r_1! r_2! r_3! r_4!}\right) p_1^{r_1} p_2^{r_2} p_3^{r_3} p_4^{r_4}$$

$$p_1 + p_2 + p_3 + p_4 = 1$$

$$r_1 + r_2 + r_3 + r_4 = N$$

$$E\{r, p, N\} = Np$$

$$\mathrm{StDv}\{r; p, N\} = (Np\bar{p})^{1/2}$$

$$\mathrm{CV}\{r; p, N\} = (\bar{p} / pN)^{1/2}$$

$$CM_3\{r; p, N\} = Np\bar{p}(\bar{p} - p)$$

$$NCM_3\{r; p, N\} = (\bar{p} - p)/(Np\bar{p})^{1/2}$$

3-3 TABLES AND CURVES

Since there are 2 parameters, the distribution is tedious and awkward to tabulate. The *pmf* is so easily calculated, it rarely is tabulated. One of the most extensive tables is Ref. 1. Refs. 2 and 5 have modest tables. Ref. 3 is reasonably extensive.

The identity in Eq. 3-5 can provide other sources of tables.

$$\sum_{s=r}^{N} \binom{N}{s} p^s \bar{p}^{N-s} = I_p(r, N-r+1) \tag{3-5}$$

where I_p is the Beta Distribution (Incomplete Beta Function), Ref. 4 (Sec. 26.5), and Chapter 10.

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TABLE 3-1

BINOMIAL DISTRIBUTION, EXAMPLES

$N = 5$ The body of the table gives the binomial $pmf\{r;p,N\}$

r	$p = 0.1$	$p = 0.2$	$p = 0.5$	$p = 0.8$	$p = 0.9$
0	0.59	0.33	0.0031	0.00032	0.000010
1	0.33	0.41	0.16	0.0064	0.00045
2	0.073	0.20	0.31	0.051	0.0081
3	0.0081	0.051	0.31	0.20	0.073
4	0.00045	0.0064	0.16	0.41	0.33
5	0.000010	0.00032	0.0031	0.33	0.59
$E\{r;p,N\}$	0.50	1.00	2.50	4.00	4.50
$\mathrm{StDv}\{r;p,N\}$	0.67	0.89	1.12	0.89	0.67
$\mathrm{CV}\{r;p,N\}$	1.34	0.89	0.45	0.22	0.15
$CM_3\{r;p,N\}$	0.36	0.48	0	−0.48	−0.36
$NCM_3\{r;p,N\}$	1.19	0.67	0	−0.67	−1.19

Note: All *pmf* terms have been rounded to 2 significant figures; that is why the terms do not sum to 1.

The Poisson approximation is useful in ordinary reliability work. If p is taken as the failure probability, it will be reasonably small (if not, very few people are interested in its exact value). The approximation is

$$\binom{N}{r} p\bar{p}^{N-r} \approx e^{-pN}\left[\frac{(pN)^r}{r!}\right] \tag{3-6}$$

Eq. 3-6 reduces the number of parameters from 2 (p, N) to 1 (pN); it is reasonably good as long as $r << N$ and the right hand side sums close to 1 for $r = 0, \ldots, N$, viz, $csqfc(2\mu, 2N + 2) \approx 1$ (see Chapter 4). For contractual situations the exact formulas ought to be used.

3-4 PARAMETER ESTIMATION

The parameter N is known. The parameter p is estimated from the data. The estimate

$$\hat{p} \equiv r/N \tag{3-7}$$

is unbiased and maximum likelihood. If $r = 0$ or $r = N$, Eq. 3-7 is esthetically displeasing to many people, although it is still quite true. Very often (where $r = 0, N$) a s-confidence limit is used in place of the point estimate, usually corresponding to about 50% s-confidence level.

s-Confidence statements are more difficult for discrete random variables than for continuous random variables. Chapter 12 discusses the matter thoroughly.

The usual s-confidence statements for p are of the forms

$$\mathrm{Conf}\{p \leq p_L\} \leq C_L \tag{3-8a}$$

$$\mathrm{Conf}\{p \leq p_U\} \geq C_U \tag{3-8b}$$

$$\mathrm{Conf}\{p_L \leq p \leq p_U\} \geq C_U - C_L \tag{3-8c}$$

where p_L and p_U are defined by

$$C_L = binfc(r; p_L, N), \text{ or}$$

$$\bar{C}_L = binfc(N - [r - 1]; \bar{p}_L, N)$$

$$C_U = 1 - binf(r; p_U, N) = binfc(r + 1; p_U, N)$$

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In this form, C_L is usually small (say 5%), and C_U is usually large (say 95%). Notation for *s*-confidence statements is not at all standard; so particular attention must be paid to the example forms. Table 3-2 and Fig. 3-1 are useful for this type of *s*-confidence statement.

Chapter 12 shows that *s*-confidence statements for p can also be of the forms

$$\mathrm{Conf}\{p \leq p_L'\} \geq C_L \qquad (3\text{-}9a)$$

$$\mathrm{Conf}\{p \leq p_U'\} \leq C_U \qquad (3\text{-}9b)$$

$$\mathrm{Conf}\{p_L' \leq p \leq p_U'\} \geq C_U - C_L \qquad (3\text{-}9c)$$

where p_L' and p_U' are defined by

$$C_L = 1 - binf(r; p_L', N) = binfc(r+1; p_L', N)$$

$$C_U = binfc(r; p_U', N)$$

In this form, as in Eq. 3-8, C_L is usually small (say, 5%), and C_U is usually large (say, 95%). p_L' and p_U' will be inside the interval p_L, p_U (for $r \neq 0, N$). Table 3-2 also can be used to find p_L' and p_U'. The procedure is to use the entry that is one position above the entry used to find the corresponding p_L and P_U and then to reverse the inequality with C. For the sample in Table 3-2 ($N = 10$, $r = 4$), write

$$\mathrm{Conf}\{p \leq 0.552\} \leq 90\%$$

$$\mathrm{Conf}\{\bar{p} \leq 0.733\} < 90\%$$

$$\rightarrow \mathrm{Conf}\{p \leq 0.267\} \geq 10\%$$

$$\mathrm{Conf}\{0.267 \leq p \leq 0.55\} \leq 80\%$$

Ref. 6 shows some interesting *s*-confidence limits that can be readily calculated (for $r \neq 0, N$).

3-5 RANDOMIZED EXACT *s*-CONFIDENCE INTERVALS

Instead of always choosing the worst case, Eq. 3-8, exact *s*-confidence limits can be found by randomly choosing a value between p_L and p_L', and/or between p_U and p_U'. There is nothing to lose and everything to gain by this procedure because it means not always choosing the worst possible case.

The equations to give the randomized limits are

$$\eta = \frac{binfc(r; p_L^*, N) - C_L}{\binom{N}{r} p_L^{*r} \bar{p}_L^{*N-r}}$$

$$= \frac{binfc(r; p_L^*, N) - C_L}{binfc(r; p_L^*, N) - binfc(r+1; p_L^*, N)} \qquad (3\text{-}10a)$$

unless (a) $r = 0$, and $\eta \leq \bar{C}_L$; then use $p_L^* = 0$

or (b) $r = N$, and $\eta \geq \bar{C}_L$; then use $p_L^* = 1$.

$$\eta = \frac{binf(r; p_U^*, N) - \bar{C}_U}{\binom{N}{r} p_U^{*r} \bar{p}_U^{*N-r}}$$

$$= \frac{C_U - binfc(r+1; p_U^*, N)}{binfc(r; p_U^*; N) - binfc(r+1; p_U^*, N)} \qquad (3\text{-}10b)$$

$$= \frac{binfc(N-r; \bar{p}_U^*, N) - \bar{C}_U}{binfc(N-r; \bar{p}_U^*, N) - binfc(N-r+1; \bar{p}_U^*, N)}$$

unless (a) $r = N$, and $\eta \leq C_U$; then use $p_U^* = 1$

or (b) $r = 0$, and $\eta \geq C_U$; then use $p_U^* = 0$.

where η is a random number from the uniform distribution: $0 \leq \eta$ 1. When $\eta = 0$, $p_L^* = p_L$ and $p_U^* = p_U$. If $\eta = 1$ (consider the least upper bound of η), $p_L^* = p_L'$ and $p_U^* = p_U'$.

If special tables which give p_L^* and p_U^* are not available, use Table 3-2 to calculate p_U, p_L and p_U', p_L'. Then use a set of tables like Ref. 1 to solve Eq. 3-10 by iteration. Table

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TABLE 3-2

1-SIDED UPPER s-CONFIDENCE LIMITS FOR p (THE BINOMIAL PROBABILITY) (ADAPTED FROM Ref. 7)

The body of the table gives p_c, a 1-sided upper s-confidence (90%, 95%, 99%) limit for p, for the form $\text{Conf}\{p < p_c\} \geq C$.

- p = probability of occurrence of labeled event
- r = number of such events in N trials, the random variable
- N = number of trials in which r events occurred, fixed
- C = s-confidence (minimum value)

Example: For $N = 10, r = 4$, $\text{Conf}\{p \leq 0.646\} \geq 90\%$

(See note 1) $\longrightarrow$ $\text{Conf}\{\bar{p} \leq 0.812\} \geq 90\% \longrightarrow \text{Conf}\{p \leq 0.188\} \leq 10\%$

$\text{Conf}\{0.188 \leq p \leq 0.646\} \geq 80\%$

r	C = 90%	95%	99%	r	C = 90%	95%	99%	r	C = 90%	95%	99%
	N = 2				N = 3				N = 4		
0	.684	.776	.900	0	.536	.632	.785-	0	.438	.527	.684
1	.949	.975-	.995-	1	.804	.865-	.941	1	.680	.751	.859
				2	.965+	.983	.997	2	.857	.902	.958
								3	.974	.987	.997
	N = 5				N = 6				N = 7		
0	.369	.451	.602	0	.319	.393	.536	0	.280	.348	.482
1	.584	.657	.778	1	.510	.582	.706	1	.453	.521	.643
2	.753	.811	.894	2	.667	.729	.827	2	.596	.659	.764
3	[illegible]	.974	.967	3	.799	.847	.915+	3	.721	.775-	.858
4	[illegible]	.990	.998	4	.907	.937	.973	4	.830	.871	.929
				5	.983	.991	.998	5	.921	.947	.977
								6	.985+	.993	.999
	N = 8				N = 9				N = 10		
0	.250	.312	.438	0	.226	.283	.401	0	.206	.259	.369
1	.406	.471	.590	1	.368	.429	.544	1	.337	.394	.504
2	.538	.600	.707	2	.490	.550	.656	2	.450	.507	.612
3	.655+	.711	.802	3	.599	.655+	.750	3	.552	.607	.703
4	.760	.807	.879	4	.699	.749	.829	4	.646	.696	.782
5	.853	.889	.939	5	.790	.831	.895-	5	.733	.778	.850
6	.931	.954	.980	6	.871	.902	.947	6	.812	.850	.907
7	.987	.994	.999	7	.939	.959	.983	7	.884	.913	.952
				8	.988	.994	.999	8	.945+	.963	.984
								9	.990	.995-	.999
	N = 11				N = 12				N = 13		
0	.189	.238	.342	0	.175-	.221	.319	0	.162	.206	.298
1	.310	.364	.470	1	.287	.339	.440	1	.260	.316	.413
2	.415+	.470	.572	2	.386	.438	.537	2	360	.410	.506
3	.511	.564	.660	3	.475+	.527	.622	3	.444	.495-	.588
4	.599	.650	.738	4	.559	.609	.698	4	[illegible]	.573	.661
5	.682	.729	[illegible]	5	.638	.685-	.765+	5	.598	.645+	.727
6	.759	.800	.866	6	.712	.755-	.825+	6	.669	.713	.787
7	.831	.865-	.916	7	.781	.819	.879	7	.736	.776	.841
8	.895+	.921	.957	8	.846	.877	.924	8	.799	.834	.889
9	.951	.967	.986	9	.904	.928	.961	9	.858	.887	.931
10	.990	.995+	.999	10	.955-	.970	.987	10	.912	.934	.964
				11	.991	.996	.999	11	.958	.972	.988
								12	.992	.996	.999

Notes:

1. If a 1-sided lower s-confidence limit is desired, use $N-r$ instead of r, $\bar{p}$ instead of p and $\bar{C}$ instead of C, and switch the inequality. See the example.
2. The +, − after a 5 indicates which way the number can be rounded to fewer decimal places.

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TABLE 3-2 (Continued)

N = 14			
0	.152	.193	.280
1	.251	.297	.389
2	.337	.385+	.478
3	.417	.466	.557
4	.492	.540	.627
5	.563	.610	.692
6	.631	.675-	.751
7	.695+	.736	.805+
8	.757	.794	.854
9	.815-	.847	.898
10	.869	.896	.936
11	.919	.939	.967
12	.961	.974	.989
13	.993	.996	.999

N = 15			
0	.142	.181	.264
1	.236	.279	.368
2	.317	.363	.453
3	.393	.440	.529
4	.464	.511	.597
5	.532	.577	.660
6	.596	.640	.718
7	.658	.700	.771
8	.718	.756	.821
9	.774	.809	.865+
10	.828	.858	.906
11	.878	.903	.941
12	.924	.943	.969
13	.964	.976	.990
14	.993	.997	.999

N = 16			
0	.134	.171	.250
1	.222	.264	.349
2	.300	.344	.430
3	.371	.417	.503
4	.439	.484	.569
5	.504	.548	.630
6	.565+	.609	.687
7	.625-	.667	.739
8	.682	.721	.788
9	.737	.773	.834
10	.790	.822	.875-
11	.839	.868	.912
12	.886	.910	.945-
13	.929	.947	.971
14	.966	.977	.990
15	.993	.997	.999

N = 17			
0	.127	.162	.237
1	.210	.250	.332
2	.284	.326	.410
3	.352	.396	.480
4	.416	.461	.543
5	.478	.522	.603
6	.537	.580	.658
7	.594	.636	.709
8	.650	.689	.758
9	.703	.740	.803
10	.754	.788	.845-
11	.803	.834	.883
12	.849	.876	.918
13	.893	.915+	.948
14	.933	.950	.973
15	.968	.979	.991
16	.994	.997	.999

N = 18			
0	.120	.153	.226
1	.199	.238	.316
2	.269	.310	.391
3	.334	.377	.458
4	.396	.439	.520
5	.455+	.498	.577
6	.512	.554	.631
7	.567	.608	.681
8	.620	.659	.729
9	.671	.709	.774
10	.721	.756	.816
11	.769	.801	.855-
12	.815-	.844	.890
13	.858	.884	.923
14	.899	.920	.951
15	.937	.953	.975-
16	.970	.980	.992
17	.994	.997	.999

N = 19			
0	.114	.146	.215+
1	.190	.226	.302
2	.257	.296	.374
3	.319	.359	.439
4	.378	.419	.498
5	.434	.476	.554
6	.489	.530	.606
7	.541	.582	.655+
8	.592	.632	.702
9	.642	.680	.746
10	.690	.726	.788
11	.737	.770	.827
12	.782	.812	.863
13	.825-	.853	.897
14	.866	.890	.927
15	.905-	.925-	.954
16	.941	.956	.976
17	.972	.981	.992
18	.994	.997	.999

N = 20			
0	.109	.139	.206
1	.181	.216	.289
2	.245-	.283	.358
3	.304	.344	.421
4	.361	.401	.478
5	.415-	.456	.532
6	.467	.508	.583
7	.518	.558	.631
8	.567	.606	.677
9	.615+	.653	.720
10	.662	.698	.761
11	.707	.741	.800
12	.751	.783	.837
13	.793	.823	.871
14	.834	.860	.902
15	.873	.896	.931
16	.910	.929	.956
17	.944	.958	.977
18	.973	.982	.992
19	.995-	.997	.999

N = 21			
0	.104	.133	.197
1	.173	.207	.277
2	.234	.271	.344
3	.291	.329	.404
4	.345+	.384	.460
5	.397	.437	.512
6	.448	.487	.561
7	.497	.536	.608
8	.544	.583	.653
9	.590	.628	.695+
10	.636	.672	.736
11	.679	.714	.774
12	.722	.755+	.811
13	.764	.794	.845+
14	.804	.832	.878
15	.842	.868	.908
16	.879	.901	.935-
17	.914	.932	.959
18	.946	.960	.978
19	.974	.983	.993
20	.995-	.998	1.000

N = 22			
0	.099	.127	.189
1	.166	.198	.266
2	.224	.259	.330
3	.279	.316	.389
4	.331	.369	.443
5	.381	.420	.493
6	.430	.468	.541
7	.477	.515+	.587
8	.523	.561	.630
9	.568	.605-	.672
10	.611	.647	.712
11	.654	.689	.750
12	.695+	.729	.786
13	.736	.767	.821
14	.775+	.804	.853
15	.813	.840	.884
16	.850	.874	.912
17	.885+	.906	.938
18	.918	.935+	.961
19	.949	.962	.979
20	.976	.984	.993
21	.995+	.998	1.000

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TABLE 3-2 (Continued)

r	90%	95%	99%	r	90%	95%	99%	r	90%	95%	99%
N = 23				N = 24				N = 25			
0	.095+	.122	.181	0	.091	.117	.175-	0	.088	.133	.168
1	.159	.190	.256	1	.153	.183	.246	1	.147	.176	.237
2	.215+	.249	.318	2	.207	.240	.307	2	.199	.231	.296
3	.268	.304	.374	3	.258	.292	.361	3	.248	.282	.349
4	.318	.355-	.427	4	.306	.342	.412	4	.295-	.330	.398
5	.366	.404	.476	5	.352	.389	.460	5	.340	.375+	.444
6	.413	.451	.522	6	.398	.435-	.505-	6	.383	.420	.488
7	.459	.496	.567	7	.442	.479	.548	7	.426	.462	.531
8	.503	.540	.609	8	.484	.521	.590	8	.467	.504	.571
9	.546	.583	.650	9	.526	.563	.630	9	.508	.544	.610
10	.589	.625-	.689	10	.567	.603	.668	10	.548	.583	.648
11	.630	.665-	.727	11	.608	.642	.705-	11	.587	.621	.684
12	.670	.704	.763	12	.647	.681	.740	12	.625-	.659	.719
13	.710	.742	.797	13	.685+	.718	.774	13	.662	.695-	.752
14	.748	.778	.829	14	.723	.754	.806	14	.699	.730	.784
15	.786	.814	.860	15	.759	.788	.837	15	.735-	.764	.815+
16	.822	.848	.889	16	.795+	.822	.867	16	.770	.798	.845+
17	.857	.880	.916	17	.830	.854	.894	17	.804	.830	.873
18	.890	.910	.941	18	.863	.885+	.920	18	.837	.861	.899
19	.922	.938	.962	19	.895+	.914	.943	19	.869	.890	.923
20	.951	.963	.980	20	.925+	.941	.964	20	.899	.918	.946
21	.977	.984	.993	21	.953	.965+	.981	21	.928	.943	.966
22	.995+	.998	1.000	22	.978	.985-	.994	22	.955+	.966	.982
				23	.996	.998	1.000	23	.979	.986	.994
								24	.996	.998	1.000

r	90%	95%	99%	r	90%	95%	99%	r	90%	95%	99%
N = 26				N = 27				N = 28			
0	.085-	.109	.162	0	.082	.105+	.157	0	.079	.101	.152
1	.142	.170	.229	1	.137	.164	.222	1	.132	.159	.215
2	.192	.223	.286	2	.185+	.215+	.277	2	.179	.208	.268
3	.239	.272	.337	3	.231	.263	.326	3	.223	.254	.316
4	.284	.318	.385-	4	.275-	.308	.373	4	.265+	.298	.361
5	.328	.363	.430	5	.317	.351	.417	5	.306	.339	.404
6	.370	.405+	.473	6	.358	.392	.458	6	.346	.380	.445-
7	.411	.447	.514	7	.397	.432	.498	7	.385-	.419	.484
8	.451	.487	.55[illegible]	8	.436	.471	.537	8	.422	.457	.521
9	.491	.526	.592	9	.475-	.509	.574	9	.459	.494	.558
10	.529	.564	.628	10	.512	.547	.610	10	.496	.530	.593
11	.567	.602	.664	11	.549	.583	.645+	11	.532	.565+	.627
12	.604	.638	.698	12	.585-	.618	.679	12	.5[illegible]	.600	.660
13	.641	.673	.731	13	.620	.653	.711	13	.601	.634	.692
14	.676	.708	.763	14	.655+	.687	.743	14	.635+	.667	.723
15	.711	.742	.794	15	.689	.720	.773	15	.669	.699	.753
16	.746	.774	.823	16	.723	.752	.802	16	.701	.731	.782
17	.779	.806	.851	17	.756	.783	.831	17	.733	.762	.810
18	.812	.837	.878	18	.788	.814	.857	18	.765-	.792	.837
19	.843	.866	.903	19	.819	.843	.883	19	.796	.821	.863
20	.874	.894	.927	20	.849	.871	.907	20	.826	.849	.888
21	.903	.921	.948	21	.879	.899	.930	21	.855+	.876	.911
22	.931	.946	.967	22	.907	.924	.950	22	.883	.902	.932
23	.957	.968	.983	23	.934	.948	.968	23	.911	.927	.952
24	.979	.986	.994	24	.958	.969	.983	24	.936	.950	.969
25	.996	.998	1.000	25	.980	.987	.994	25	.960	.970	.984
				26	.996	.998	1.000	26	.981	.987	.995-
								27	.996	.998	1.000

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TABLE 3-2 (Continued)

r	90%	95%	99%	r	90%	95%	99%
	N = 29				N = 30		
0	.076	.098	.147	0	.074	.095+	.142
1	.128	.153	.208	1	.124	.149	.202
2	.173	.202	.260	2	.168	.195+	.252
3	.216	.246	.307	3	.209	.239	.298
4	.257	.288	.350	4	.249	.280	.340
5	.297	.329	.392	5	.287	.319	.381
6	.335-	.368	.432	6	.325-	.357	.420
7	.372	.406	.470	7	.361	.394	.457
8	.409	.443	.507	8	.397	.430	.493
9	.445+	.479	.542	9	.432	.465+	.527
10	.481	.514	.577	10	.466	.499	.561
11	.515+	.549	.610	11	.500	.533	.594
12	.550	.583	.643	12	.533	.566	.626
13	.583	.616	.674	13	.566	.598	.657
14	.616	.648	.705-	14	.599	.630	.667
15	.649	.680	.734	15	.630	.661	.716
16	.681	.711	.763	16	.662	.692	.744
17	.712	.741	.791	17	.692	.721	.772
18	.743	.771	.818	18	.723	.750	.799
19	.774	.800	.843	19	.752	.779	.824
20	.803	.828	.868	20	.782	.807	.849
21	.832	.855-	.892	21	.810	.834	.873
22	.860	.881	.914	22	.838	.860	.896
23	.888	.906	.935-	23	.865+	.885+	.917
24	.914	.930	.954	24	.891	.907	.937
25	.938	.951	.970	25	.917	.932	.955+
26	.961	.971	.985-	26	.941	.953	.972
27	.982	.988	.995-	27	.963	.972	.985+
28	.996	.998	1.000	28	.982	.988	.995-
				29	.996	.998	1.000

3-3 is a copy of pages 10, 579, & 580, Ref. 1, and is used to illustrate the procedure of finding p_L^* and p_U^*.

A handy random number generator is a coin, flipped several times. Decide whether heads is to be 0 or 1; tails is the reverse. Then multiply the result of the first flip by 0.5, the second flip by 0.25, the third by 0.125, etc. (the numbers are $2^{-\text{flip}}$), as fine as desired. Then add the numbers. Usually 5 or 6 flips give a sufficiently continuous random variable. (For example, heads is 0, tails is 1; the sequence is H, T, H, H, T, H. Add 0.25 + 0.03125 = 0.28125; truncate to 0.281 for convenience.)

Use the example in Table 3-2: N = 10, r = 4, C_U = 90%, C_L = 10%. It is shown in Table 3-2 that p_U = 0.646, p_L = 0.188. It is shown just following Eq. 3-9c that p'_U = 0.552, p_L 0.267. Suppose the random number is η = 0.28125 (same as in the example in the paragraph immediately above). Linear interpolation (applied several times) will be used to solve Eq. 3-10. The forms of Eq. 3-10 using the rightmost expressions are most suitable for using Table 3-3. They are written as

$$f_L(x_L) \equiv \frac{binfc(r, x_L, N) - C_L}{binfc(r, x_L, N) - binfc(r+1, x_L, N)} - \eta \geq 0 \qquad (3\text{-}11a)$$

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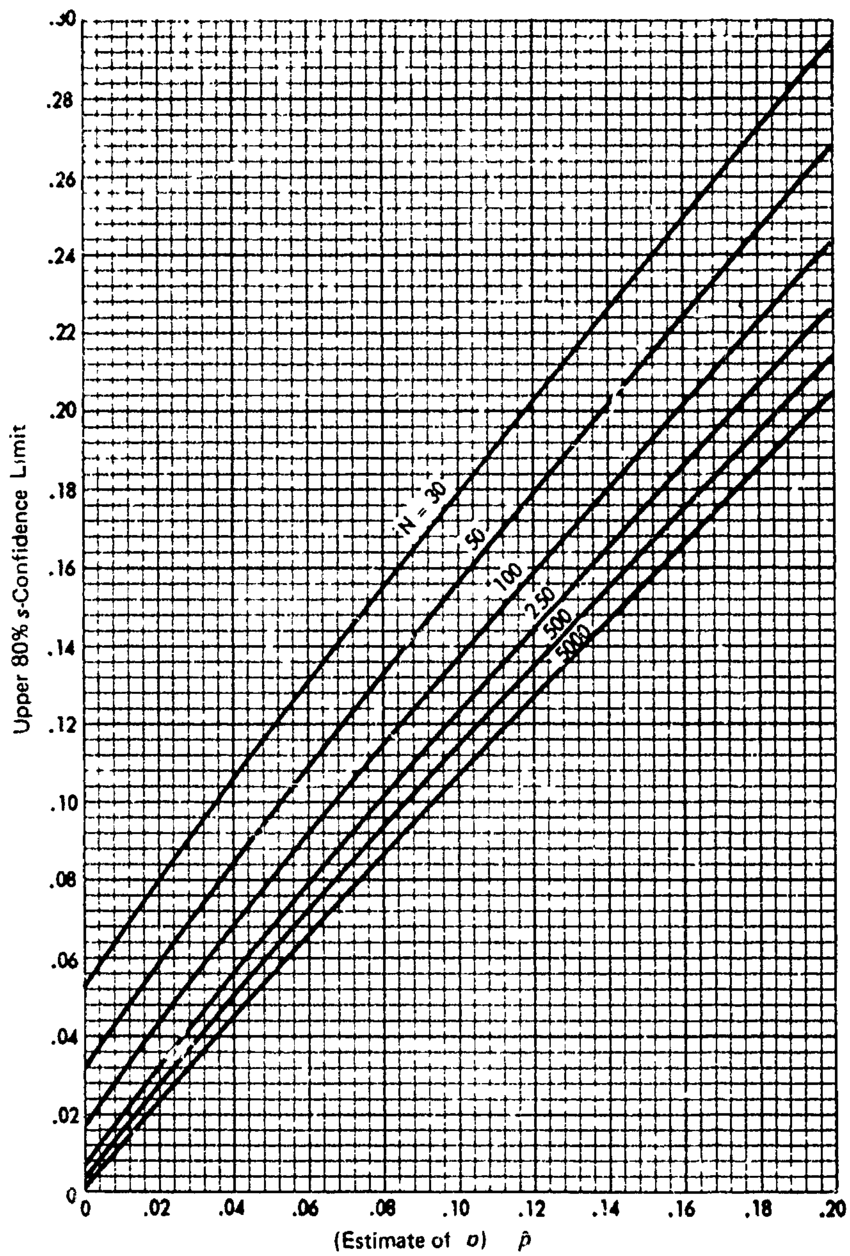

Figure 3-1(A). 1-sided Upper s-Confidence Limit (80%) for p (adapted from Ref. 7)

The graph gives the 1-sided upper *s*-confidence limit for p where

p = probability of occurrence of labeled event
r = number of such events in N trials, the random variable
N = number of trials in which r events occurred, fixed
$\hat{p}$ = r/N

Note. If a 1-sided lower *s*-confidence limit is desired, use $N - r$ instead of r, $\bar{p}$ instead of p, $\bar{C}$ instead of C, and switch the inequality. See the example in Table 3-2.

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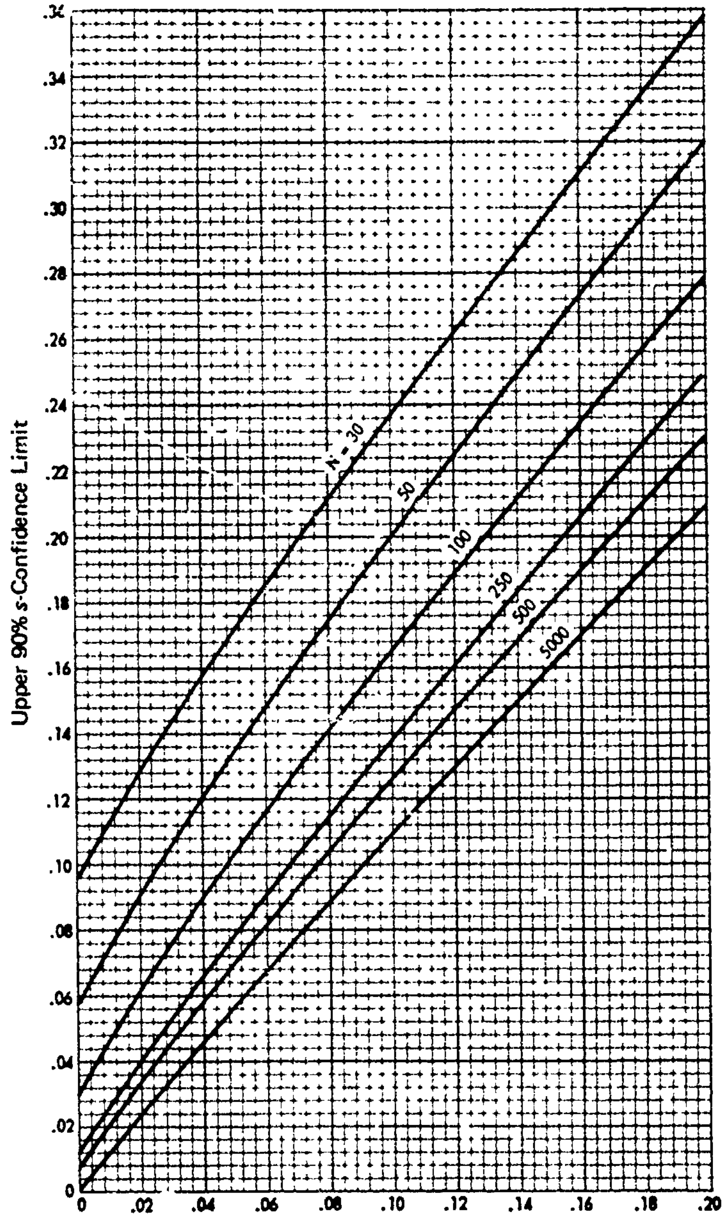

Figure 3-1(B). 1-sided Upper s-Confidence Limit (90%) for p (adapted from Ref. 7)

The graph gives the 1-sided upper *s*-confidence limit for p where

p = probability of occurrence of labeled event
r = number of such events in N trials, the random variable
N = number of trials in which r events occurred, fixed
$\hat{p}$ = r/N

Note: If a 1-sided lower *s*-confidence limit is desired, use $N-r$ instead of r, $\bar{p}$ instead of p, $\bar{C}$ instead of C, and switch the inequality. See the example in Table 3-2.

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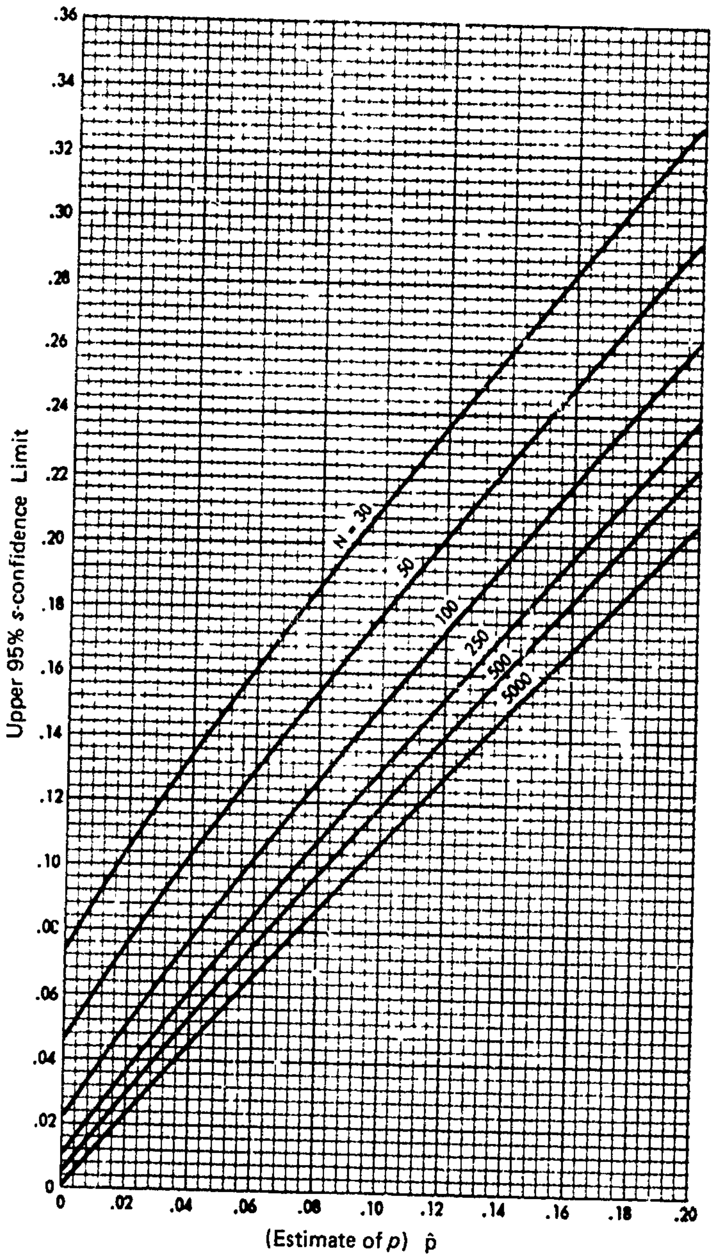

Figure 3-1(C). 1-sided Upper s-Confidence Limit (95%) for p *(adapted from Ref. 7)*

The graph gives the 1-sided upper *s*-confidence limit for p where

p = probability of occurrence of labeled event
r = number of such events in N trials, the random variable
N = number of trials in which r events occurred, fixed
$\hat{p}$ = r/N

Note: If a 1-sided lower *s*-confidence limit is desired, use $N-r$ instead of r, $\bar{p}$ instead of p, $\bar{C}$ instead of C, and switch the inequality. See the example in Table 3-2.

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$$f_U(x_U) \equiv \frac{binfc(N-r, x_U, N) - \bar{C}_U}{binfc([N-r], x_U, N) - binfc([N-r]+1, x_U, N)} - \eta = 0 \tag{3-11b}$$

The solution to Eq. 3-11a is $x_L = p_L^*$. The solution to Eq. 3-11b is $x_U = \bar{p}_U^*$.

The formula for linear interpolation is

$$x_{new} = \frac{x_- f_+ - x_+ f_-}{f_+ - f_-} \tag{3-12}$$

where x_- and x_+ are the smaller and larger values of x, respectively; and $f_+ \equiv f(x_+)$ and $f_- \equiv f(x_-)$.

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Procedure / **Example**

1. Solve Eq. 3-11a first. Use $x_- = p_L$ and $x_+ = p_L'$. The values of f are known from the definition of p_L, p_L'. Use Eq. 3-12 to find x_{new}, round off to 2 decimal places. Solve Eq. 3-11a using $x = 0.21$.

1. $C_L = 0.10$

x	f
0.188	$-0.281 = (-\eta)$
0.267	$+0.719 = (1-\eta)$
0.210	– –
0.21	+0.113 (new)

2. Make a new chart, discarding the old pair (0.267, 0.719) with the same sign as f_{new}. Repeat the linear interpolation and round off.

2.

x	f
0.188	−0.281
0.21	+0.113
0.204	– –
0.20	−0.044 (new)

3. x is now isolated to be between 2 consecutive entries in the table. Repeat the linear interpolation but do not round off. The answer is $p_L^* = 0.203$.

3.

x	f
0.20	−0.044
0.21	+0.113
0.203	– –

4. Solve Eq. 3-11b next. Use $x_- = \bar{p}_U$, $x_+ = \bar{p}_U'$. Proceed as in Step 1. $N - r = 10 - 4 = 6$.

4. $\bar{C}_U = 0.10$

x	f
0.354	$-0.281 = (-\eta)$
0.478	$+0.719 = (1-\eta)$
0.389	– –
0.39	+0.208 (new)

5. Make new chart and repeat Step 1.

5.

x	f
0.354	−0.281
0.39	+0.208
0.3747	– –
0.37	−0.0394 (new)

6. Make new chart and repeat Step 1. Round "up", to bracket the true value.

6.

x	f
0.37	−0.0394
0.39	+0.208
0.373	– –
0.38	+0.0910 (new)

7. Repeat Step 3. The answer is $\bar{p}_U^* = 0.373$; $p_U^* = 0.627$.

7.

x	f
0.37	−0.0394
0.38	+0.0910
0.373	– –

8. Make the final s-confidence statement.

8. Conf $\{0.203 \leqslant p \leqslant 0.627\} = 80\%$

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TABLE 3-3

SAMPLE PAGE FROM A BINOMIAL DISTRIBUTION (Ref. 1)

$$binfc\,(c,p,N) = \sum_{i=c}^{N} \binom{N}{i} p_i \bar{p}^{\,N-i}$$

$N = 10$

c \ p	.01	.02	.03	.04	.05	.06	.07	.08
1	.0956179	.1829272	.2625759	.3351674	.4012631	.4613849	.5160177	.5656115
2	.0042662	.0161776	.0345066	.0581538	.0861384	.1175880	.1517299	.1878825
3	.0001138	.0008639	.0027649	.0062137	.0115036	.0188378	.0283421	.0400754
4	.0000020	.0000305	.0001471	.0004426	.0010285	.0020293	.0035761	.0058013
5		.0000007	.0000084	.0000218	.0000637	.0001517	.0003139	.0005857
6			.0000001	.0000007	.0000028	.0000079	.0000193	.0000416
7					.0000001	.0000003	.0000008	.0000020
8								.0000001

c \ p	.09	.10	.11	.12	.13	.14	.15	.16
1	.6105839	.6513216	.6881828	.7214990	.7515766	.7786984	.8031256	.8250988
2	.2254471	.2639011	.3027908	.3417250	.3803692	.4184400	.4557002	.4919536
3	.0540400	.0701908	.0884435	.1086818	.1307642	.1545298	.1798035	.2064005
4	.0088338	.0127952	.0177972	.0239388	.0313048	.0399642	.0499698	.0613577
5	.0010096	.0016349	.0025170	.0037161	.0052967	.0073263	.0098741	.0130101
6	.0000810	.0001469	.0002507	.0004069	.0006332	.0009505	.0013832	.0019593
7	.0000045	.0000091	.0000173	.0000308	.0000525	.0000856	.0001346	.0002051
8	.0000002	.0000004	.0000008	.0000015	.0000029	.0000051	.0000087	.0000142
9					.0000001	.0000002	.0000003	.0000006

c \ p	.17	.18	.19	.20	.21	.22	.23	.24
1	.8448396	.8625520	.8784233	.8926258	.9053172	.9166422	.9267332	.9357111
2	.5270412	.5608368	.5932435	.6241904	.6536289	.6815306	.7078843	.7326936
3	.2341305	.2628010	.2922204	.3222005	.3525586	.3831197	.4137173	.4441949
4	.0741472	.0883411	.1039261	.1208739	.1391418	.1586739	.1794024	.2012487
5	.0168038	.0213229	.0266325	.0327935	.0398624	.0478897	.0569196	.0669890
6	.0027098	.0036694	.0048757	.0063694	.0081935	.0103936	.0130167	.0161116
7	.0003042	.0004401	.0006229	.0008644	.0011783	.0015804	.0020885	.0027228
8	.0000226	.0000350	.0000528	.0000779	.0001127	.0001599	.0002232	.0003068
9	.0000010	.0000017	.0000027	.0000042	.0000064	.0000097	.0000143	.0000207
10			.0000001	.0000001	.0000002	.0000003	.0000004	.0000006

c \ p	.25	.26	.27	.28	.29	.30	.31	.32
1	.9436865	.9507601	.9570237	.9625609	.9674476	.9717525	.9755381	.9788608
2	.7559748	.7777550	.7980705	.8169646	.8344869	.8506917	.8656366	.8793821
3	.4744072	.5042200	.5335112	.5621710	.5901015	.6172172	.6434445	.6687212
4	.2241249	.2479349	.2725761	.2979405	.3239164	.3503893	.3772433	.4043626
5	.0781269	.0903542	.1036831	.1181171	.1336503	.1502683	.1679475	.1866554
6	.0197277	.0239148	.0287224	.0341994	.0403932	.0473490	.0551097	.0637149
7	.0035057	.0044618	.0056181	.0070039	.0086507	.0105921	.0128637	.0155029
8	.0004158	.0005562	.0007350	.0009605	.0012420	.0015904	.0020179	.0025384
9	.0000296	.0000416	.0000577	.0000791	.0001072	.0001437	.0001906	.0002505
10	.0000010	.0000014	.0000021	.0000030	.0000042	.0000059	.0000082	.0000113

c \ p	.33	.34	.35	.36	.37	.38	.39	.40
1	.9817716	.9843166	.9865373	.9884708	.9901507	.9916070	.9928666	.9939534
2	.8919901	.9035235	.9140456	.9236190	.9323056	.9401661	.9472594	.9536426
3	.6929966	.7162304	.7383926	.7594627	.7794292	.7982887	.8160453	.8327102
4	.4316390	.4589388	.4861730	.5132284	.5400038	.5664030	.5923361	.6177194
5	.2063514	.2269866	.2485045	.2708415	.2939277	.3176870	.3420385	.3668967
6	.0732005	.0835979	.0949341	.1072304	.1205026	.1347603	.1500068	.1662386
7	.0185489	.0220422	.0260243	.0305376	.0356252	.0413301	.0476949	.0547619
8	.0031673	.0039219	.0048213	.0058864	.0071403	.0086079	.0103163	.0122946
9	.0003265	.0004214	.0005399	.0006865	.0008668	.0010871	.0013546	.0016777
10	.0000153	.0000206	.0000276	.0000366	.0000481	.0000628	.0000814	.0001049

c \ p	.41	.42	.43	.44	.45	.46	.47	.48
1	.9948888	.9956920	.9963797	.9969669	.9974670	.9978917	.9982511	.9985544
2	.9593705	.9644958	.9690684	.9731358	.9767429	.9799319	.9827422	.9852109
3	.8483007	.8628393	.8763538	.8888757	.9004403	.9110859	.9208530	.9297839
4	.6424762	.6665372	.6898401	.7123307	.7339621	.7546952	.7744985	.7933480
5	.3921728	.4177749	.4436094	.4695813	.4955954	.5215571	.5473730	.5729517
6	.1834452	.2016092	.2207058	.2407033	.2615627	.2832382	.3056772	.3288205
7	.0625719	.0711643	.0805763	.0908427	.1019949	.1140612	.1270655	.1410272
8	.0145738	.0171871	.0201696	.0235583	.0273918	.0317105	.0365560	.0419713
9	.0020658	.0025295	.0030809	.0037335	.0045022	.0054040	.0064574	.0076828
10	.0001342	.0001708	.0002161	.0002720	.0003405	.0004242	.0005260	.0006493

c \ p	.49	.50
1	.9988096	.9990234
2	.9873722	.9892578
3	.9379222	.9453125
4	.8112268	.8281250
5	.5982047	.6230469
6	.3525028	.3769531
7	.1559607	.1718750
8	.0480003	.0546875
9	.0091028	.0107422
10	.0007979	.0009766

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Table 3-4 lists randomized 2-sided *s*-confidence limits which have special statistical properties. They are not equal-tailed *s*-confidence limits; they cannot be used separately for upper and lower *s*-confidence limits. See Ref. 8 for a more complete discussion. In general, the *s*-confidence limits in Table 3-4 will be different from those calculated using the methods in this chapter. It is difficult to say that one set is better than the other except in the narrow statistical sense stated for Table 3-4.

3-6 CHOOSING A *s*-CONFIDENCE LEVEL

Choosing an appropriate *s*-confidence level is always troublesome. Suppose the labeled events are failures; so p is the probability of failure and $\bar{p}$ is the *s*-reliability. There is obviously little point in having a very high *s*-confidence that the *s*-reliability is very low, or a very low *s*-confidence that the *s*-reliability is very high. A reasonable compromise is to choose a *s*-confidence level which is approximately the fraction of success in the sample (unless that fraction is 100%).

If there are no failures, one can reasonably choose the *s*-confidence level equal to the minimum 1-sided lower *s*-confidence limit on *s*-reliability. Fig. 3-2 shows the graph for this situation. One could also use the Poisson approximation for this case.

3-7 EXAMPLES

3-7.1 EXAMPLE NO. 1

Ten tests were run with 1 failure. Find the 5% and 95% 1-sided *s*-confidence limits on the failure probability p; combine them for a 2-sided *s*-confidence statement.

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Procedure	Example
1. The labeled events are failures. State N, r, C_U, C_L, and $\hat{p} = r/N$ (by Eq. 3-7).	1. $N = 10, r = 1$ $C_U = 95\%, C_L = 5\%$ $\hat{p} = 0.10$
2. Use Table 3-2 to find p_U and p_L. Then make the *s*-confidence statements.	2. $p_U = 0.394$ $p_L = 1 - 0.995 = 0.005$ $\text{Conf}\{p < 0.394\} > 95\%$ $\text{Conf}\{p < 0.005\} < 5\%$ $\text{Conf}\{0.005 < p < 0.394\} = 90\%$
3. Use Table 3-2 to find p'_U and p'_L. Then make the *s*-confidence statements.	3. $p'_U = 0.259$ $p'_L = 1 - 0.963 = 0.037$ $\text{Conf}\{p < 0.259\} < 95\%$ $\text{Conf}\{p < 0.037\} > 5\%$ $\text{Conf}\{0.037 < p < 0.259\} < 90\%$
4. Again choose a random number by the coin flipping method. Take heads = 0, tails = 1. Use linear interpolation in Ref. 1 to find the exact randomized *s*-confidence interval. Use Table 3-3.	4. Result is H, H, T, T, T, H (unusual, but true) $\rightarrow 0.125 + 0.0625 + 0.03125 = 0.21875 \rightarrow 0.219 = \eta$. p^*_L lies between 0.006 and 0.007; $p^*_L = 0.0066$. p^*_U lies between 0.37 and 0.38; $p^*_U = 0.378$.
5. Make the exact *s*-confidence statements.	5. $\text{Conf}\{p < 0.0066\} = 5\%$ $\text{Conf}\{p < 0.378\} = 95\%$ $\text{Conf}\{0.0066 < p < 0.378\} = 90\%$

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TABLE 3-4(A)

NEYMAN-SHORTEST UNBIASED 95% *s*-CONFIDENCE INTERVALS FOR *p* (ADAPTED FROM Ref. 8)

$n = 2$		$n = 3$		$n = 4$		$n = 5$		$n = 6$		$n = 7$		$n = 8$		$n = 9$		$n = 10$		$r + \eta$
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	50	00	29	00	23	00	18	00	15	00	13	00	11	00	10	00	09	0·1
00	75	00	50	00	37	00	31	00	26	00	23	00	20	00	18	00	16	·2
00	83	00	59	00	45	00	38	00	32	00	28	00	25	00	22	00	20	·3
00	87	00	65	00	50	00	42	00	35	00	31	00	28	00	25	00	23	·4
00	90	00	68	00	54	00	44	00	38	00	34	00	30	00	27	00	25	·5
00	92	00	71	00	56	00	46	00	41	00	36	00	32	00	29	00	26	·6
00	93	00	73	00	59	00	48	00	42	00	37	00	33	00	30	00	27	·7
00	94	00	75	00	60	00	50	00	44	00	39	00	34	00	31	00	29	·8
00	94	00	76	00	62	00	51	00	45	00	40	00	35	00	32	00	30	·9
00	95	00	78	00	63	00	53	00	46	00	41	00	36	00	33	00	30	1·0
00	1·0	00	85	00	69	00	58	00	50	00	44	00	40	00	36	00	33	1·1
00	1·0	00	89	00	73	00	62	00	53	00	47	00	42	00	38	00	35	1·2
00	1·0	00	92	00	77	00	65	00	56	00	50	00	45	00	40	00	37	1·3
00	1·0	00	94	00	79	00	67	00	58	00	52	00	46	00	42	00	39	1·4
00	1·0	00	95	00	81	00	69	00	60	00	53	00	48	00	44	00	40	1·5
		00	96	00	83	00	71	00	61	00	55	00	49	00	45	00	41	1·6
		00	96	00	84	00	72	00	63	00	56	00	50	00	46	00	42	1·7
		00	97	00	85	00	73	00	64	00	57	00	51	00	47	00	43	1·8
		00	97	00	86	00	74	00	65	00	58	00	52	00	48	00	44	1·9
		03	97	02	86	01	75	01	66	01	58	01	53	01	48	01	44	2·0
				02	90	01	79	01	69	01	61	01	55	01	50	01	46	2·1
				02	93	02	81	01	71	01	63	01	57	01	52	01	48	2·2
				02	95	02	83	01	73	01	65	01	59	01	54	01	50	2·3
				03	96	02	85	02	75	01	67	01	60	01	55	01	51	2·4
				03	97	03	86	02	77	02	68	01	62	01	57	01	52	2·5
						03	88	02	78	02	69	02	63	02	58	01	53	2·6
						04	88	03	79	03	70	02	63	02	59	02	54	2·7
						05	89	04	80	03	71	03	64	03	59	02	55	2·8
						07	90	06	80	05	72	04	65	03	60	03	55	2·9
						10	90	08	81	06	73	05	66	05	61	04	56	3·0
								08	84	07	75	06	68	05	62	04	58	3·1
								09	86	07	77	06	70	05	64	05	59	3·2
								09	87	07	79	06	71	06	65	05	60	3·3
								10	88	08	80	07	73	06	66	05	62	3·4
								11	89	09	81	07	74	06	67	06	63	3·5
										09	82	08	75	07	68	06	63	3·6
										11	83	09	75	08	69	07	64	3·7
										12	84	10	76	09	70	08	65	3·8
										13	84	11	77	10	70	09	65	3·9
										15	85	13	77	11	71	10	66	4·0
												13	79	12	73	10	67	4·1
												14	81	12	74	11	69	4·2
												14	82	12	75	11	70	4·3
												15	83	13	77	11	71	4·4
												16	84	14	77	12	72	4·5
														14	78	13	72	4·6
														15	79	14	73	4·7
														16	80	14	74	4·8
														18	80	16	74	4·9
														19	81	17	75	5·0
																17	76	5·1
																18	78	5·2
																18	79	5·3
																19	79	5·4
																20	80	5·5

Notes:

1. The pairs of figures are lower and upper *s*-confidence limits for *p* given to 2 decimal places.
2. Notation
   - n = sample size (in place of N in the text)
   - r = number of labeled events
   - p = probability of labeled event
   - η = random number from the uniform distribution on (0, 1)
3. For tabular convenience, r, η is listed as $r + \eta$.

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TABLE 3-4(A) (Continued)

n = 11		n = 12		n = 13		n = 14		n = 15		n = 16		n = 17		n = 18		n = 19		r + η
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	01	00	0	0
00	08	00	08	00	07	00	07	00	06	00	06	00	05	00	05	00	05	0·1
00	15	00	14	00	13	00	12	00	11	00	10	00	10	00	09	00	09	·2
00	19	00	17	00	16	00	15	00	14	00	13	00	12	00	12	00	11	·3
00	21	00	19	00	18	00	17	00	16	00	15	00	14	00	13	00	13	·4
00	23	00	21	00	19	00	18	00	17	00	16	00	15	00	14	00	14	·5
00	24	00	22	00	21	00	19	00	18	00	17	00	16	00	15	00	15	·6
00	25	00	23	00	22	00	20	00	19	00	18	00	17	00	16	00	15	·7
00	26	00	24	00	23	00	21	00	20	00	19	00	18	00	17	00	16	·8
00	27	00	25	00	23	00	22	00	21	00	19	00	18	00	17	00	17	·9
00	28	00	26	00	24	00	22	00	21	00	20	00	19	00	18	00	17	1·0
00	32	00	30	00	28	00	26	00	25	00	23	00	22	00	21	00	20	1·2
00	36	00	33	00	31	00	29	00	27	00	26	00	24	00	23	00	22	1·4
00	38	00	35	00	33	00	31	00	29	00	27	00	26	00	25	00	23	1·6
00	40	00	37	00	34	00	32	00	30	00	29	00	27	00	26	00	25	1·8
01	41	00	38	00	36	00	34	00	32	00	30	00	28	00	27	00	26	2·0
01	45	01	41	01	39	00	36	00	34	00	32	00	31	00	29	00	28	2·2
01	47	01	44	01	41	01	39	01	36	01	34	01	32	00	31	00	29	2·4
01	49	01	46	01	43	01	40	01	38	01	36	01	34	01	32	01	31	2·6
02	51	02	47	02	44	02	42	01	39	01	37	01	35	01	34	01	32	2·8
04	52	03	49	03	45	03	43	03	40	02	38	02	36	02	34	02	33	3·0
04	55	04	51	03	48	03	45	03	43	03	40	03	38	02	36	02	35	3·2
05	57	04	53	04	50	04	47	03	44	03	42	03	40	03	38	03	36	3·4
06	59	05	55	05	52	04	49	04	46	04	44	03	41	03	39	03	38	3·6
07	60	06	56	06	53	05	50	05	47	04	45	04	42	04	40	04	39	3·8
09	61	08	58	07	54	07	51	06	48	06	46	05	43	05	41	05	39	4·0
09	64	08	60	08	56	07	53	07	50	06	48	06	45	05	43	05	41	4·2
10	66	09	62	08	58	08	55	07	52	07	49	06	47	06	45	06	43	4·4
11	68	10	63	09	60	09	56	08	53	07	51	07	48	06	46	06	44	4·6
13	69	12	65	11	61	10	57	09	54	08	52	08	49	07	47	07	45	4·8
15	70	14	66	12	62	11	58	10	55	10	53	09	50	08	48	08	46	5·0
16	72	14	68	13	64	12	60	11	57	10	54	10	52	09	49	08	47	5·2
17	74	15	70	14	66	13	62	12	59	11	56	10	53	10	51	09	48	5·4
18	76	16	71	15	67	14	63	13	60	12	57	11	54	10	52	10	50	5·6
20	77	18	72	16	68	15	64	14	61	13	58	12	55	11	53	11	50	5·8
22	78	20	73	18	69	17	65	15	62	14	59	13	56	12	54	12	51	6·0
		21	75	19	71	17	67	16	64	15	61	14	58	13	55	12	53	6·2
		22	77	20	73	18	69	17	65	16	62	14	59	14	56	13	54	6·4
		23	78	21	74	19	70	18	66	16	63	15	60	14	58	14	55	6·6
				23	75	21	71	19	67	18	64	16	61	15	58	14	56	6·8
				24	76	22	72	21	68	19	65	18	62	17	59	16	57	7·0
						23	74	21	70	20	67	18	63	17	61	16	58	7·2
						24	75	22	71	21	68	19	65	18	62	17	59	7·4
						25	76	23	72	21	69	20	66	19	63	18	60	7·6
								24	73	23	70	21	67	20	64	18	61	7·8
								26	74	24	71	22	68	21	65	20	62	8·0
										25	72	23	69	21	66	20	63	8·2
										26	73	24	70	22	67	21	64	8·4
										27	74	25	71	23	68	22	65	8·6
												26	72	24	69	23	66	8·8
												27	73	26	70	24	67	9·0
														26	71	25	68	9·2
														27	72	25	69	9·4
														28	73	26	70	9·6
																27	71	9·8
																28	72	10·0

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TABLE 3-4(A) (Continued)

$n = 20$		$n = 21$		$n = 22$		$n = 23$		$n = 24$		$n = 26$		$n = 28$		$n = 30$		$n = 32$		$r + \eta$
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	05	00	04	00	04	00	04	00	04	00	04	00	03	00	03	00	03	0·1
00	08	00	08	00	08	00	07	00	07	00	07	00	06	00	06	00	05	·2
00	11	00	10	00	10	00	09	00	09	00	08	00	08	00	07	00	07	·3
00	12	00	11	00	11	00	10	00	10	00	09	00	09	00	08	00	08	·4
00	13	00	12	00	12	00	11	00	11	00	10	00	09	00	09	00	08	·5
00	14	00	13	00	13	00	12	00	12	00	11	00	10	00	09	00	09	·6
00	15	00	14	00	13	00	13	00	12	00	11	00	11	00	10	00	09	·7
00	15	00	15	00	14	00	13	00	13	00	12	00	11	00	10	00	10	·8
00	16	00	15	00	14	00	14	00	13	00	12	00	12	00	11	00	10	·9
00	16	00	16	00	15	00	14	00	14	00	13	00	12	00	11	00	10	1·0
00	19	00	18	00	17	00	17	00	16	00	15	00	14	00	13	00	12	1·2
00	21	00	20	00	19	00	18	00	18	00	16	00	15	00	14	00	13	1·4
00	22	00	21	00	20	00	20	00	19	00	18	00	16	00	15	00	14	1·6
00	24	00	23	00	22	00	21	00	20	00	19	00	17	00	16	00	15	1·8
00	24	00	23	00	22	00	22	00	21	00	19	00	18	00	17	00	16	2·0
00	27	00	25	00	24	00	23	00	22	00	21	00	20	00	18	00	17	2·2
00	28	00	27	00	26	00	25	00	24	00	22	00	21	00	19	00	18	2·4
01	29	01	28	01	27	01	26	01	25	00	23	00	22	00	20	00	19	2·6
01	31	01	29	01	28	01	27	01	26	01	24	01	23	01	21	01	20	2·8
02	31	02	30	02	29	02	28	02	27	01	25	01	23	01	22	01	20	3·0
02	33	02	32	02	31	02	29	02	28	02	26	01	25	01	23	01	22	3·2
02	35	02	33	02	32	02	31	02	29	02	27	02	26	02	24	01	23	3·4
03	36	03	34	03	33	02	32	02	31	02	28	02	27	02	25	02	23	3·6
03	37	03	35	03	34	03	33	03	31	03	29	02	27	02	26	02	24	3·8
04	38	04	36	04	35	04	33	04	32	03	30	03	28	03	26	03	25	4·0
05	39	05	38	04	36	04	35	04	34	04	31	03	29	03	27	03	26	4·2
05	41	05	39	05	37	04	36	04	35	04	32	04	30	03	28	03	27	4·4
06	42	05	40	05	39	05	37	05	36	04	33	04	31	04	29	03	28	4·6
07	43	06	41	06	39	06	38	05	36	05	34	05	32	04	30	04	28	4·8
08	44	07	42	07	40	06	39	06	37	06	35	05	32	05	31	05	29	5·0
08	45	08	43	07	42	07	40	07	39	06	36	06	34	05	32	05	30	5·2
08	46	08	45	08	43	07	41	07	40	06	37	06	35	05	33	05	31	5·4
09	47	09	46	08	44	08	42	07	41	07	38	06	35	06	33	05	31	5·6
10	48	09	46	09	45	09	43	08	41	07	39	07	36	06	34	06	32	5·8
11	49	10	47	10	45	09	44	09	42	08	39	08	37	07	35	07	33	6·0
12	52	12	50	11	48	11	46	10	45	09	42	09	39	08	37	07	35	6·5
15	54	14	52	13	50	13	48	12	47	11	44	10	41	09	38	09	36	7·0
16	57	15	55	15	53	14	51	13	49	12	46	11	43	10	41	10	38	7·5
18	59	17	57	17	55	16	53	15	51	14	48	13	45	12	42	11	40	8·0
20	62	19	60	18	58	17	56	16	54	15	50	14	47	13	44	12	42	8·5
23	64	21	62	20	60	19	57	18	55	17	52	15	49	14	46	13	43	9·0
24	67	23	64	22	62	21	60	20	58	18	54	17	51	15	48	14	45	9·5
27	69	25	66	24	64	23	62	22	60	20	56	18	52	17	49	16	47	10·0
28	72	27	69	25	66	24	64	23	62	21	58	19	55	18	51	17	49	10·5
		29	71	28	68	26	66	25	64	23	60	21	56	20	53	18	50	11·0
				29	71	28	68	27	66	24	62	22	58	21	55	19	52	11·5
						30	70	29	67	26	63	24	60	22	56	21	53	12·0
								30	70	28	65	25	62	23	58	22	55	12·5
								33	71	30	67	27	63	25	60	23	56	13·0
										31	69	29	65	26	61	25	58	13·5
										33	70	30	66	28	63	26	59	14·0
												32	68	29	65	27	61	14·5
												34	70	31	66	29	63	15
														34	69	32	65	16
																35	68	17

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TABLE 3-4(A) (Continued)

n = 34		n = 36		n = 38		n = 40		n = 42		n = 44		n = 46		n = 48		n = 50		r + η
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	05	00	05	00	05	00	04	00	04	00	04	00	04	00	04	00	03	0·2
00	07	00	07	00	06	00	06	00	06	00	06	00	05	00	05	00	05	·4
00	08	00	08	00	08	00	07	00	07	00	07	00	06	00	06	00	06	·6
00	09	00	09	00	08	00	08	00	08	00	07	00	07	00	07	00	06	·8
00	10	00	09	00	09	00	08	00	08	00	08	00	07	00	07	00	07	1·0
00	12	00	11	00	10	00	10	00	09	00	09	00	09	00	08	00	08	1·2
00	13	00	12	00	11	00	11	00	10	00	10	00	10	00	09	00	09	1·4
00	14	00	13	00	12	00	12	00	11	00	11	00	10	00	10	00	09	1·6
00	14	00	14	00	13	00	12	00	12	00	11	00	11	00	10	00	10	1·8
00	15	00	14	00	13	00	13	00	12	00	12	00	11	00	11	00	10	2·0
00	18	00	17	00	16	00	15	00	15	00	14	00	13	00	13	00	12	2·5
01	19	01	18	01	17	01	17	01	16	01	15	01	15	01	14	01	13	3·0
01	22	01	21	01	20	01	19	01	18	01	17	01	16	01	16	01	15	3·5
03	23	02	22	02	21	02	20	02	19	02	18	02	18	02	17	02	16	4·0
03	26	03	24	03	23	03	22	02	21	02	20	02	19	02	19	02	18	4·5
04	27	04	26	04	25	04	23	03	22	03	21	03	21	03	20	03	19	5·0
05	29	05	28	04	27	04	25	04	24	04	23	04	22	03	21	03	21	5·5
06	31	06	29	05	28	05	27	05	25	05	24	04	23	04	22	04	22	6·0
07	33	07	31	06	30	06	28	06	27	05	26	05	25	05	24	05	23	6·5
08	34	08	33	07	31	07	30	07	28	06	27	06	26	06	25	05	24	7·0
09	36	09	35	08	33	08	31	07	30	07	29	07	28	06	27	06	26	7·5
10	38	10	36	09	34	09	33	08	31	08	30	06	29	07	28	07	27	8·0
11	40	10	38	10	36	09	34	09	33	08	31	08	30	08	29	07	28	8·5
12	41	12	39	11	37	10	36	10	34	09	33	09	31	09	30	08	29	9·0
13	43	13	41	12	39	11	37	11	36	10	34	10	33	09	31	09	30	9·5
15	44	14	42	13	40	12	38	12	37	11	35	11	34	10	33	10	31	10·0
16	46	15	44	14	42	13	40	12	38	12	37	11	35	11	34	10	33	10·5
17	47	16	45	15	43	14	41	14	39	13	38	12	36	12	35	11	34	11·0
18	49	17	47	16	45	15	43	14	41	14	39	13	38	12	36	12	35	11·5
19	51	18	48	17	46	16	44	15	42	15	40	14	39	13	37	13	36	12·0
20	52	19	50	18	47	17	45	16	43	15	42	15	40	14	39	13	37	12·5
22	54	21	51	19	49	18	47	17	45	16	43	16	41	15	40	14	38	13·0
23	55	22	53	20	50	19	48	18	46	17	44	16	42	16	41	15	39	13·5
24	57	23	54	22	51	20	49	19	47	18	45	18	43	17	42	16	40	14·0
25	58	24	55	22	53	21	51	20	48	19	47	18	45	17	43	17	42	14·5
27	59	25	57	24	54	22	52	21	50	20	48	19	46	18	44	18	42	15·0
28	61	26	58	25	56	23	53	22	51	21	49	20	47	19	45	18	44	15·5
30	62	28	59	26	57	25	54	23	52	22	50	21	48	20	46	19	45	16·0
31	64	29	61	27	58	26	56	24	53	23	51	22	49	21	47	20	46	16·5
32	65	30	62	28	59	27	57	25	54	24	52	23	50	22	48	21	47	17·0
33	67	31	64	29	31	28	58	26	56	25	54	24	52	23	50	22	48	17·5
35	68	32	65	31	62	29	59	28	57	26	55	25	52	24	51	23	49	18·0
		34	66	32	63	30	61	28	58	27	56	26	54	25	52	23	50	18·5
		35	67	33	64	31	62	30	59	28	57	27	55	26	53	24	51	19·0
				34	66	32	63	31	60	29	58	28	56	26	54	25	52	19·5
				36	67	34	64	32	61	30	59	29	57	27	55	26	53	20·0
						35	65	33	63	31	60	30	58	28	56	27	54	20·5
						36	66	34	64	32	61	31	59	29	57	28	55	21·0
								35	65	33	62	32	60	30	58	29	56	21·5
								36	66	34	63	33	61	31	59	30	57	22·0
										35	65	34	62	32	60	31	58	22·5
										37	66	35	63	33	61	32	59	23
												37	65	35	63	34	61	24
														37	65	35	63	25
																37	65	26

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TABLE 3-4(B)

NEYMAN-SHORTEST UNBIASED 99% *s*-CONFIDENCE INTERVALS FOR *p* (ADAPTED FROM Ref. 8)

$n = 2$		$n = 3$		$n = 4$		$n = 5$		$n = 6$		$n = 7$		$n = 8$		$n = 9$		$n = 10$		$r + \eta$
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	90	00	68	00	54	00	44	00	37	00	33	00	29	00	26	00	24	0·1
00	95	00	78	00	63	00	53	00	45	00	40	00	36	00	32	00	30	·2
00	97	00	82	00	68	00	57	00	49	00	44	00	39	00	36	00	33	·3
00	98	00	84	00	71	00	60	00	52	00	46	00	42	00	38	00	35	·4
00	98	00	86	00	73	00	62	00	54	00	48	00	43	00	39	00	36	·5
00	98	00	87	00	74	00	64	00	56	00	49	00	45	00	41	00	38	·6
00	99	00	88	00	76	00	66	00	57	00	51	00	46	00	42	00	39	·7
00	99	00	89	00	77	00	67	00	58	00	52	00	47	00	43	00	39	·8
00	99	00	89	00	78	00	68	00	59	00	53	00	48	00	44	00	40	·9
00	99	00	90	00	78	00	68	00	60	00	54	00	49	00	44	00	41	1·0
00	1·0	00	98	00	84	00	74	00	65	00	58	00	52	00	48	00	44	1·1
00	1·0	00	98	00	88	00	77	00	68	00	61	00	55	00	51	00	47	1·2
00	1·0	00	98	00	90	00	79	00	71	00	63	00	57	00	52	00	48	1·3
00	1·0	00	99	00	91	00	81	00	72	00	65	00	59	00	54	00	50	1·4
00	1·0	00	99	00	92	00	82	00	74	00	66	00	60	00	55	00	51	1·5
		00	99	00	92	00	83	00	75	00	67	00	61	00	56	00	52	1·6
		00	99	00	93	00	84	00	76	00	68	00	62	00	57	00	53	1·7
		00	99	00	93	00	85	00	77	00	69	00	63	00	58	00	54	1·8
		00	99	00	94	00	85	00	77	00	70	00	64	00	58	00	54	1·9
		01	99	00	94	00	86	00	78	00	71	00	64	00	59	00	55	2·0
				00	97	00	89	00	81	00	74	00	67	00	62	00	57	2·1
				00	98	00	91	00	83	00	76	00	69	00	64	00	59	2·2
				00	99	00	93	00	85	00	77	00	71	00	65	00	60	2·3
				01	99	00	93	00	86	00	79	00	72	00	66	00	61	2·4
				01	99	01	94	00	87	00	80	00	73	00	67	00	62	2·5
						01	95	00	88	00	80	00	74	00	68	00	63	2·6
						01	95	01	88	01	81	00	75	00	69	00	64	2·7
						01	95	01	89	01	82	01	75	01	70	01	65	2·8
						02	96	02	89	01	82	01	76	01	70	01	65	2·9
						04	96	03	89	03	83	02	76	02	71	02	66	3·0
								03	92	03	85	02	79	02	73	02	68	3·1
								04	93	03	87	03	80	02	74	02	69	3·2
								04	94	03	88	03	81	02	76	02	70	3·3
								04	95	03	89	03	82	03	77	02	71	3·4
								05	95	04	89	03	83	03	77	02	72	3·5
										04	90	04	84	03	78	03	73	3·6
										05	90	04	84	03	79	03	74	3·7
										05	91	05	85	04	79	04	74	3·8
										07	91	06	85	05	80	04	75	3·9
										08	92	07	86	06	80	05	75	4·0
												07	88	06	82	06	77	4·1
												08	89	07	83	06	78	4·2
												08	90	07	84	[illegible]	79	4·3
												08	91	07	85	06	80	4·4
												09	91	08	86	07	81	4·5
														08	86	07	81	4·6
														09	87	08	82	4·7
														10	87	08	82	4·8
														11	88	09	83	4·9
														12	88	11	83	5·0
																11	84	5·1
																11	85	5·2
																11	86	5·3
																12	87	5·4
																12	88	5·5

Notes:

1. The pairs of figures are lower and upper *s*-confidence limits for *p*, given to 2 decimal places.
2. Notation
   - n = sample size (in place of N in the text)
   - r = number of labeled events
   - p = probability of labeled event
   - η = random number from the uniform distribution on [0, 1)
3. For tabular convenience, r, η is listed as $r + \eta$.

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TABLE 3-4(B) (Continued)

n = 11		n = 12		n = 13		n = 14		n = 15		n = 16		n = 17		n = 18		n = 19		r ÷ n
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	22	00	20	00	19	00	18	00	17	00	16	00	15	00	14	00	13	0·1
00	27	00	25	00	23	00	22	00	21	00	19	00	18	00	17	00	17	·2
00	30	00	28	00	26	00	24	00	23	00	22	00	20	00	19	00	18	·3
00	32	00	30	00	28	00	26	00	24	00	23	00	22	00	21	00	20	·4
00	34	00	31	00	29	00	27	00	26	00	24	00	23	00	22	00	21	·5
00	35	00	32	00	30	00	28	00	27	00	25	00	24	00	23	00	22	·6
00	36	00	33	00	31	00	29	00	27	00	26	00	25	00	23	00	22	·7
00	37	00	34	00	32	00	30	00	28	00	27	00	25	00	24	00	23	·8
00	37	00	35	00	32	00	31	00	29	00	27	00	26	00	25	00	23	·9
00	38	00	35	00	33	00	31	00	29	00	28	00	26	00	25	00	24	1·0
00	43	00	40	00	38	00	35	00	33	00	32	00	30	00	29	00	27	1·2
00	46	00	43	00	41	00	38	00	36	00	34	00	32	00	31	00	29	1·4
00	48	00	45	00	42	00	40	00	38	00	36	00	34	00	32	00	31	1·6
00	50	00	47	00	44	00	41	00	39	00	37	00	35	00	33	00	32	1·8
00	51	00	48	00	45	00	42	00	40	00	38	00	36	00	34	00	33	2·0
00	55	00	51	00	48	00	46	00	43	00	41	00	39	00	37	00	35	2·2
00	57	00	54	00	51	00	48	00	45	00	43	00	41	00	39	00	37	2·4
00	59	00	55	00	52	00	49	00	47	00	44	00	42	00	40	00	38	2·6
00	60	00	57	00	53	00	50	00	48	00	45	00	43	00	41	00	39	2·8
02	61	01	58	01	54	01	51	01	49	01	46	01	44	01	42	01	40	3·0
02	64	02	61	01	57	01	54	01	51	01	49	01	46	01	44	01	42	3·2
02	67	02	63	02	59	02	56	01	53	01	50	01	48	01	46	01	44	3·4
02	68	02	64	02	61	02	57	02	54	02	52	01	49	01	47	01	45	3·6
03	69	03	65	03	62	02	58	02	55	02	53	02	50	02	48	02	46	3·8
05	70	04	66	04	63	04	59	03	56	03	54	03	51	03	49	03	47	4·0
05	73	05	69	04	65	04	62	04	59	03	56	03	53	03	51	03	49	4·2
06	75	05	71	05	67	04	63	04	60	04	57	03	55	03	52	03	50	4·4
06	76	06	72	05	68	05	65	04	61	04	59	04	56	04	53	03	51	4·6
07	77	07	73	06	69	06	66	05	62	05	59	04	57	04	54	04	52	4·8
09	78	08	74	08	70	07	66	06	63	06	60	06	58	05	55	05	53	5·0
10	81	09	76	08	72	07	68	07	65	06	62	06	59	05	57	05	55	5·2
11	82	09	78	09	74	08	70	07	67	07	64	06	61	06	58	06	56	5·4
11	83	10	79	09	75	09	71	08	68	07	65	07	62	06	59	06	57	5·6
13	84	11	80	10	76	10	72	09	69	08	66	08	63	07	60	07	58	5·8
15	85	13	81	12	77	11	73	10	69	09	66	09	63	08	61	08	58	6·0
		14	83	13	79	12	75	11	71	10	68	09	65	09	63	08	60	6·2
		15	84	13	80	12	76	11	73	10	69	10	66	09	64	08	61	6·4
		16	85	14	81	13	77	12	74	11	70	10	67	10	65	09	62	6·6
				16	82	14	78	13	74	12	71	11	68	10	65	10	63	6·8
				17	83	16	79	15	75	13	72	13	69	12	66	11	64	7·0
						16	81	15	77	14	74	13	71	12	68	11	65	7·2
						17	82	16	78	15	75	14	72	13	69	12	66	7·4
						18	83	17	79	15	76	14	73	13	70	12	67	7·6
								18	80	16	77	15	73	14	71	13	68	7·8
								19	81	18	77	17	74	16	71	15	68	8·0
										19	79	17	76	16	73	15	70	8·2
										19	80	18	77	17	74	16	71	8·4
										20	81	19	78	17	75	16	72	8·6
												20	78	18	75	17	73	8·8
												21	79	20	76	18	73	9·0
														20	77	19	74	9·2
														21	78	20	75	9·4
														22	79	20	76	9·6
																21	77	9·8
																23	77	10·0

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TABLE 3-4(B) (Continued)

$n = 20$		$n = 21$		$n = 22$		$n = 23$		$n = 24$		$n = 26$		$n = 28$		$n = 30$		$n = 32$		$r + \eta$
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	13	00	12	00	12	00	11	00	11	00	10	00	09	00	09	00	08	0·1
00	16	00	15	00	14	00	14	00	13	00	12	00	12	00	11	00	10	·2
00	18	00	17	00	16	00	16	00	15	00	14	00	13	00	12	00	11	·3
00	19	00	18	00	17	00	17	00	16	00	15	00	14	00	13	00	12	·4
00	20	00	19	00	18	00	17	00	17	00	16	00	15	00	14	00	13	·5
00	21	00	20	00	19	00	18	00	17	00	16	00	15	00	14	00	13	·6
00	21	00	20	00	20	00	19	00	18	00	17	00	16	00	15	00	14	·7
00	22	00	21	00	20	00	19	00	19	00	17	00	16	00	15	00	14	·8
00	22	00	21	00	21	00	20	00	19	00	18	00	16	00	15	00	15	·9
00	23	00	22	00	21	00	20	00	19	00	18	00	17	00	16	00	15	1·0
00	26	00	25	00	24	00	23	00	22	00	21	00	19	00	18	00	17	1·2
00	28	00	27	00	26	00	25	00	24	00	22	00	21	00	20	00	18	1·4
00	29	00	[illegible]	00	27	00	26	00	25	00	23	00	22	00	21	00	19	1·6
00	31	00	29	00	28	00	27	00	26	00	24	00	23	00	21	00	20	1·8
00	31	00	30	00	29	00	28	00	27	00	25	00	23	00	22	00	21	2·0
00	34	00	32	00	31	00	30	00	29	00	27	00	25	00	24	00	22	2·2
00	36	00	34	00	33	00	32	00	30	00	28	00	27	00	25	00	24	2·4
00	37	00	35	00	34	00	33	00	31	00	29	00	27	00	26	00	24	2·6
00	38	00	36	00	35	00	34	00	32	00	30	00	28	00	27	00	25	2·8
01	39	01	37	01	36	01	34	01	33	01	31	01	29	01	27	00	26	3·0
01	41	01	39	01	37	01	[illegible]	01	35	01	32	01	30	01	29	01	27	3·2
01	42	01	40	01	39	01	37	01	36	01	34	01	32	01	30	01	28	3·4
01	43	01	42	01	40	01	38	01	37	01	35	01	32	01	31	01	29	3·6
02	44	02	42	01	41	01	39	01	38	01	35	01	33	01	31	01	30	3·8
02	45	02	43	02	42	02	40	02	[illegible]	02	36	02	34	02	32	01	30	4·0
03	47	02	45	02	43	02	42	02	40	02	38	02	35	02	33	02	31	4·2
03	48	03	46	03	44	02	43	02	41	02	39	02	36	02	34	02	32	4·4
03	49	03	47	03	46	03	44	03	42	02	40	02	37	02	35	02	33	4·6
04	51	04	48	03	46	03	45	03	43	03	40	03	38	02	36	02	34	4·8
05	51	04	49	04	47	04	45	04	44	03	41	03	38	03	36	03	34	5·0
05	52	05	50	04	49	04	47	04	45	04	42	03	40	03	37	03	35	5·2
05	54	05	52	05	50	04	48	04	46	04	43	04	41	03	38	03	36	5·4
06	55	05	53	05	51	05	49	05	47	04	44	04	41	04	39	03	37	5·6
06	55	06	53	06	51	05	50	05	48	05	45	04	42	04	40	04	38	5·8
07	56	07	54	07	52	06	50	06	49	05	45	05	43	05	40	04	38	6·0
08	59	08	57	07	55	07	53	07	51	06	48	06	45	05	43	05	40	6·5
10	61	10	59	09	57	09	55	08	53	08	50	07	47	07	44	06	42	7·0
11	64	11	62	10	60	10	58	09	56	08	52	08	49	07	47	07	44	7·5
14	66	13	64	12	61	12	59	11	57	10	54	09	51	09	48	08	45	8·0
15	69	14	66	13	64	13	62	12	60	11	56	10	53	09	50	09	48	8·5
17	70	16	68	15	66	15	64	14	62	13	58	12	55	11	52	10	49	9·0
19	73	18	71	17	68	16	66	15	64	14	60	13	57	12	54	11	51	9·5
21	75	20	72	19	70	18	68	17	66	16	62	14	58	13	55	12	52	10·0
23	77	21	75	20	72	19	70	18	68	17	64	15	60	14	57	13	54	10·5
		24	76	22	74	21	72	20	69	18	65	17	62	16	58	15	56	11·0
				24	76	23	74	21	72	19	68	18	64	17	60	15	57	11·5
						25	75	24	73	21	69	20	65	18	62	17	59	12·0
								25	75	23	71	21	67	19	64	18	61	12·5
								27	76	25	72	22	68	21	65	19	62	13·0
										26	74	24	70	22	67	20	64	13·5
										28	75	25	72	23	68	22	65	14·0
												27	73	24	70	23	66	14·5
												28	75	26	71	24	68	15
														29	74	27	70	16
																30	73	17

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TABLE 3-4(B) (Continued)

n = 34		n = 36		n = 38		n = 40		n = 42		n = 44		n = 46		n = 48		n = 50		r + η
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	0
00	10	00	09	00	09	00	08	00	08	00	07	00	07	00	07	00	07	·2
00	12	00	11	00	10	00	10	00	09	00	09	00	09	00	08	00	08	·4
00	13	00	12	00	11	00	11	00	10	00	10	00	09	00	09	00	09	·6
00	13	00	13	00	12	00	12	00	11	00	11	00	10	00	10	00	09	·8
00	14	00	13	00	13	00	12	00	12	00	11	00	11	00	10	00	10	1·0
00	16	00	15	00	15	00	14	00	13	00	13	00	12	00	12	00	11	1·2
00	17	00	16	00	16	00	15	00	14	00	14	00	13	00	13	00	12	1·4
00	18	00	17	00	17	00	16	00	15	00	14	00	14	00	13	00	13	1·6
00	19	00	18	00	17	00	16	00	16	00	15	00	14	00	14	00	13	1·8
00	20	00	19	00	18	00	17	00	16	00	15	00	15	00	14	00	14	2·0
00	23	00	22	00	20	00	20	00	19	00	18	00	17	00	16	00	16	2·5
00	24	00	23	00	22	00	21	00	20	00	19	00	18	00	18	00	17	3·0
01	27	01	26	01	24	01	23	01	22	00	21	00	20	00	20	00	19	3·5
01	29	01	27	01	26	01	25	01	24	01	23	01	22	01	21	01	20	4·0
02	31	02	29	01	28	01	27	01	26	01	25	01	24	01	23	01	22	4·5
03	32	02	31	02	29	02	28	02	27	02	26	02	25	02	24	02	23	5·0
03	35	03	33	03	32	03	30	02	29	02	28	02	27	02	26	02	25	5·5
04	36	04	34	04	33	03	31	03	30	03	29	03	28	03	27	03	26	6·0
05	38	04	37	04	35	04	33	04	32	03	31	03	29	03	28	03	27	6·5
06	40	05	38	05	36	05	35	05	33	04	32	04	30	04	29	04	28	7·0
06	42	06	40	06	38	05	36	05	35	05	33	05	32	04	31	04	30	7·5
07	43	07	41	07	39	06	38	06	36	06	35	05	33	05	32	05	31	8·0
08	45	08	43	07	41	07	39	07	38	06	36	06	35	06	33	05	32	8
09	47	09	44	08	42	08	40	07	39	07	37	07	36	06	34	06	33	9·0
10	48	10	46	09	44	08	42	08	40	08	39	07	37	07	36	07	35	9·5
11	50	11	47	10	45	10	43	09	42	09	40	08	38	08	37	08	36	10·0
12	52	11	49	11	47	10	45	10	43	09	41	09	40	08	38	08	37	10·5
14	53	13	50	12	48	11	46	11	44	10	42	10	41	09	39	09	38	11·0
14	55	14	52	13	50	12	48	11	46	11	44	10	42	10	41	09	39	11·5
16	56	15	53	14	51	13	49	12	47	12	45	11	43	11	42	10	40	12·0
17	58	15	55	15	53	14	50	13	48	12	46	12	45	11	43	11	42	12·5
18	59	17	56	16	54	15	51	14	49	13	47	13	46	12	44	12	43	13·0
19	61	18	58	17	55	16	53	15	51	14	49	13	47	13	45	12	44	13·5
20	62	19	59	18	56	17	54	16	52	15	50	14	48	14	46	13	45	14·0
21	63	20	61	19	58	18	56	17	53	16	51	15	49	14	48	14	46	14·5
23	64	21	62	20	59	19	57	18	54	17	52	16	50	15	49	15	47	15·0
24	66	22	63	21	61	20	58	19	56	18	54	17	52	16	50	15	48	15·5
25	67	23	64	22	62	21	59	20	57	19	55	18	53	17	51	16	49	16·0
26	69	24	66	23	63	22	61	20	58	19	56	18	54	18	52	17	50	16·5
28	70	26	67	24	64	23	61	22	59	20	57	19	55	19	53	18	51	17·0
29	71	27	68	25	66	24	63	22	60	21	58	20	56	19	54	18	52	17·5
30	72	28	69	26	67	25	64	24	61	22	59	21	57	20	55	19	53	18·0
		29	71	27	68	26	65	24	63	23	60	22	58	21	56	20	54	18·5
		31	72	29	69	27	66	26	64	24	61	23	59	22	57	21	55	19·0
				30	70	28	68	26	65	25	63	24	60	23	58	22	56	19·5
				31	71	20	68	28	66	26	64	25	61	24	59	23	57	20·0
						30	70	25	67	27	65	26	62	24	60	23	58	20·5
						32	71	30	68	28	66	27	63	26	61	24	59	21·0
								31	69	29	67	28	65	26	62	25	60	21·5
								32	70	30	68	20	65	27	63	26	61	22·0
										31	69	29	67	28	64	27	62	22·5
										32	70	31	67	20	65	28	63	23
												33	69	31	67	30	65	24
														33	69	31	67	25
																33	69	26

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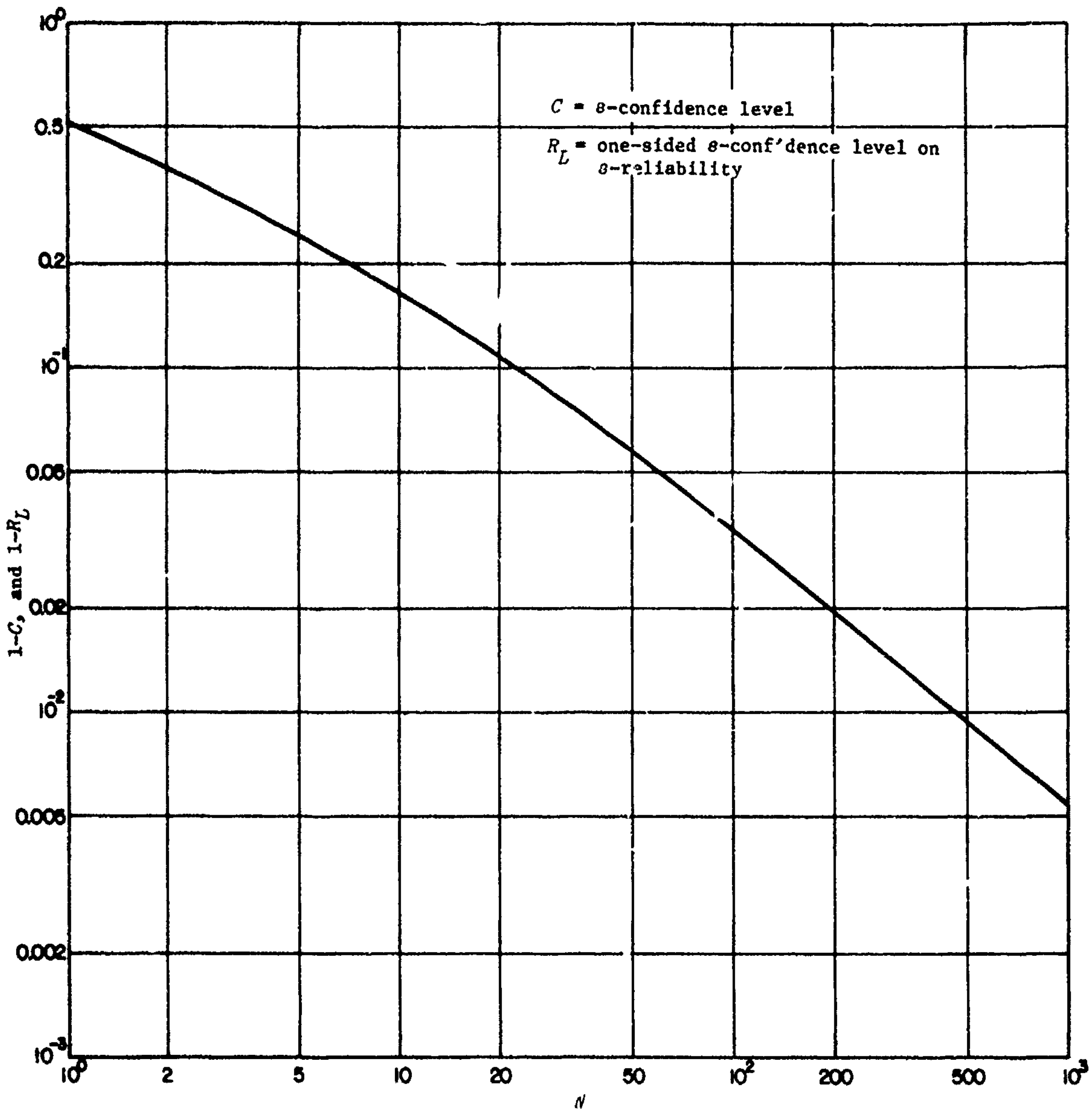

Figure 3-2. Special Case for No Failures in **N** *Trials and* **C** = **R_L**

All the *s*-confidence intervals are discouragingly broad. Statistics shows us how little we know from the experiment.

If N is too large for Table 3-2, use Fig. 3-1 or use the Poisson approximation. Ref. 1 can be used to find the inside and outside limits by setting $\eta = 1$ and 0 and solving for the appropriate value of p.

3-7.2 EXAMPLE NO. 2

Thirty tests were run, there were no failures. Find the lower 1-sided *s*-confidence limit C on the *s*-reliability R such that $C = R$.

$\hat{R} = 30/30 = 1.$

Enter Fig. 3-2 with $N = 30$; then
$1 - C = 1 - R = 0.08 = 1 - 0.92$;
Conf $\{R \geq 0.92\} \geq 92\%$.

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CHAPTER 4

POISSON DISTRIBUTION

4-0 LIST OF SYMBOLS

C	=	*s*-Confidence
Cdf	=	Cumulative distribution function
C, L, U	=	subscripts that imply a *s*-confidence level; C is general, L is lower, U is upper
$CM_i\{\ \}$	=	ith central moment
Conf{ }	=	*s*-Confidence level
csqfc	=	*Sf* for chi-square distribution
CV{ }	=	coefficient of variation: StDv{ }/E{ }
$E\{\ \}$	=	*s*-Expected value
gaufc	=	*Sf* for Gaussian distribution
$M_i\{\ \}$	=	ith moment about the origin
$NCM_i\{\ \}$	=	normalized ith central moment; $CM_i\{\ \}/[\text{StDv}\{\ \}]^i$
pdf	=	probability density function
pmf	=	probability mass function
poi	=	base name for Poisson distribution
poif	=	*Cdf* for Poisson distribution
poifc	=	*Sf* for Poisson distribution
Pr { }	=	Probability
PrD	=	Probability distribution
R	=	*s*-Reliability
r	=	random variable, discrete
s-	=	denotes statistical definition
Sf	=	Survivor function
StDv{ }	=	standard deviation
Var{ }	=	variance
μ	=	parameter
$\{\cdot;\cdot\},(\cdot;\cdot)$	=	the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon
–	=	the complement, e.g., $\overline{\phi} \equiv 1 - \phi$ where ϕ is any probability

4-1 INTRODUCTION

The Poisson distribution relates the actual number of events in a given interval to the true average number of events in that interval, when the process is Poisson. (Poisson was French—the name is pronounced pwah-ssohn.) It is often a good approximation to the binomial distribution.

The base name *poi* is given to the Poisson distribution (for Poisson). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf*. The *Cdf* and *Sf* are not complementary because the random variable is discrete.

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4-2 FORMULAS

μ = Poisson parameter (true average number of events in the interval) $\mu \geqslant 0$

r = random variable, $r = 0, 1, 2, \ldots$ (actual number of events in the interval)

$$pmf\{r;\mu\} = \exp(-\mu)\mu^r/r! \qquad (4\text{-}1)$$

$$Cdf\{r;\mu\} = poif(r;\mu) = \sum_{i=0}^{r} \exp(-\mu)\mu^i/i! \qquad (4\text{-}2)$$

$$Sf\{r;\mu\} = poifc(r;\mu) = \sum_{i=r}^{\infty} \exp(-\mu)\mu^i/i! \qquad (4\text{-}3)$$

$$E\{r,\mu\} = \mu$$

$$StDp\{r,\mu\} = \mu^{1/2}$$

$$CV\{r,\mu\} = \mu^{-1/2}$$

$$CM_3\{r,\mu\} = \mu$$

$$NCM_3\{r,\mu\} = \mu^{-1/2}$$

Table 4-1 shows a few examples of the Poisson *pmf*. If a Poisson process has a rate λ (events per unit measure of τ), then

$$\mu = \lambda\tau \qquad (4\text{-}4)$$

4-3 TABLES AND CURVES

The *pmf* is easily calculated and so is rarely tabulated; Ref. 1 (Table 39 and Sec. 21) gives some tables of individual terms. The *Cdf* is available in tables such as Ref. 1 (Table 7 and Sec. 3), Ref. 2 (Table V and p. 24), Ref. 3 (Table 26.7), Ref. 4, and almost any statistics/probability/quality control textbook.

Eq. 4-5 relates the Poisson and chi-square distributions.

$$poif(r;\mu) = csqfc[2\mu; 2(r+1)] \qquad (4\text{-}5a)$$

$$poifc(r;\mu) = csqf(2\mu, 2r) \qquad (4\text{-}5b)$$

For reasonably large ν (say $\nu \geqslant 5$), Eq. 4-6 is sufficiently accurate.

$$\chi^2_{\nu,Q} \approx \nu\left[1 - \left(\frac{2}{9\nu}\right) + z_Q\left(\frac{2}{9\nu}\right)^{1/2}\right]^3 \qquad (4\text{-}6)$$

where

$$csqfc(\chi^2_{\nu,Q};\nu) = Q$$

$$gaufc(z_Q) = Q$$

Figure 4-1 is a graph of the *poif* from Eq. 4-2.

4-4 PARAMETER ESTIMATION

The estimator

$$\hat{\mu} \equiv r \qquad (4\text{-}7)$$

is unbiased and maximum likelihood. If $r = 0$, it is esthetically displeasing, although still quite true. Very often (when $r = 0$) a *s*-confidence limit is used in place of $\hat{\mu}$ usually corresponding to about 50% *s*-confidence level. If $\mu = \lambda\tau$, and λ is to be estimated, merely divide all estimates and *s*-confidence limits for μ by τ.

s-Confidence statements are more difficult for discrete random variables than for continuous random variables. Chapter 12 discusses the matter thoroughly. The usual *s*-confidence intervals for μ are of the forms

$$\mathrm{Conf}\{\mu \le \mu_L\} \le C_L \qquad (4\text{-}8a)$$

$$\mathrm{Conf}\{\mu \le \mu_U\} \ge C_U \qquad (4\text{-}8b)$$

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TABLE 4-1

POISSON DISTRIBUTION, EXAMPLES

$pmf\left\{r;\mu\right\}$

r	$\mu = 0.01$	$\mu = 0.1$	$\mu = 0.5$	$\mu = 1$	$\mu = 2$	$\mu = 5$	$\mu = 10$
0	0.99	0.90	0.61	0.37	0.14	0.0067	0.45×10^{-4}
1	$.99 \times 10^{-2}$	.090	.30	.37	.27	.034	.0045
2	$.50 \times 10^{-4}$	.0045	.076	.18	.27	.084	.0023
3	$.17 \times 10^{-6}$	.00015	.013	.061	.18	.14	.0076
4	$.041 \times 10^{-8}$	$.38 \times 10^{-5}$	.0016	.015	.090	.18	.019
5	•	$.75 \times 10^{-7}$	.00016	.0031	.036	.18	.038
6	•	$.13 \times 10^{-8}$	13×10^{-4}	.00051	.012	.15	.063
7	•	•	$.94 \times 10^{-6}$	$.73 \times 10^{-4}$	.0034	.10	.090
8		•	$.59 \times 10^{-7}$	$.91 \times 10^{-5}$	.00086	.065	.11
9		•	$.33 \times 10^{-8}$	1.0×10^{-6}	.00019	.036	.13
10			$.16 \times 10^{-9}$	1.0×10^{-7}	$.38 \times 10^{-4}$	.018	.13
11			•	$.92 \times 10^{-8}$	$.69 \times 10^{-5}$	.0082	.11
12			•	$.77 \times 10^{-9}$	$.12 \times 10^{-5}$	.0034	.095
13			•	•	$.18 \times 10^{-6}$	.0013	.073
14				•	$.25 \times 10^{-7}$	.00047	.052
15				•	$.34 \times 10^{-8}$	.00016	.035
16					$.42 \times 10^{-9}$	$.49 \times 10^{-4}$	.022
17					•	$.14 \times 10^{-4}$	.013
18					•	$.40 \times 10^{-5}$	.0071
19					•	$.11 \times 10^{-5}$	.0037
20						$.26 \times 10^{-6}$	.0019
⋮						⋮	⋮
25						$.13 \times 10^{-9}$	$.29 \times 10^{-4}$
⋮						⋮	⋮
30							$.17 \times 10^{-6}$
⋮							⋮
35							$.44 \times 10^{-9}$
							⋮
$E\left\{r;\mu\right\}$	0.01	0.1	0.5	1.0	2.0	5.0	10.0
$StDv\left\{r;\mu\right\}$	0.10	0.32	0.71	1.0	1.4	2.2	3.2
$CV\left\{r;\mu\right\}$	10.0	3.2	1.4	1.0	.71	.45	.32
$CM_3\left\{r;\mu\right\}$	0.01	0.1	0.5	1.0	2.0	5.0	10.0
$NCM_3\left\{r;\mu\right\}$	10.0	3.2	1.4	1.0	.71	.45	.32

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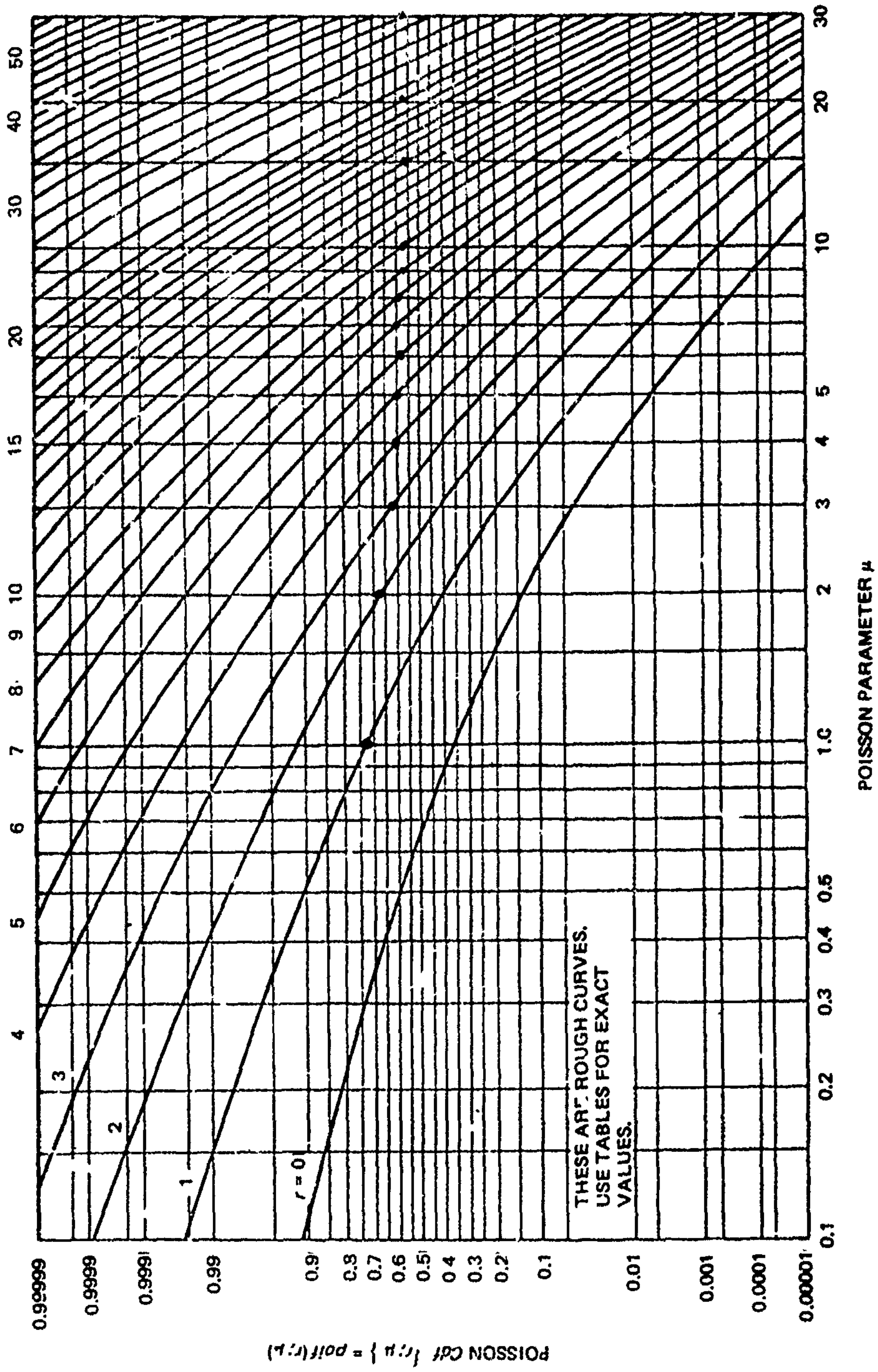

Figure 4-1. Poisson Cumulative Distribution Function

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$$\text{Conf}\{\mu_L \leq \mu \leq \mu_U\} \geq C_U - C_L \tag{4-8c}$$

where μ_L and μ_U are defined by

$$C_U = 1 - poif(r; \mu_U) = csqf(2\mu_U; 2r+2)$$

$$C_L = poifc(r; \mu_L) = csqf(2\mu_L; 2r), r \neq 0$$

$$\mu_L = 0, r = 0.$$

In this form, C_L is usually small (say 5%) and C_U is usually large (say 95%). Notation for *s*-confidence statements is not at all standard; so particular attention must be paid to the example forms. Table 6-1 and Fig. 4-1 are useful for this type of *s*-confidence statement.

Chapter 12 shows that *s*-confidence statements for μ can also be of the forms

$$\text{Conf}\{\mu \leq \mu_L'\} \geq C_L \tag{4-9a}$$

$$\text{Conf}\{\mu \leq \mu_U'\} \leq C_U \tag{4-9b}$$

$$\text{Conf}\{\mu_L' \leq \mu \leq \mu_U'\} \leq C_U - C_L \tag{4-9c}$$

where μ_L' and μ_U' are defined by

$$C_U = poifc(r; \mu_U') = csqf(2\mu_U'; 2r), r \neq 0.$$

$$\mu_U' = 0, r = 0.$$

$$C_L = 1 - poif(r; \mu_L') = csqf(2\mu_L'; 2r+2)$$

In this form, as in Eq. 4-8, C_L is usually small (say 5%), and C_U is usually large (say 95%). μ_L' and μ_U' will be inside the interval (μ_L, μ_U) (for $r \neq 0$). Table 6-1 and Fig. 4-1 are useful for this type of *s*-confidence statement also.

Example. In a 1000-hr life test, there are 3 failures. Find the 5% and 95% 1-sided *s*-confidence limits on the true mean (for the 1000 hr) μ; also make the associated 2-sided *s*-confidence statement. Find the corresponding limits on the failure rate $\lambda(\lambda = \mu/1000\text{-hr})$.

Procedure	Example
1. Find μ_U, μ_L from Eq. 4-8 and Table 6-1.	1. $csqf(15.5;8) = 0.95$ $\mu_U = 7.75$ $csqf(1.64;6) = 0.05$ $\mu_L = 0.82$
2. Find μ_U', μ_L' from Eq. 4-9 and Table 6-1.	2. $csqf(12.6;6) = 0.95$ $\mu_U' = 6.3$ $csqf(2.73;8) = 0.05$ $\mu_L' = 1.37$
3. Make the *s*-confidence statements from Steps 1 and 2.	3. $\text{Conf}\{\mu \leqslant 0.82\} \leqslant 5\%$ $\text{Conf}\{\mu \leqslant 7.75\} \geqslant 95\%$ $\text{Conf}\{0.82 \leqslant \mu \leqslant 7.75\} \geqslant 90\%$ $\text{Conf}\{\mu \leqslant 1.37\} \geqslant 5\%$ $\text{Conf}\{\mu \leqslant 6.3\} \leqslant 95\%$ $\text{Conf}\{1.37 \leqslant \mu \leqslant 6.3\} \leqslant 90\%$
4. Make the corresponding *s*-confidence statements about λ.	4. $\text{Conf}\{\lambda \leqslant 0.82/1000\text{-hr}\} \leqslant 5\%$ $\text{Conf}\{\lambda \leqslant 7.8/1000\text{-hr}\} \leqslant 95\%$ $\text{Conf}\{0.82/1000\text{-hr} \leqslant \lambda \leqslant 7.8/1000\text{-hr}\} \geqslant 90\%$ $\text{Conf}\{\lambda \leqslant 1.37/1000\text{-hr}\} \geqslant 5\%$ $\text{Conf}\{\lambda \leqslant 6.3/1000\text{-hr}\} \leqslant 95\%$ $\text{Conf}\{1.37/1000\text{-hr} \leqslant \lambda \leqslant 6.3/1000\text{-hr}\} \leqslant 90\%$

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The statements about λ and μ are discouragingly wide. This is due to the small number of failures.

4-5 RANDOMIZED EXACT *s*-CONFIDENCE INTERVALS

Traditionally, the *s*-confidence statement (Eq. 4-8)–worst case–is made to be on the safe side. Instead of always choosing this worst case, one can get exact *s*-confidence limits by randomly choosing a value between μ_L and μ_L', or between μ_U and μ_U'. There is nothing to lose and everything to gain by this procedure because it means not always choosing the worst possible case.

The equations to give the randomized limits are

$$\eta = \frac{poifc(r;\mu_L^{\dagger}) - C_L}{\exp(-\mu_L^{\dagger})(\mu_L^{\dagger})^r/r!}$$

$$= \frac{poifc(r;\mu_L^{\dagger}) - C_L}{poifc(r;\mu_L^{\dagger}) - poifc(r+1;\mu_L^{\dagger})}$$

$$= \frac{\overline{C}_L - poif(r-1;\mu_L^{\dagger})}{poif(r;\mu_L^{\dagger}) - poif(r-1;\mu_L^{\dagger})}$$

(unless $r = 0$, and $\eta \leq \overline{C}_L$;

then use $\mu_L^{\dagger} = 0$) (4-10a)

$$\eta = \frac{poif(r;\mu_U^{\dagger}) - \overline{C}_U}{\exp(-\mu_U^{\dagger})(\mu_U^{\dagger})^r/r!}$$

$$= \frac{C_U - poifc(r+1;\mu_U^{\dagger})}{poifc(r;\mu_U^{\dagger}) - poifc(r+1;\mu_U^{\dagger})}$$

$$= \frac{poif(r;\mu_U^{\dagger}) - \overline{C}_U}{poif(r;\mu_U^{\dagger}) - poif(r-1;\mu_U^{\dagger})}$$

(unless $r = 0$, and $\eta \geq C_U$,

then use $\mu_U^{\dagger} = 0$) (4-10b)

where η is a random number from the uniform distribution $0 \leq \eta < 1$. When $\eta = 0$, $\mu_L^* = \mu_L$ and $\mu_U^* = \mu_U$. If $\eta = 1$ (consider the least upper bound of η), $\mu_L^* = \mu_L'$ and $\mu_U^{\dagger} = \mu_U'$.

If special tables are not available for $\mu_L^{\dagger}$ and $\mu_U^{\dagger}$ use Eqs. 3-8 and 3-9 with Table 6-1 to find μ_U, μ_L and μ_U', μ_L'. Then use Eq. 3-10 to find $\mu_L^{\dagger}, \mu_U^{\dagger}$ by an iterative process.

A handy random number generator is a coin, flipped several times. Decide whether heads is to be 0 or 1; tails is the reverse. Then multiply the result of the first flip by 0.5, the second flip by 0.25, the third flip by 0.125, etc. (the numbers are $2^{-\text{flip}}$), as fine as desired. Then add the numbers. Usually 5 or 6 flips give a sufficiently continuous random variable. (For example, heads is 0, tails is 1, the sequence is H, T, H, H, T, H. Add 0.25 + 0.03125 = 0.28125; truncate to 0.281 for convenience.)

Example. Use the data and solution from the example in par. 4-4. Find the randomized exact 5% and 95% *s*-confidence limits. Suppose η = 0.281 (the random number in the paragraph immediately above). Since $r = 3$ is not very large, direct calculation of $poif(r;\mu)$ will be used. (This is reasonable on the electronic calculators with engineering functions. An HP-45 was used for this example.)

The forms of Eq. 4-10 suitable for direct calculation with interpolation are

$$f_L(x_L) \equiv \frac{\overline{C}_L - poif(r-1;x_L)}{\exp(-x_L)x_L^r/r!} - \eta = 0$$

(unless $r = 0$, and $\eta \leq \overline{C}_L$;

then use $\mu_L^{\dagger} = 0$) (4-11a)

$$f_U(x_U) \equiv \frac{poif(r;x_U) - \overline{C}_U}{\exp(-x_U)x_U^r/r!} - \eta = 0$$

(unless $r = 0$, and $\eta \geq C_U$;

then use $\mu_U^{\dagger} = 0$). (4-11b)

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The solution to Eq. 4-11a is $x_L = \mu_L^*$. The solution to Eq. 4-11b is $x_U = \mu_U^*$.

The formula for linear interpolation is

$$x_{new} = \frac{x_- f_+ - x_+ f_-}{f_+ - f_-} \tag{4-12}$$

where x_- and x_+ are the smaller and larger values of x, respectively; and $f_- \equiv f(x_-)$ and $f_+ \equiv f(x_+)$.

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Procedure / **Example**

1. Solve Eq. 4-11a first. Use $x_- = \mu_L'$, and $x_+ = \mu_L$. The values of f are known from the definitions of μ_L', μ_L. Use Eq. 4-12 to find x_{new}. Solve Eq. 3-11a using x_{new}. Use Eq. 4-2 to calculate $poi(r;\mu)$.

Example 1. $\bar{C}_L = 0.95, r = 3$

x	f
0.82	$-0.281 = (-\eta)$
1.37	$+0.719 = (1-\eta)$
0.975	-0.138 (new)

2. Make a new chart, discarding the old pair (from Step 1) which is farthest from the solution. Repeat the linear interpolation and calculation of f_{new}.

Example 2.

x	f
0.82	-0.281
0.975	-0.138
1.125	-0.015 (new)

3. Repeat Step 2. Try 1.144 to be sure the solution is bracketed. This is close enough, $\mu_L^* = x_L = 1.14$.

Example 3.

x	f
0.975	-0.138
1.125	-0.015
1.143	-0.0004
1.144	$+0.0001$

4. Solve Eq. 4-11b next. Use $x_- = \mu_U'$, $x_+ = \mu_U$. The values of f are known from the definitions of μ_U', μ_U.

Example 4. $\bar{C}_U = 0.05, r = 3$

x	f
6.3	$+0.719 = (1-\eta)$
7.75	$-0.281 = (-\eta)$
7.34	$+0.0834$ (new)

5. Repeat Step 2.

Example 5.

x	f
7.34	$+0.0834$
7.75	-0.281
7.434	$+0.0099$ (new)

6. Repeat Step 2.

Example 6.

x	f
7.34	$+0.0834$
7.434	$+0.0099$
7.4467	-0.0007 (new)

7. Repeat Step 2. Try 7.445 to be sure the solution is bracketed.

$\mu_U^* = x_U = 7.45.$

Example 7.

x	f
7.434	$+0.0099$
7.4467	-0.0007
7.44586	-0.000057
7.445	$+0.00066$

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The randomized exact *s*-confidence statements are

$$\text{Conf}\left\{\mu \leq 1.14\right\} = 5\%$$
$$\text{Conf}\left\{\mu \leq 7.45\right\} = 95\%$$
$$\text{Conf}\left\{1.14 \leq \mu \leq 7.45\right\} = 90\%$$

The corresponding statements for λ are

$$\text{Conf}\left\{\lambda \leq 1.14/1000\text{-hr}\right\} = 5\%$$
$$\text{Conf}\left\{\lambda \leq 7.45/1000\text{-hr}\right\} = 95\%$$
$$\text{Conf}\left\{1.14/1000\text{-hr} \leq \lambda \leq 7.45/1000\text{-hr}\right\} = 90\%.$$

Table 4-2 lists randomized 2-sided *s*-confidence limits that have special statistical properties. They are not equal-tailed *s*-confidence limits; they cannot be used separately for upper and lower *s*-confidence limits. See Ref. 6 for a more complete discussion. In general, the *s*-confidence limits in Table 4-2 will be different from those calculated using the methods in this chapter. It is difficult to say that one set is better than the other except in the narrow statistical sense stated for Table 4-2.

4-6 CHOOSING A *s*-CONFIDENCE LEVEL

Choosing an appropriate *s*-confidence level is always troublesome. There is obviously little point in having a very high *s*-confidence that the *s*-reliability is very low, or a very low *s*-confidence that the *s*-reliability is very high. A reasonable compromise is to choose a *s*-confidence level that is approximately the point estimate of the *s*-reliability; for $r = 0$ the *s*-confidence level can be chosen to be equal to the lower *s*-confidence limit on *s*-reliability.

4-7 EXAMPLE, LIFE TEST RESULTS

On a life test there were no failures. Find 1%, 99%, 1-sided *s*-confidence intervals for μ, and then find the associated 2-sided *s*-confidence interval.

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TABLE 4-2

NEYMAN-SHORTEST UNBIASED 95% AND 99% s-CONFIDENCE INTERVALS FOR μ (ADAPTED FROM Ref. 6)

$r+\eta$	95%		99%		$r+\eta$	95%		99%		$r+\eta$	95%		99%	
0·01	0	0	0	0	3·6	0·5	8·1	0·2	10·3	9·1	4·0	15·8	2·9	18·6
·02	0	0	0	·9	3·7	·6	8·2	·3	10·5	9·2	4·0	15·9	3·0	18·8
·03	0	0	0	1·3	3·8	·6	8·4	·3	10·6	9·3	4·1	16·1	3·0	18·9
·04	0	0	0	1·7	3·9	·7	8·5	·4	10·7	9·4	4·1	16·2	3·1	19·1
·05	0	0	0	1·9	4·0	·8	8·6	·4	10·8	9·5	4·2	16·3	3·1	19·2
0·06	0	0·3	0	2·1	4·1	0·8	8·8	0·5	11·0	9·6	4·3	16·5	3·2	19·4
·07	0	·5	0	2·3	4·2	·9	9·0	·5	11·3	9·7	4·4	16·6	3·3	19·5
·08	0	·7	0	2·4	4·3	·9	9·1	·5	11·5	9·8	4·4	16·7	3·3	19·6
·09	0	·8	0	2·5	4·4	1·0	9·3	·5	11·6	9·9	4·5	16·8	3·4	19·7
·10	0	·9	0	2·7	4·5	1·0	9·5	·5	11·8	10·0	4·6	16·9	3·5	19·8
0·12	0	1·2	0	2·9	4·6	1·1	9·6	0·6	11·9	10·2	4·7	17·2	3·6	20·2
·14	0	1·3	0	3·0	4·7	1·1	9·7	·6	12·1	10·4	4·8	17·5	3·7	20·5
·16	0	1·5	0	3·2	4·8	1·2	9·8	·7	12·2	10·6	5·0	17·8	3·9	20·8
·18	0	1·6	0	3·3	4·9	1·3	10·0	·7	12·3	10·8	5·1	18·0	4·0	21·0
·20	0	1·7	0	3·4	5·0	1·4	10·1	·8	12·4	11·0	5·3	18·2	4·1	21·2
0·25	0	2·0	0	3·6	5·1	1·4	10·3	0·8	12·6	11·2	5·4	18·5	4·2	21·6
·30	0	2·2	0	3·8	5·2	1·4	10·4	·9	12·9	11·4	5·6	18·8	4·3	21·9
·35	0	2·4	0	4·0	5·3	1·5	10·6	·9	13·0	11·6	5·7	19·1	4·4	22·1
·40	0	2·5	0	4·1	5·4	1·5	10·8	·9	13·2	11·8	5·8	19·3	4·6	22·4
·45	0	2·6	0	4·3	5·5	1·6	10·9	1·0	13·4	12·0	6·0	19·6	4·8	22·6
0·50	0	2·8	0	4·4	5·6	1·6	11·0	1·0	13·5	12·2	6·1	19·8	4·9	22·9
·55	0	2·9	0	4·5	5·7	1·7	11·2	1·1	13·6	12·4	6·3	20·1	5·0	23·3
·60	0	3·0	0	4·6	5·8	1·8	11·3	1·1	13·8	12·6	6·4	20·3	5·1	23·5
·65	0	3·0	0	4·7	5·9	1·9	11·4	1·2	13·9	12·8	6·6	20·6	5·2	23·8
·70	0	3·1	0	4·7	6·0	2·0	11·5	1·3	14·0	13·0	6·8	20·8	5·4	24·0
0·75	0	3·2	0	4·8	6·1	2·0	11·7	1·3	14·2	13·2	6·9	21·1	5·5	24·3
·80	0	3·3	0	4·9	6·2	2·1	11·8	1·3	14·4	13·4	7·0	21·4	5·6	24·6
·85	0	3·3	0	4·9	6·3	2·1	12·0	1·4	14·6	13·6	7·1	21·6	5·7	24·9
·90	0	3·4	0	5·0	6·4	2·2	12·2	1·4	14·7	13·8	7·3	21·9	5·9	25·1
1·0	0	3·5	0	5·1	6·5	2·2	12·3	1·4	14·9	14·0	7·5	22·1	6·0	25·3
1·1	0	3·8	0	5·5	6·6	2·3	12·4	1·5	15·0	14·2	7·6	22·4	6·1	25·7
1·2	0	4·1	0	5·9	6·7	2·4	12·6	1·6	15·1	14·4	7·7	22·6	6·3	26·0
1·3	0	4·3	0	6·1	6·8	2·4	12·7	1·6	15·3	14·6	7·9	22·9	6·4	26·2
1·4	0	4·5	0	6·4	6·9	2·5	12·8	1·7	15·4	14·8	8·0	23·1	6·5	26·4
1·5	0	4·7	0	6·6	7·0	2·6	12·9	1·8	15·5	15·0	8·2	23·3	6·7	26·7
1·6		4·9	0	6·7	7·1	2·7	13·1	1·8	15·7	15·2	8·3	23·6	6·8	27·0
1·7		5·0	0	6·9	7·2	2·7	13·2	1·9	15·9	15·4	8·5	23·9	6·9	27·3
1·8	0	5·2	0	7·0	7·3	2·8	13·4	1·9	16·0	15·6	8·6	24·1	7·1	27·6
1·9	0	5·3	0	7·1	7·4	2·8	13·5	1·9	16·2	15·8	8·8	24·4	7·2	27·8
2·0	0	5·4	0	7·2	7·5	2·9	13·7	2·0	16·4	16·0	9·0	24·6	7·4	28·0
2·1	0·1	5·6	0	7·6	7·6	2·9	13·8	2·0	16·5	16·2	9·1	24·9	7·5	28·3
2·2	·1	5·9	0	7·8	7·7	3·0	13·9	2·1	16·6	16·4	9·2	25·1	7·6	28·6
2·3	·1	6·1	0	8·1	7·8	3·1	14·0	2·2	16·7	16·6	9·4	25·4	7·7	28·9
2·4	·1	6·2	0	8·3	7·9	3·2	14·1	2·2	16·8	16·8	9·5	25·6	7·9	29·1
2·5	·1	6·4	0	8·4	8·0	3·3	14·3	2·3	17·0	17·0	9·7	25·8	8·1	29·3
2·6	0·1	6·6	0	8·6	8·1	3·3	14·4	2·4	17·2	17·2	9·9	26·1	8·2	29·6
2·7	·2	6·7	0	8·7	8·2	3·4	14·6	2·4	17·3	17·4	10·0	26·4	8·3	29·9
2·8	·2	6·8	0	8·8	8·3	3·4	14·7	2·4	17·5	17·6	10·2	26·6	8·4	30·3
2·9	·3	6·9	0·1	9·0	8·4	3·5	14·9	2·5	17·7	17·8	10·3	26·8	8·6	30·4
3·0	·4	7·1	·1	9·1	8·5	3·6	15·0	2·5	17·8	18·0	10·5	27·1	8·8	30·6
3·1	0·4	7·3	0·2	9·4	8·6	3·6	15·1	2·6	17·9	18·2	10·6	27·3	8·9	30·9
3·2	·4	7·5	·2	9·6	8·7	3·7	15·3	2·7	18·1	18·4	10·8	27·6	9·0	31·2
3·3	·4	7·6	·2	9·8	8·8	3·8	15·4	2·7	18·2	18·6	10·9	27·8	9·1	31·5
3·4	·5	7·8	·2	10·0	8·9	3·8	15·5	2·8	18·3	18·8	11·1	28·1	9·3	31·7
3·5	·5	8·0	·2	10·2	9·0	3·9	15·6	2·9	18·4	19·0	11·3	28·3	9·5	31·9

Notes:

1. The pairs of figures under each s-confidence heading are lower and upper s-confidence limits for μ.
2. Notation:
   - μ = Poisson parameter
   - r = number of events observed in sample
   - η = random number from the uniform distribution on [0, 1)
3. For tabular convenience, r, η is listed as $r + \eta$.

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TABLE 4-2 (Continued)

$r+\eta$	95%		99%		$r+\eta$	95%		99%		$r+\eta$	95%		99%	
86	69	105	64	112	141	119	165	112	173	196	169	224	162	234
87	70	106	65	113	142	119	166	113	174	197	170	225	163	235
88	70	107	66	114	143	120	167	114	176	198	171	226	163	236
89	71	108	66	115	144	121	168	115	177	199	172	227	164	237
90	72	109	67	116	145	122	169	116	178	200	173	228	165	238
91	73	110	68	117	146	123	170	117	179	201	174	230	166	239
92	74	112	69	118	147	124	172	117	180	202	175	231	167	240
93	75	113	70	120	148	125	173	118	181	203	176	232	168	241
94	76	114	71	121	149	126	174	119	182	204	177	233	169	242
95	77	115	72	122	150	127	175	120	183	205	178	234	170	244
96	78	116	72	123	151	128	176	121	184	206	179	235	171	245
97	78	117	73	124	152	129	177	122	185	207	180	236	172	246
98	79	118	74	125	153	130	178	123	187	208	181	237	173	247
99	80	119	75	126	154	130	179	124	188	209	181	238	173	248
100	81	120	76	127	155	131	180	125	189	210	182	239	174	249
101	82	121	77	129	156	132	181	126	190	211	183	240	175	250
102	83	123	78	130	157	133	182	126	191	212	184	241	176	251
103	84	124	79	131	158	134	183	127	192	213	185	242	177	252
104	85	125	79	132	159	135	184	128	193	214	186	243	178	253
105	86	126	80	133	160	136	186	129	194	215	187	245	179	254
106	87	127	81	134	161	137	187	130	195	216	188	246	180	256
107	88	128	82	135	162	138	188	131	196	217	189	247	181	257
108	88	129	83	136	163	139	189	132	198	218	190	248	182	258
109	89	130	84	138	164	140	190	133	199	219	191	249	183	259
110	90	131	85	139	165	141	191	134	200	220	192	250	184	260
111	91	132	86	140	166	142	192	135	201	221	193	251	184	261
112	92	134	86	141	167	142	193	135	202	222	194	252	185	262
113	93	135	87	142	168	143	194	136	203	223	195	253	186	263
114	94	136	88	143	169	144	195	137	204	224	195	254	187	264
115	95	137	89	144	170	145	196	138	205	225	196	255	188	265
116	96	138	90	145	171	146	197	139	206	226	197	256	189	266
117	97	139	91	147	172	147	198	140	207	227	198	257	190	268
118	98	140	92	148	173	148	200	141	209	228	199	258	191	269
119	98	141	93	149	174	149	201	142	210	229	200	259	192	270
120	99	142	93	150	175	150	202	143	211	230	201	260	193	271
121	100	143	94	151	176	151	203	144	212	231	202	262	194	272
122	101	144	95	152	177	152	204	144	213	232	203	263	194	273
123	102	146	96	153	178	153	205	145	214	233	204	264	195	274
124	103	147	97	154	179	154	206	146	215	234	205	265	196	275
125	104	148	98	156	180	154	207	147	216	235	206	266	197	276
126	105	149	99	157	181	155	208	148	217	236	207	267	198	277
127	106	150	100	158	182	156	209	149	218	237	208	268	199	278
128	107	151	101	159	183	157	210	150	220	238	209	269	200	279
129	108	152	101	160	184	158	211	151	212	239	209	270	201	281
130	108	153	102	161	185	159	212	152	222	240	210	271	202	282
131	109	154	103	162	186	160	213	153	223	241	211	272	203	283
132	110	155	104	163	187	161	215	153	225	242	212	273	204	284
133	111	156	105	164	188	162	216	154	225	243	213	274	205	285
134	112	157	106	166	189	163	217	155	226	244	214	275	205	286
135	113	159	107	167	190	164	218	156	227	245	215	276	206	287
136	114	160	108	168	191	165	219	157	228	246	216	278	207	288
137	115	161	109	169	192	166	220	158	229	247	217	279	208	289
138	116	162	109	170	193	167	221	159	230	248	218	280	209	290
139	117	163	110	171	194	167	222	160	232	249	219	281	210	291
140	118	164	111	172	195	168	223	161	233	250	220	282	211	292

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TABLE 4-2 (Continued)

$r+\eta$	95% lower	95% upper	99% lower	99% upper	$r+\eta$	95% lower	95% upper	99% lower	99% upper	$r+\eta$	95% lower	95% upper	99% lower	99% upper
19·2	11·4	28·6	9·6	32·2	30·2	20·2	41·7	17·7	46·1	43·0	31·0	56·6	27·8	61·6
19·4	11·5	28·8	9·7	32·5	30·4	20·4	42·0	17·9	46·4	43·5	31·3	57·2	28·2	62·2
19·6	11·7	29·1	9·8	32·8	30·6	20·5	42·2	18·0	46·6	44·0	31·8	57·8	28·6	62·8
19·8	11·9	29·3	10·0	33·0	30·8	20·7	42·4	18·2	46·8	44·5	32·2	58·4	29·0	63·4
20·0	12·0	29·5	10·2	33·2	31·0	20·9	42·7	18·4	47·0	45·0	32·7	58·9	29·4	64·0
20·2	12·2	29·8	10·3	33·5	31·2	21·0	42·9	18·5	47·3	45·5	33·1	59·5	29·8	64·6
20·4	12·3	30·0	10·4	33·8	31·4	21·2	43·2	18·6	47·6	46·0	33·5	60·0	30·2	65·2
20·6	12·5	30·3	10·6	34·1	31·6	21·4	43·4	18·8	47·8	46·5	33·9	60·6	30·6	65·8
20·8	12·6	30·5	10·7	34·3	31·8	21·6	43·6	18·9	48·1	47·0	34·4	61·2	31·1	66·4
21·0	12·8	30·7	10·9	34·5	32·0	21·7	43·8	19·1	48·3	47·5	34·8	61·8	31·4	67·0
21·2	13·0	31·0	11·0	34·8	32·2	21·9	44·1	19·3	48·6	48·0	35·2	62·3	31·9	67·5
21·4	13·1	31·3	11·1	35·1	32·4	22·0	44·3	19·4	48·8	48·5	35·6	62·9	32·3	68·2
21·6	13·3	31·5	11·3	35·3	32·6	22·2	44·6	19·6	49·1	49·0	36·1	63·5	32·7	68·7
21·8	13·4	31·7	11·4	35·6	32·8	22·4	44·8	19·7	49·3	49·5	36·5	64·1	33·1	69·4
22·0	13·6	31·9	11·6	35·8	33·0	22·5	45·0	19·9	49·5	50·0	36·9	64·6	33·5	69·9
22·2	13·7	32·2	11·7	36·1	33·2	22·7	45·3	20·0	49·8	50·5	37·3	65·2	33·9	70·6
22·4	13·9	32·5	11·9	36·4	33·4	22·8	45·5	20·2	50·0	51·0	37·8	65·8	34·3	71·1
22·6	14·1	32·7	12·0	36·6	33·6	23·0	45·7	20·3	50·3	51·5	38·2	66·4	34·7	71·7
22·8	14·2	32·9	12·2	36·8	33·8	23·2	46·0	20·5	50·5	52·0	38·7	66·9	35·1	72·3
23·0	14·4	33·1	12·4	37·0	34·0	23·4	46·2	20·7	50·7	52·5	39·1	67·5	35·5	72·9
23·2	14·5	33·4	12·5	37·3	34·2	23·5	46·4	20·8	51·0	53·0	39·5	68·0	36·0	73·5
23·4	14·7	33·7	12·6	37·6	34·4	23·7	46·7	21·0	51·3	53·5	39·9	68·6	36·3	74·1
23·6	14·8	33·9	12·7	37·9	34·6	23·8	46·9	21·1	51·5	54·0	40·4	69·2	36·8	74·6
23·8	15·0	34·1	12·9	38·1	34·8	24·0	47·1	21·3	51·7	54·5	40·8	69·8	37	75
24·0	15·2	34·3	13·1	38·3	35·0	24·2	47·3	21·5	51·9	55·0	41·3	70·3	38	76
24·2	15·3	34·6	13·2	38·6	35·2	24·4	47·6	21·6	52·2	56	42·1	71·4	38	77
24·4	15·5	34·9	13·3	38·9	35·4	24·5	47·8	21·7	52·5	57	43·0	72·6	39	78
24·6	15·7	35·1	13·5	39·1	35·6	24·7	48·1	21·9	52·7	58	43·9	73·7	40	79
24·8	15·8	35·3	13·6	39·4	35·8	24·9	48·3	22·1	52·9	59	44·7	74·8	41	80
25·0	16·0	35·5	13·8	39·6	36·0	25·0	48·5	22·3	53·2	60	46	76	42	82
25·2	16·1	35·8	14·0	39·9	36·2	25·2	48·8	22·4	53·4	61	46	77	43	83
25·4	16·3	36·1	14·1	40·1	36·4	25·3	49·0	22·5	53·7	62	47	78	43	84
25·6	16·5	36·3	14·2	40·4	36·6	25·5	49·2	22·7	53·9	63	48	79	44	85
25·8	16·6	36·5	14·4	40·6	36·8	25·7	49·5	22·9	54·2	64	49	80	45	86
26·0	16·8	36·7	14·6	40·8	37·0	25·9	49·7	23·0	54·4	65	50	82	46	87
26·2	17·0	37·0	14·7	41·1	37·2	26·0	49·9	23·2	54·6	66	51	83	47	89
26·4	17·1	37·3	14·8	41·4	37·4	26·2	50·2	23·3	54·9	67	52	84	48	90
26·6	17·3	37·5	15·0	41·6	37·6	26·4	50·4	23·5	55·1	68	53	85	48	91
26·8	17·4	37·7	15·1	41·9	37·8	26·5	50·6	23·6	55·4	69	54	86	49	92
27·0	17·6	37·9	15·3	42·1	38·0	26·7	50·8	23·8	55·6	70	54	87	50	93
27·2	17·8	38·2	15·5	42·4	38·2	26·9	51·1	24·0	55·9	71	55	88	51	94
27·4	17·9	38·4	15·6	42·6	38·4	27·0	51·3	24·1	56·1	72	56	89	52	96
27·6	18·1	38·7	15·7	42·9	38·6	27·2	51·6	24·3	56·4	73	57	91	53	97
27·8	18·3	38·9	15·9	43·1	38·8	27·4	51·8	24·4	56·6	74	58	92	54	98
28·0	18·4	39·1	16·1	43·3	39·0	27·6	52·0	24·6	56·8	75	59	93	54	99
28·2	18·6	39·4	16·2	43·6	39·2	27·7	52·2	24·8	57·1	76	60	94	55	100
28·4	18·7	39·6	16·3	43·9	39·4	27·9	52·5	24·9	57·3	77	61	95	56	101
28·6	18·9	39·9	16·5	44·1	39·6	28·0	52·7	25·1	57·6	78	62	96	57	102
28·8	19·1	40·1	16·7	44·4	39·8	28·2	52·9	25·2	57·8	79	62	97	58	104
29·0	19·3	40·3	16·8	44·6	40·0	28·4	53·1	25·4	58·0	80	63	98	59	105
29·2	19·4	40·6	17·0	44·9	40·5	28·8	53·8	25·8	58·6	81	64	99	60	106
29·4	19·6	40·8	17·1	45·1	41·0	29·3	54·3	26·2	59·2	82	65	101	60	107
29·6	19·7	41·0	17·2	45·4	41·5	29·6	54·9	26·6	59·8	83	66	102	61	108
29·8	19·9	41·3	17·4	45·6	42·0	30·1	55·5	27·0	60·4	84	67	103	62	109
30·0	20·1	41·5	17·6	45·8	42·5	30·5	56·1	27·4	61·0	85	68	104	63	110

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Procedure	Example
1. Use Eqs. 4-8 and 4-9 along with Table 6-1.	1. $r = 0$, $C_L = 1\%$, $C_U = 99\%$ $csqf(9.21;2) = 99\%$ $csqf(0.0201;2) = 1\%$ $\mu_U = 4.61$ $\mu_L' = 0.0101$ $\mu_U' = \mu_L = 0$
2. Make the feasible *s*-confidence statements.	2. Conf $\{\mu \leq 4.61\} \geq 99\%$ Conf $\{\mu \leq 0.0101\} \geq 1\%$
3. Find the randomized exact *s*-confidence limits. Use the method near the end of par. 4-5 to find the random number (Let heads $\rightarrow$ 0, tails $\rightarrow$ 1).	3. The coin flipping sequence is T, H, T, T, H, T which gives the number 0.703125. $\eta = 0.703$ (truncated)
4. Solve Eq. 4-11a. Since $r = 0$, check the condition "$\eta \leq \bar{C}_L$".	4. $\eta = 0.703$, $\bar{C}_L = 0.99$ $\eta < \bar{C}_L$; so $\mu_L^* = 0$
5. Solve Eq. 4-11b. Since $r = 0$, check the condition "$\eta \geq C_U$". Since $r = 0$, Eq. 4-11b becomes $\mu_U^* = \ell n(\bar{\eta}/\bar{C}_U)$	5. $\eta = 0.703$, $C_U = 0.99$ $\eta < C_U$; so $\mu_U^* \neq 0$. $\mu_U^* = 3.39$

The randomized exact *s*-confidence statement is Conf $\{0 \leq \mu \leq 3.39\} = 98\%$.

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CHAPTER 5

GAUSSIAN (*s*-NORMAL) DISTRIBUTION

5-0 LIST OF SYMBOLS

C = *s*-Confidence

Cdf = Cumulative distribution function

C, L, U = subscripts that imply a *s*-confidence level; C is general, L is lower, U is upper

$CM_i\{\ \}$ = ith central moment

Conf$\{\ \}$ = *s*-Confidence level

csn = base name for chi-square/nu distribution

CV$\{\ \}$ = coefficient of variation: StDv$\{\ \}/E\{\ \}$

$E\{\ \}$ = *s*-Expected value

gau = base name for Gaussian (*s*-normal) distribution

gaud = *pdf* for Gaussian *s*-normal distribution

gauf = *Cdf* for Gaussian *s*-normal distribution

gaufc = *Sf* for Gaussian *s*-normal distribution

gauhr = hazard rate (failure rate) for Gaussian *s*-normal distribution

$M_i\{\ \}$ = ith moment about the origin

N = sample size

$NCM_i\{\ \}$ = normalized ith central moment: $CM_i\{\ \}/[\mathrm{StDv}\{\ \}]^i$

pdf = probability density function

pmf = probability mass function

$Pr\{\ \}$ = Probability

PrD = Probability distribution

R = *s*-Reliability

s = s statistic

s- = denotes statistical definition

t = t statistic

Sf = Survivor function

StDv$\{\ \}$ = standard deviation

stu = base name for Student's *t*-distribution

Var$\{\ \}$ = variance

x = random variable

$\bar{x}$ = sample mean

z, Z = $(x-\mu)/\sigma$

μ = location parameter

ν = degrees of freedom

σ = scale parameter

χ^2/ν = (χ^2/ν) statistic

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$\{\cdot;\cdot\}, (\cdot;\cdot)$ = the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon

$\bar{\ }$ = the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

5-1 INTRODUCTION

The Gaussian distribution is a good approximation to the central portion of many distributions, and often is used to describe the random behavior of product performance. The base name *gau* is given to the Gaussian standard (*s*-normal) distribution (for gaussian). The suffix *f* implies the *Cdf*, the suffix *fc* implies the *Sf* (complement of the *Cdf*), the suffix *hr* implies the failure rate (hazard rate)

5-2 FORMULAS

μ = location parameter

σ = scale parameter, $\sigma>0$

x = random variable, it may take any value

z = $(x-\mu)/\sigma$, standard *s*-normal variate

$$pdf\{x;\mu,\sigma\} = (1/\sqrt{2\pi}\sigma)\exp\left[-\left(\frac{x-\mu}{\sigma}\right)^2\Big/2\right]$$

$$= (1/\sigma)\,gaud[(x-\mu)/\sigma] \quad (5\text{-}1)$$

$$Cdf\{x,\mu,\sigma\} = gauf[(x-\mu)/\sigma] \quad (5\text{-}2)$$

$$Sf\{x;\mu,\sigma\} = gaufc[(x-\mu)/\sigma] \quad (5\text{-}3)$$

$$\text{failure rate}\{x;\mu,\sigma\} = (1/\sigma)\,gauhr[(x-\mu)/\sigma] \quad (5\text{-}4)$$

$E\{x;\mu,\sigma\} = \mu$ $\quad E\{z\} = 0$

$\mathrm{StDv}\{x,\mu,\sigma\} = \sigma$ $\quad \mathrm{StDv}\{z\} = 1$

$\mathrm{CV}\{x;\mu,\sigma\} = \sigma/\mu$, for $\mu > 0$

$CM_3\{x;\mu,\sigma\} = 0$ $\quad CM_3\{z\} = 0$

$NCM_3\{x;\mu,\sigma\} = 0$ $\quad NCM_3\{z\} = 0$

$NCM_4\{x;\mu,\sigma\} = 3$ $\quad NCM_4\{z\} = 3$

$\mathrm{mode}\{x;\mu,\sigma\} = \mu$ $\quad \mathrm{mode}\{z\} = 0$

$\mathrm{median}\{\tilde{x};\mu,\sigma\} = \tilde{\mu}$ $\quad \mathrm{median}\{z\} = 0$

Figure 5-1 shows some curves of the *pdf* and failure rate. The random variable *x* always can be scaled to *z* so that curves for all values of (μ,σ) become the same. The *PrD* for *z* is called the Gaussian or standard *s*-normal distribution.

The *s*-normal distribution often is applied to characteristics which are inherently non negative such as length, weight, strength, and time-to-failure. In order that there be no conceptual difficulties, the coefficient of variation ought to be at least 3; then the negative fraction is quite negligible. Truncated (on the left) *s*-normal distributions can be used (the theory is straightforward) but the extra complication is rarely justified. Where the truncation would be necessary, one often tries a Weibull or lognormal distribution instead.

5-3 TABLES AND CURVES

The *pdf* is calculated readily, not often needed, and tabulated in many places; so it is not given here. See Ref. 1 (Table 1 and Sec. 1) and Ref. 2 (Table 1). The *Cdf* is given in Tables 5-1 and 5-2. It also is given in virtually every probability/statistics/quality control book and set of mathematical/statistical tables. The failure rate is given in Table 5-3 Formulas for calculating these and related functions are given in Ref. 3 (Sec. 26.2).

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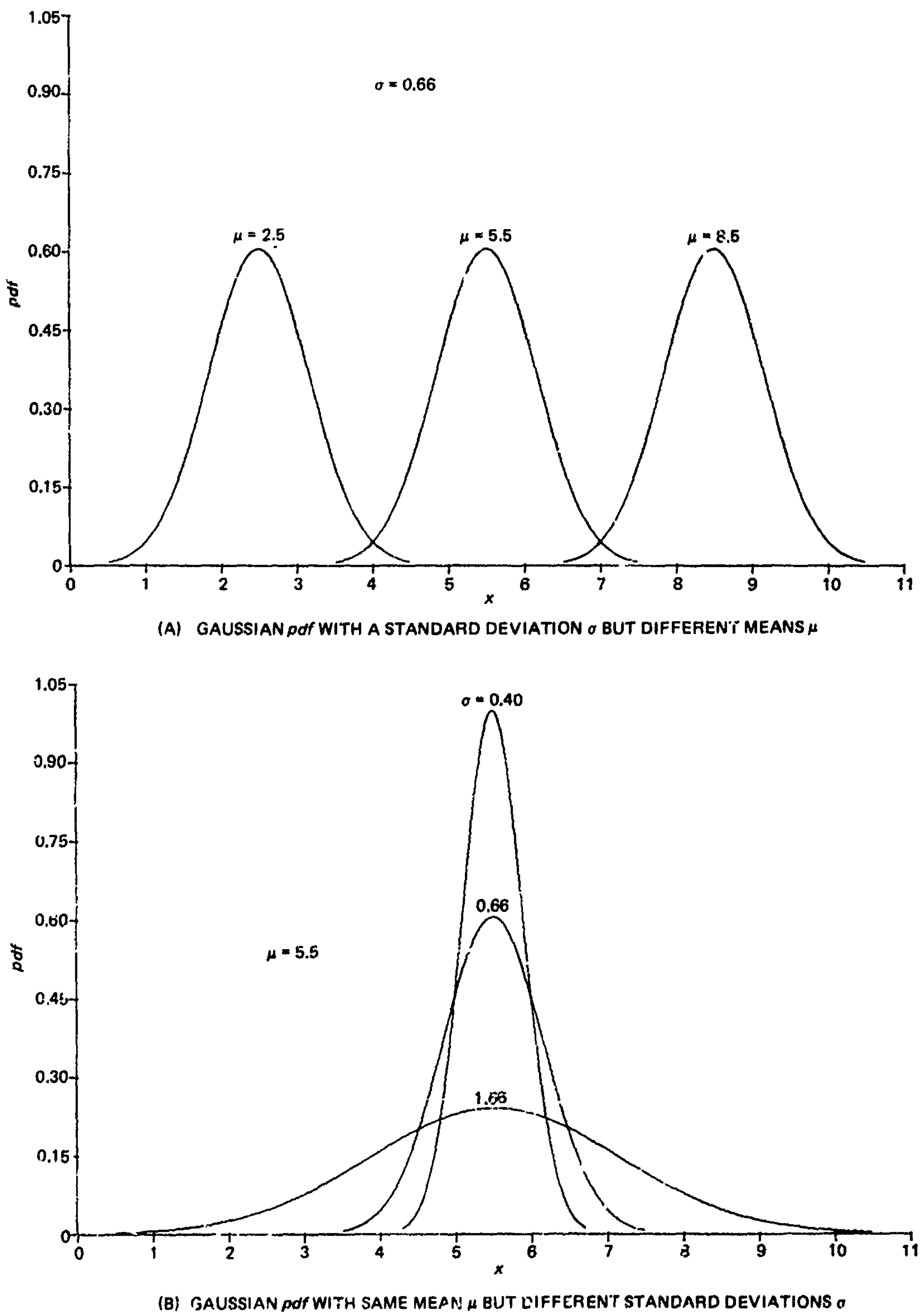

Figure 5-1. Curves for Gaussian Distributions

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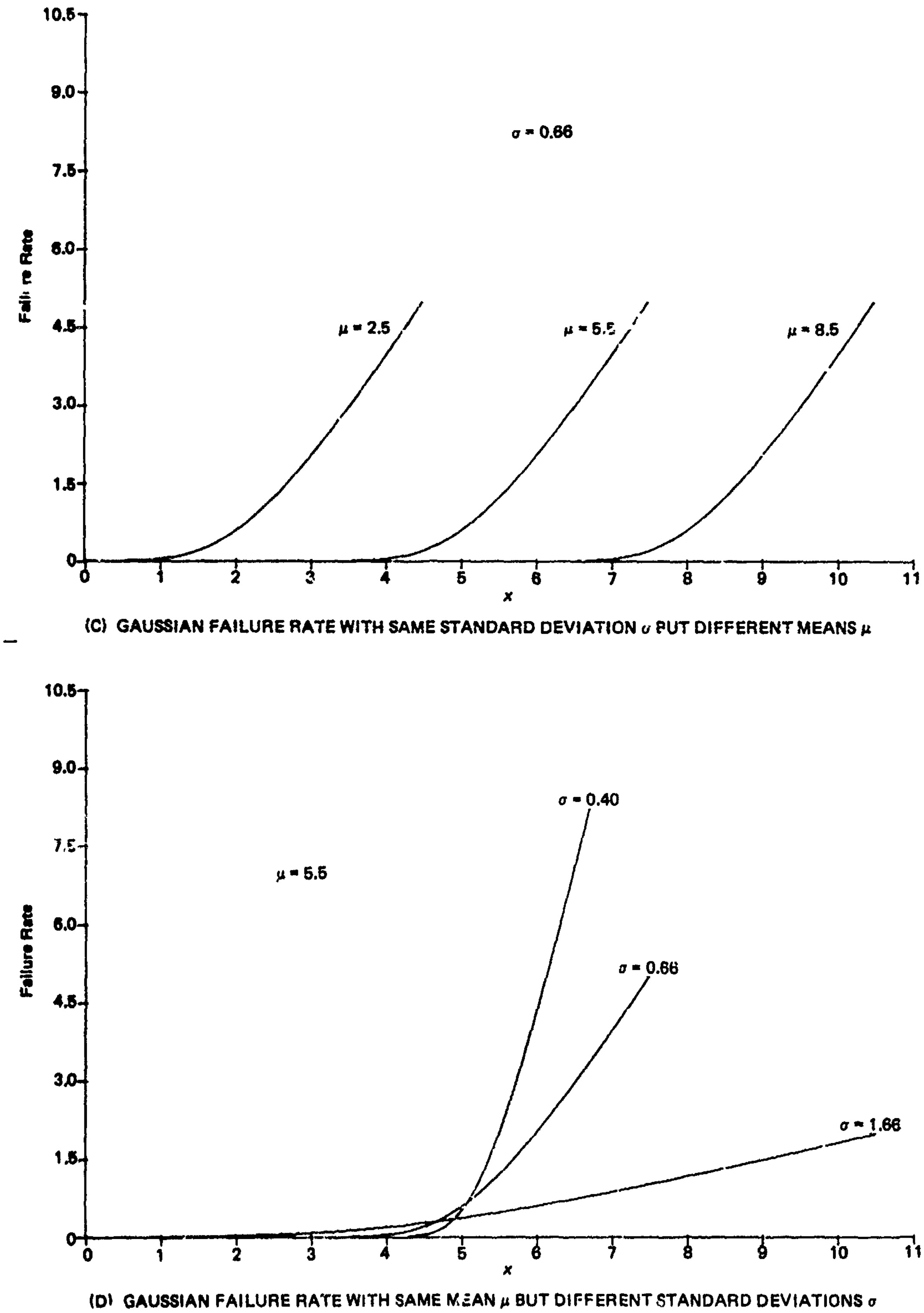

Figure 5-1. Curves for Gaussian Distributions

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TABLE 5-1

STANDARD *s*-NORMAL (GAUSSIAN) *Cdf* — *gauf* (Z)

The body of the table gives the *Cdf* for the standard *s*-normal (Gaussian) distribution, i.e., $gauf(Z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} \exp(-z^2/2)\, dz$. It is the area under the left-side of the pdf. For example, $gauf(-1.63) = 0.51551 \times 10^{-1}$; the E-notation gives the power of 10. For greater accuracy for large values of Z, use the identity: $gauf(Z) \equiv 1 - gauf(-Z)$. To find $gaufc(Z)$, the area under the right-side of the *pdf*, use the identity: $gaufc(Z) \equiv 1 - gauf(Z) \equiv gauf(-Z)$.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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TABLE 5-1

STANDARD *s*-NORMAL (GAUSSIAN) *Cdf* (Continued)

Z	0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
-0.0	0.50000E 00	0.49601E 00	0.49202E 00	0.48803E 00	0.48405E 00	0.48006E 00	0.47608E 00	0.47210E 00	0.46812E 00	0.46414E 00
-0.1	0.46017E 00	0.45620E 00	0.45224E 00	0.44828E 00	0.44433E 00	0.44038E 00	0.43644E 00	0.43251E 00	0.42858E 00	0.42465E 00
-0.2	0.42074E 00	0.41683E 00	0.41294E 00	0.40905E 00	0.40517E 00	0.40129E 00	0.39743E 00	0.39358E 00	0.38974E 00	0.38591E 00
-0.3	0.38209E 00	0.37828E 00	0.37448E 00	0.37070E 00	0.36693E 00	0.36317E 00	0.35942E 00	0.35569E 00	0.35197E 00	0.34827E 00
-0.4	0.34458E 00	0.34090E 00	0.33724E 00	0.33360E 00	0.32997E 00	0.32636E 00	0.32276E 00	0.31918E 00	0.31561E 00	0.31207E 00
-0.5	0.30854E 00	0.30503E 00	0.30153E 00	0.29806E 00	0.29460E 00	0.29116E 00	0.28774E 00	0.28434E 00	0.28096E 00	0.27760E 00
-0.6	0.27425E 00	0.27093E 00	0.26763E 00	0.26435E 00	0.26109E 00	0.25785E 00	0.25463E 00	0.25143E 00	0.24825E 00	0.24510E 00
-0.7	0.24196E 00	0.23885E 00	0.23576E 00	0.23270E 00	0.22965E 00	0.22663E 00	0.22363E 00	0.22065E 00	0.21770E 00	0.21476E 00
-0.8	0.21186E 00	0.20897E 00	0.20611E 00	0.20327E 00	0.20045E 00	0.19766E 00	0.19489E 00	0.19215E 00	0.18943E 00	0.18673E 00
-0.9	0.18406E 00	0.18141E 00	0.17879E 00	0.17619E 00	0.17361E 00	0.17106E 00	0.16853E 00	0.16602E 00	0.16354E 00	0.16109E 00
-1.0	0.15866E 00	0.15625E 00	0.15386E 00	0.15150E 00	0.14917E 00	0.14686E 00	0.14457E 00	0.14231E 00	0.14007E 00	0.13786E 00
-1.1	0.13567E 00	0.13350E 00	0.13136E 00	0.12924E 00	0.12714E 00	0.12507E 00	0.12302E 00	0.12100E 00	0.11900E 00	0.11702E 00
-1.2	0.11507E 00	0.11314E 00	0.11123E 00	0.10935E 00	0.10749E 00	0.10565E 00	0.10383E 00	0.10204E 00	0.10027E 00	0.98525E-01
-1.3	0.96800E-01	0.95098E-01	0.93417E-01	0.91759E-01	0.90123E-01	0.88508E-01	0.86915E-01	0.85343E-01	0.83793E-01	0.82264E-01
-1.4	0.80757E-01	0.79270E-01	0.77804E-01	0.76358E-01	0.74934E-01	0.73529E-01	0.72145E-01	0.70781E-01	0.69437E-01	0.68112E-01
-1.5	0.66807E-01	0.65522E-01	0.64255E-01	0.63008E-01	0.61780E-01	0.60571E-01	0.59380E-01	0.58208E-01	0.57053E-01	0.55917E-01
-1.6	0.54799E-01	0.53699E-01	0.52616E-01	0.51551E-01	0.50503E-01	0.49471E-01	0.48457E-01	0.47460E-01	0.46479E-01	0.45514E-01
-1.7	0.44565E-01	0.43633E-01	0.42716E-01	0.41815E-01	0.40930E-01	0.40059E-01	0.39204E-01	0.38364E-01	0.37538E-01	0.36727E-01
-1.8	0.35930E-01	0.35148E-01	0.34380E-01	0.33625E-01	0.32884E-01	0.32157E-01	0.31443E-01	0.30742E-01	0.30054E-01	0.29379E-01
-1.9	0.28717E-01	0.28067E-01	0.27429E-01	0.26803E-01	0.26190E-01	0.25588E-01	0.24998E-01	0.24419E-01	0.23852E-01	0.23295E-01
-2.0	0.22750E-01	0.22216E-01	0.21692E-01	0.21178E-01	0.20675E-01	0.20182E-01	0.19699E-01	0.19226E-01	0.18763E-01	0.18309E-01
-2.1	0.17864E-01	0.17429E-01	0.17003E-01	0.16586E-01	0.16177E-01	0.15778E-01	0.15386E-01	0.15003E-01	0.14629E-01	0.14262E-01
-2.2	0.13903E-01	0.13553E-01	0.13209E-01	0.12874E-01	0.12545E-01	0.12224E-01	0.11911E-01	0.11604E-01	0.11304E-01	0.11011E-01
-2.3	0.10724E-01	0.10444E-01	0.10170E-01	0.99031E-02	0.96419E-02	0.93867E-02	0.91375E-02	0.88940E-02	0.86563E-02	0.84242E-02
-2.4	0.81975E-02	0.79763E-02	0.77603E-02	0.75494E-02	0.73436E-02	0.71428E-02	0.69469E-02	0.67557E-02	0.65691E-02	0.63872E-02
-2.5	0.62097E-02	0.60366E-02	0.58677E-02	0.57031E-02	0.55426E-02	0.53861E-02	0.52336E-02	0.50849E-02	0.49400E-02	0.47988E-02
-2.6	0.46612E-02	0.45271E-02	0.43965E-02	0.42692E-02	0.41453E-02	0.40246E-02	0.39070E-02	0.37926E-02	0.36811E-02	0.35726E-02
-2.7	0.34670E-02	0.33642E-02	0.32641E-02	0.31667E-02	0.30720E-02	0.29798E-02	0.28901E-02	0.28028E-02	0.27179E-02	0.26354E-02
-2.8	0.25551E-02	0.24771E-02	0.24012E-02	0.23274E-02	0.22557E-02	0.21860E-02	0.21182E-02	0.20524E-02	0.19884E-02	0.19262E-02
-2.9	0.18658E-02	0.18071E-02	0.17502E-02	0.16948E-02	0.16411E-02	0.15889E-02	0.15382E-02	0.14890E-02	0.14412E-02	0.13949E-02
-3.0	0.13499E-02	0.13062E-02	0.12639E-02	0.12228E-02	0.11829E-02	0.11442E-02	0.11067E-02	0.10703E-02	0.10350E-02	0.10008E-02
-3.1	0.96760E-03	0.93544E-03	0.90426E-03	0.87403E-03	0.84474E-03	0.81635E-03	0.78885E-03	0.76219E-03	0.73638E-03	0.71136E-03
-3.2	0.68714E-03	0.66367E-03	0.64095E-03	0.61895E-03	0.59765E-03	0.57702E-03	0.55706E-03	0.53774E-03	0.51904E-03	0.50094E-03
-3.3	0.48342E-03	0.46648E-03	0.45009E-03	0.43423E-03	0.41889E-03	0.40406E-03	0.38971E-03	0.37584E-03	0.36243E-03	0.34946E-03
-3.4	0.33693E-03	0.32481E-03	0.31311E-03	0.30179E-03	0.29086E-03	0.28029E-03	0.27009E-03	0.26023E-03	0.25071E-03	0.24151E-03
-3.5	0.23263E-03	0.22405E-03	0.21577E-03	0.20778E-03	0.20006E-03	0.19262E-03	0.18543E-03	0.17849E-03	0.17180E-03	0.16534E-03
-3.6	0.15911E-03	0.15310E-03	0.14730E-03	0.14171E-03	0.13632E-03	0.13112E-03	0.12611E-03	0.12128E-03	0.11662E-03	0.11213E-03
-3.7	0.10780E-03	0.10363E-03	0.99611E-04	0.95740E-04	0.92010E-04	0.88417E-04	0.84957E-04	0.81624E-04	0.78414E-04	0.75324E-04
-3.8	0.72348E-04	0.69483E-04	0.66726E-04	0.64072E-04	0.61517E-04	0.59059E-04	0.56694E-04	0.54418E-04	0.52228E-04	0.50122E-04
-3.9	0.48096E-04	0.46148E-04	0.44274E-04	0.42473E-04	0.40741E-04	0.39076E-04	0.37475E-04	0.35936E-04	0.34458E-04	0.33037E-04

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TABLE 5-1

STANDARD *s*-NORMAL (GAUSSIAN) *Cdf* (Continued)

Z	0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
-4.0	0.31671E-04	0.30359E-04	0.29099E-04	0.27888E-04	0.26726E-04	0.25609E-04	0.24536E-04	0.23507E-04	0.22518E-04	0.21569E-04
-4.1	0.20658E-04	0.19783E-04	0.18944E-04	0.18138E-04	0.17365E-04	0.16624E-04	0.15912E-04	0.15230E-04	0.14575E-04	0.13948E-04
-4.2	0.13346E-04	0.12769E-04	0.12215E-04	0.11685E-04	0.11176E-04	0.10689E-04	0.10221E-04	0.97736E-05	0.93447E-05	0.89337E-05
-4.3	0.85399E-05	0.81627E-05	0.78015E-05	0.74555E-05	0.71241E-05	0.68069E-05	0.65031E-05	0.62123E-05	0.59340E-05	0.56675E-05
-4.4	0.54125E-05	0.51685E-05	0.49350E-05	0.47117E-05	0.44979E-05	0.42935E-05	0.40980E-05	0.39110E-05	0.37322E-05	0.35612E-05
-4.5	0.33977E-05	0.32414E-05	0.30920E-05	0.29492E-05	0.28127E-05	0.26823E-05	0.25577E-05	0.24386E-05	0.23249E-05	0.22162E-05
-4.6	0.21125E-05	0.20133E-05	0.19187E-05	0.18283E-05	0.17420E-05	0.16597E-05	0.15810E-05	0.15060E-05	0.14344E-05	0.13660E-05
-4.7	0.1300[illegible]05	0.12386E-05	0.11792E-05	0.11226E-05	0.10686E-05	0.10171E-05	0.96796E-06	0.92113E-06	0.87648E-06	0.83391E-06
-4.8	0.7933[illegible]E-06	0.75465E-06	0.71779E-06	0.68267E-06	0.64920E-06	0.61731E-06	0.58693E-06	0.55799E-06	0.53043E-06	0.50418E-06
-4.9	0.47918E-06	0.45538E-06	0.43272E-06	0.41115E-06	0.39061E-06	0.37107E-06	0.35247E-06	0.33476E-06	0.31792E-06	0.30190E-06
-5.0	0.28665E-06	0.27215E-06	0.25836E-06	0.24524E-06	0.23277E-06	0.22091E-06	0.20963E-06	0.19891E-06	0.18872E-06	0.17903E-06
-5.1	0.16983E-06	0.16108E-06	0.15277E-06	0.14487E-06	0.13737E-06	0.13024E-06	0.12347E-06	0.11705E-06	0.11094E-06	0.10515E-06
-5.2	0.99644E-07	0.94420E-07	0.89462E-07	0.84755E-07	0.80288E-07	0.76050E-07	0.72028E-07	0.68212E-07	0.64592E-07	0.61158E-07
-5.3	0.57901E-07	0.54813E-07	0.51884E-07	0.49106E-07	0.46473E-07	0.43977E-07	0.41611E-07	0.39368E-07	0.37243E-07	0.35229E-07
-5.4	0.33320E-07	0.31512E-07	0.29800E-07	0.28177E-07	0.26640E-07	0.25185E-07	0.23807E-07	0.22502E-07	0.21266E-07	0.20097E-07
-5.5	0.18990E-07	0.17942E-07	0.16950E-07	0.16012E-07	0.15124E-07	0.14283E-07	0.13489E-07	0.12737E-07	0.12026E-07	0.11353E-07
-5.6	0.10718E-07	0.10116E-07	0.95479E-08	0.90105E-08	0.85025E-08	0.80224E-08	0.75686E-08	0.71399E-08	0.67347E-08	0.63520E-08
-5.7	0.59904E-08	0.56468E-08	0.53262E-08	0.50215E-08	0.47338E-08	0.44622E-08	0.42057E-08	0.39636E-08	0.37350E-08	0.35193E-08
-5.8	0.33157E-08	0.31236E-08	0.29424E-08	0.27714E-08	0.26100E-08	0.24579E-08	0.23143E-08	0.21790E-08	0.20513E-08	0.19310E-08
-5.9	0.18175E-08	0.17105E-08	0.16097E-08	0.15147E-08	0.14251E-08	0.13407E-08	0.12612E-08	0.11863E-08	0.11157E-08	0.10492E-08
-6.0	0.98659E-09	0.92762E-09	0.87209E-09	0.81980E-09	0.77057E-09	0.72423E-09	0.68061E-09	0.63955E-09	0.60091E-09	0.56455E-09
-6.1	0.53034E-09	0.49816E-09	0.46788E-09	0.43940E-09	0.41261E-09	0.38741E-09	0.36372E-09	0.34145E-09	0.32051E-09	0.30082E-09
-6.2	0.28232E-09	0.26492E-09	0.24858E-09	0.23322E-09	0.21879E-09	0.20523E-09	0.19249E-09	0.18052E-09	0.16929E-09	0.15873E-09
-6.3	0.14882E-09	0.13952E-09	0.[illegible]3078E-09	0.12258E-09	0.11488E-09	0.10766E-09	0.10088E-09	0.94514E-10	0.88544E-10	0.82943E-10
-6.4	0.77688E-10	0.72760E-10	0.68137E-10	0.63802E-10	0.59737E-10	0.55925E-10	0.52351E-10	0.49001E-10	0.45861E-10	0.42918E-10
-6.5	0.40160E-10	0.37575E-10	0.35154E-10	0.32885E-10	0.30759E-10	0.28769E-10	0.26904E-10	0.25158E-10	0.23522E-10	0.21991E-10
-6.6	0.20558E-10	0.19216E-10	0.17960E-10	0.16784E-10	0.15684E-10	0.14655E-10	0.13691E-10	0.12790E-10	0.11947E-10	0.11159E-10
-6.7	0.10421E-10	0.97312E-11	0.90862E-11	0.84832E-11	0.79193E-11	0.73923E-11	0.68996E-11	0.64391E-11	0.60088E-11	0.56067E-11
-6.8	0.52310E-11	0.48799E-11	0.45520E-11	0.42457E-11	0.39597E-11	0.36925E-11	0.34430E-11	0.32101E-11	0.29926E-11	0.27896E-11
-6.9	0.26001E-11	0.24233E-11	0.22582E-11	0.21042E-11	0.19605E-11	0.18264E-11	0.17014E-11	0.15847E-11	0.14759E-11	0.13744E-11
-7.0	0.12798E-11	0.11916E-11	0.11093E-11	0.10327E-11	0.96120E-12	0.89459E-12	0.83251E-12	0.77467E-12	0.72077E-12	0.67056E-12
-7.1	0.62378E-12	0.58021E-12	0.53964E-12	0.50184E-12	0.46665E-12	0.43389E-12	0.40339E-12	0.37499E-12	0.34856E-12	0.32396E-12
-7.2	0.30106E-12	0.27976E-12	0.25994E-12	0.24150E-12	0.22434E-12	0.20839E-12	0.19355E-12	0.17974E-12	0.16691E-12	0.15498E-12
-7.3	0.14388E-12	0.13357E-12	0.122[illegible]9E-12	0.11508E-12	0.10680E-12	0.99103E-13	0.91955E-13	0.85314E-13	0.79145E-13	0.73414E-13
-7.4	0.68092E-13	0.63150E-13	0.58590E-13	0.54299E-13	0.50343E-13	0.46670E-13	0.43261E-13	0.40097E-13	0.37161E-13	0.34437E-13
-7.5	0.31909E-13	0.29564E-13	0.27388E-13	0.25370E-13	0.23499E-13	0.21763E-13	0.20153E-13	0.18661E-13	0.17278E-13	0.15995E-13
-7.6	0.14807E-13	0.13705E-13	0.12684E-13	0.11738E-13	0.10861E-13	0.10049E-13	0.92967E-14	0.85998E-14	0.79544E-14	0.73567E-14
-7.7	0.68033E-14	0.62909E-14	0.58165E-14	0.53773E-14	0.49708E-14	0.45946E-14	0.42465E-14	0.39243E-14	0.36262E-14	0.33505E-14
-7.8	0.30954E-14	0.28594E-14	0.26412E-14	0.24393E-14	0.22527E-14	0.20802E-14	0.19207E-14	0.17732E-14	0.16369E-14	0.15109E-14
-7.9	0.13945E-14	0.12869E-14	0.11876E-14	0.10957E-14	0.10109E-14	0.93256E-15	0.86020E-15	0.79337E-15	0.73167E-15	0.67469E-15

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TABLE 5-2

GAUSSIAN (STANDARD s-NORMAL) *Cdf* – *gauf* (*z*)

Body of the table is *z*, the standard *s*-normal variate, corresponding to *gauf* (*z*).

gauf (*z*)	.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
.00	–	−2.33	−2.05	−1.88	−1.75	−1.64	−1.55	−1.48	−1.41	−1.34
.10	−1.28	−1.23	−1.18	−1.13	−1.08	−1.04	−0.99	−0.95	−0.92	−0.88
.20	−0.84	−0.81	−0.77	−0.74	−0.71	−0.67	−0.64	−0.61	−0.58	−0.55
.30	−0.52	−0.50	−0.47	−0.44	−0.41	−0.39	−0.36	−0.33	−0.31	−0.28
.40	−0.25	−0.23	−0.20	−0.18	−0.15	−0.13	−0.10	−0.08	−0.05	−0.03
.50	0.00	−0.03	0.05	0.08	0.13	0.13	0.15	0.18	0.20	0.23
.60	0.25	0.28	0.31	0.33	0.36	0.39	0.41	0.44	0.47	0.50
.70	0.52	0.55	0.58	0.61	0.64	0.67	0.71	0.74	0.77	0.81
.80	0.84	0.88	0.92	0.95	0.99	1.04	1.08	1.13	1.18	1.23
.90	1.28	1.34	1.41	1.48	1.55	1.64	1.75	1.88	2.05	2.33

Special Values

gauf (*z*)	0.001	0.005	0.010	0.025	0.050	0.100
z	−3.090	−2.576	−2.326	−1.960	−1.645	−1.282
gauf (*z*)	0.999	0.995	0.990	0.975	0.950	0.900
z	3.090	2.576	2.326	1.960	1.645	1.282

TABLE 5-3

STANDARD s-NORMAL (GAUSSIAN) FAILURE RATE (HAZARD RATE)

The body of the table gives the failure rate (hazard rate) for the standard s-normal (Gaussian) distribution; i.e., $gauhr(Z) \equiv pdf\{Z\}/Sf\{Z\}$. For example, $gauhr(-2.19) = 0.36787 \times 10^{-1}$; the E-notation gives the power of 10. The failure rate for a nonstandardized variable X (with mean μ and standard deviation σ) is $(1/\sigma)\, gauhr(Z)$, where $Z \equiv (X - \mu)/\sigma$.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.7979	0.8043	0.8107	0.8171	0.8235	0.8300	0.8365	0.8430	0.8495	0.8560
0.1	0.8626	0.8692	0.8758	0.8824	0.8891	0.8958	0.9025	0.9092	0.9159	0.9226
0.2	0.9294	0.9362	0.9430	0.9498	0.9567	0.9636	0.9704	0.9773	0.9843	0.9912
0.3	0.9982	1.0051	1.0121	1.0192	1.0262	1.0332	1.0403	1.0474	1.0545	1.0616
0.4	1.0688	1.0759	1.0831	1.0903	1.0975	1.1047	1.1119	1.1192	1.1265	1.1338
0.5	1.1411	1.1484	1.1557	1.1631	1.1705	1.1779	1.1853	1.1927	1.2001	1.2076
0.6	1.2150	1.2225	1.2300	1.2375	1.2453	1.2526	1.2601	1.2677	1.2753	1.2829
0.7	1.2905	1.2981	1.3058	1.3134	1.3211	1.3288	1.3365	1.3442	1.3519	1.3596
0.8	1.3674	1.3752	1.3829	1.3907	1.3985	1.4064	1.4142	1.4220	1.4299	1.4378
0.9	1.4456	1.4535	1.4614	1.4694	1.4773	1.4852	1.4932	1.5012	1.5091	1.5171
1.0	1.5251	1.5331	1.5412	1.5492	1.5573	1.5653	1.5734	1.5815	1.5896	1.5977
1.1	1.6058	1.6139	1.6221	1.6302	1.6384	1.6465	1.6547	1.6629	1.6711	1.6793
1.2	1.6876	1.6958	1.7040	1.7123	1.7205	1.7288	1.7371	1.7454	1.7537	1.7620
1.3	1.7703	1.7787	1.7870	1.7953	1.8037	1.8121	1.8205	1.8288	1.8372	1.8456
1.4	1.8541	1.8625	1.8709	1.8794	1.8878	1.8963	1.9047	1.9132	1.9217	1.9302
1.5	1.9387	1.9472	1.9557	1.9642	1.9728	1.9813	1.9899	1.9984	2.0070	2.0155
1.6	2.0241	2.0327	2.0413	2.0499	2.0585	2.0671	2.0758	2.0844	2.0931	2.1017
1.7	2.1104	2.1190	2.1277	2.1364	2.1451	2.1537	2.1624	2.1712	2.1799	2.1886
1.8	2.1973	2.2060	2.2148	2.2235	2.2323	2.2410	2.2498	2.2586	2.2674	2.2762
1.9	2.2849	2.2937	2.3026	2.3114	2.3202	2.3290	2.3378	2.3467	2.3555	2.3644
2.0	2.3732	2.3821	2.3909	2.3998	2.4087	2.4176	2.4265	2.4354	2.4443	2.4532
2.1	2.4621	2.4710	2.4799	2.4888	2.4978	2.5067	2.5157	2.5246	2.5336	2.5425
2.2	2.5515	2.5605	2.5694	2.5784	2.5874	2.5964	2.6054	2.6144	2.6234	2.6324
2.3	2.6414	2.6505	2.6595	2.6685	2.6775	2.6866	2.6956	2.7047	2.7137	2.7228
2.4	2.7319	2.7409	2.7500	2.7591	2.7682	2.7772	2.7863	2.7954	2.8045	2.8136
2.5	2.8227	2.8319	2.8410	2.8501	2.8592	2.8683	2.8775	2.8866	2.8958	2.9049
2.6	2.9141	2.9232	2.9324	2.9415	2.9507	2.9599	2.9690	2.9782	2.9874	2.9966
2.7	3.0058	3.0150	3.0242	3.0334	3.0426	3.0518	3.0610	3.0702	3.0794	3.0886
2.8	3.0979	3.1071	3.1163	3.1256	3.1348	3.1440	3.1533	3.1625	3.1718	3.1811
2.9	3.1903	3.1996	3.2088	3.2181	3.2274	3.2367	3.2459	3.2552	3.2645	3.2738
3.0	3.2831	3.2924	3.3017	3.3110	3.3203	3.3296	3.3389	3.3482	3.3576	3.3669
3.1	3.3762	3.3855	3.3949	3.4042	3.4135	3.4229	3.4322	3.4415	3.4509	3.4602
3.2	3.4696	3.4789	3.4883	3.4977	3.5070	3.5164	3.5258	3.5351	3.5445	3.5539
3.3	3.5633	3.5726	3.5820	3.5914	3.6008	3.6102	3.6196	3.6290	3.6384	3.6478
3.4	3.6572	3.6666	3.6760	3.6854	3.6948	3.7043	3.7137	3.7231	3.7325	3.7420
3.5	3.7514	3.7608	3.7703	3.7797	3.7891	3.7986	3.8080	3.8175	3.8269	3.8364
3.6	3.8458	3.8553	3.8647	3.8742	3.8836	3.8931	3.9026	3.9120	3.9215	3.9310
3.7	3.9405	3.9499	3.9594	3.9689	3.9784	3.9879	3.9973	4.0068	4.0163	4.0258
3.8	4.0353	4.0448	4.0543	4.0638	4.0733	4.0828	4.0923	4.1018	4.1113	4.1209
3.9	4.1304	4.1399	4.1494	4.1589	4.1684	4.1780	4.1875	4.1970	4.2065	4.2161

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TABLE 5-3

STANDARD s-NORMAL (GAUSSIAN) FAILURE RATE (HAZARD RATE) (Continued)

Z	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
4,0	4,2256	4,2351	4,2447	4,2542	4,2638	4,2733	4,2828	4,2924	4,3019	4,3115
4,1	4,3210	4,3306	4,3401	4,3497	4,3592	4,3688	4,3784	4,3879	4,3975	4,4071
4,2	4,4166	4,4262	4,4358	4,4453	4,4549	4,4645	4,4740	4,4836	4,4932	4,5028
4,3	4,5124	4,5220	4,5315	4,5411	4,5507	4,5603	4,5699	4,5795	4,5891	4,5987
4,4	4,6083	4,6179	4,6275	4,6371	4,6467	4,6563	4,6659	4,6755	4,6851	4,6947
4,5	4,7043	4,7139	4,7235	4,7332	4,7428	4,7524	4,7620	4,7716	4,7813	4,7909
4,6	4,8005	4,8101	4,8198	4,8294	4,8390	4,8486	4,8583	4,8679	4,8776	4,8872
4,7	4,8968	4,9065	4,9161	4,9257	4,9354	4,9450	4,9547	4,9643	4,9740	4,9836
4,8	4,9933	5,0029	5,0126	5,0222	5,0319	5,0415	5,0512	5,0608	5,0705	5,0802
4,9	5,0898	5,0995	5,1092	5,1188	5,1285	5,1382	5,1478	5,1575	5,1672	5,1768
5,0	5,1865	5,1962	5,2059	5,2155	5,2252	5,2349	5,2446	5,2542	5,2639	5,2736
5,1	5,2833	5,2930	5,3027	5,3123	5,3220	5,3317	5,3414	5,3511	5,3608	5,3705
5,2	5,3802	5,3899	5,3996	5,4093	5,4190	5,4287	5,4384	5,4481	5,4578	5,4675
5,3	5,4772	5,4869	5,4966	5,5063	5,5160	5,5257	5,5354	5,5451	5,5548	5,5645
5,4	5,5742	5,5840	5,5937	5,6034	5,6131	5,6228	5,6325	5,6423	5,6520	5,6617
5,5	5,6714	5,6811	5,6909	5,7006	5,7103	5,7200	5,7298	5,7395	5,7492	5,7589
5,6	5,7687	5,7784	5,7881	5,7979	5,8076	5,8173	5,8271	5,8368	5,8465	5,8563
5,7	5,8660	5,8757	5,8855	5,8952	5,9050	5,9147	5,9244	5,9342	5,9439	5,9537
5,8	5,9634	5,9732	5,9829	5,9927	6,0024	6,0122	6,0219	6,0317	6,0414	6,0512
5,9	6,0609	6,0707	6,0804	6,0902	6,0999	6,1097	6,1194	6,1292	6,1390	6,1487
6,0	6,1585	6,1682	6,1780	6,1878	6,1975	6,2073	6,2171	6,2268	6,2366	6,2464
6,1	6,2561	6,2659	6,2757	6,2854	6,2952	6,3050	6,3147	6,3245	6,3343	6,3440
6,2	6,3538	6,3636	6,3734	6,3831	6,3929	6,4027	6,4125	6,4222	6,4320	6,4418
6,3	6,4516	6,4614	6,4711	6,4809	6,4907	6,5005	6,5103	6,5201	6,5298	6,5396
6,4	6,5494	6,5592	6,5690	6,5788	6,5886	6,5984	6,6081	6,6179	6,6277	6,6375
6,5	6,6473	6,6571	6,6669	6,6767	6,6865	6,6963	6,7061	6,7159	6,7257	6,7354
6,6	6,7452	6,7550	6,7648	6,7746	6,7844	6,7942	6,8040	6,8138	6,8236	6,8334
6,7	6,8432	6,8530	6,8628	6,8727	6,8825	6,8923	6,9021	6,9119	6,9217	6,9315
6,8	6,9413	6,9511	6,9609	6,9707	6,9805	6,9903	7,0001	7,0100	7,0198	7,0296
6,9	7,0394	7,0492	7,0590	7,0688	7,0786	7,0885	7,0983	7,1081	7,1179	7,1277
7,0	7,1375	7,1474	7,1572	7,1670	7,1768	7,1866	7,1965	7,2063	7,2161	7,2259
7,1	7,2357	7,2456	7,2554	7,2652	7,2750	7,2849	7,2947	7,3045	7,3143	7,3242
7,2	7,3340	7,3438	7,3536	7,3635	7,3733	7,3831	7,3930	7,4028	7,4126	7,4224
7,3	7,4323	7,4421	7,4519	7,4618	7,4716	7,4814	7,4913	7,5011	7,5109	7,5208
7,4	7,5306	7,5404	7,5503	7,5601	7,5699	7,5798	7,5896	7,5995	7,6093	7,6191
7,5	7,6290	7,6388	7,6486	7,6585	7,6683	7,6782	7,6880	7,6978	7,7077	7,7175
7,6	7,7274	7,7372	7,7471	7,7569	7,7667	7,7766	7,7864	7,7963	7,8061	7,8160
7,7	7,8258	7,8357	7,8455	7,8554	7,8652	7,8751	7,8849	7,8948	7,9046	7,9145
7,8	7,9243	7,9341	7,9440	7,9539	7,9637	7,9736	7,9834	7,9933	8,0031	8,0130
7,9	8,0228	8,0327	8,0425	8,0524	8,0622	8,0721	8,0819	8,0918	8,1017	8,1115

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TABLE 5-3

STANDARD s-NORMAL (GAUSSIAN) FAILURE RATE (HAZARD RATE) (Continued)

Z	0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
0.0	0.79788E 00	0.79153E 00	0.78520E 00	0.77888E 00	0.77260E 00	0.76633E 00	0.76008E 00	0.75386E 00	0.74766E 00	0.74149E 00
-0.1	0.73533E 00	0.72920E 00	0.72309E 00	0.71701E 00	0.71095E 00	0.70491E 00	0.69889E 00	0.69290E 00	0.68694E 00	0.68099E 00
-0.2	0.67507E 00	0.66918E 00	0.66331E 00	0.65746E 00	0.65164E 00	0.64584E 00	0.64007E 00	0.63432E 00	0.62859E 00	0.62289E 00
-0.3	0.61722E 00	0.61157E 00	0.60595E 00	0.60035E 00	0.59478E 00	0.58923E 00	0.58371E 00	0.57821E 00	0.57274E 00	0.56730E 00
-0.4	0.56188E 00	0.55649E 00	0.55113E 00	0.54579E 00	0.54047E 00	0.53519E 00	0.52993E 00	0.52470E 00	0.51949E 00	0.51431E 00
-0.5	0.50916E 00	0.50404E 00	0.49894E 00	0.49387E 00	0.48883E 00	0.48381E 00	0.47882E 00	0.47386E 00	0.46893E 00	0.46402E 00
-0.6	0.45915E 00	0.45430E 00	0.44948E 00	0.44468E 00	0.43992E 00	0.43518E 00	0.43047E 00	0.42579E 00	0.42114E 00	0.41652E 00
-0.7	0.41192E 00	0.40736E 00	0.40282E 00	0.39831E 00	0.39383E 00	0.38938E 00	0.38496E 00	0.38057E 00	0.37620E 00	0.37187E 00
-0.8	0.36756E 00	0.36328E 00	0.35904E 00	0.35482E 00	0.35063E 00	0.34647E 00	0.34234E 00	0.33824E 00	0.33416E 00	0.33012E 00
-0.9	0.32611E 00	0.32213E 00	0.31817E 00	0.31425E 00	0.31035E 00	0.30649E 00	0.30265E 00	0.29884E 00	0.29507E 00	0.29132E 00
-1.0	0.28760E 00	0.28391E 00	0.28025E 00	0.27662E 00	0.27302E 00	0.26945E 00	0.26591E 00	0.26240E 00	0.25892E 00	0.25547E 00
-1.1	0.25205E 00	0.24865E 00	0.24529E 00	0.24196E 00	0.23865E 00	0.23538E 00	0.23213E 00	0.22891E 00	0.22572E 00	0.22257E 00
-1.2	0.21944E 00	0.21634E 00	0.21327E 00	0.21022E 00	0.20721E 00	0.20423E 00	0.20127E 00	0.19834E 00	0.19545E 00	0.19258E 00
-1.3	0.18973E 00	0.18692E 00	0.18414E 00	0.18138E 00	0.17866E 00	0.17596E 00	0.17328E 00	0.17064E 00	0.16803E 00	0.16544E 00
-1.4	0.16288E 00	0.16035E 00	0.15785E 00	0.15537E 00	0.15292E 00	0.15050E 00	0.14810E 00	0.14573E 00	0.14339E 00	0.14108E 00
-1.5	0.13879E 00	0.13653E 00	0.13429E 00	0.13209E 00	0.12990E 00	0.12775E 00	0.12562E 00	0.12351E 00	0.12143E 00	0.11938E 00
-1.6	0.11735E 00	0.11535E 00	0.11337E 00	0.11142E 00	0.10949E 00	0.10759E 00	0.10571E 00	0.10385E 00	0.10202E 00	0.10022E 00
-1.7	0.98436E-01	0.96677E-01	0.94943E-01	0.93231E-01	0.91543E-01	0.89878E-01	0.88235E-01	0.86616E-01	0.85019E-01	0.83445E-01
-1.8	0.81893E-01	0.80362E-01	0.78854E-01	0.77368E-01	0.75903E-01	0.74459E-01	0.73037E-01	0.71635E-01	0.70255E-01	0.68895E-01
-1.9	0.67556E-01	0.66237E-01	0.64938E-01	0.63659E-01	0.62399E-01	0.61160E-01	0.59939E-01	0.58738E-01	0.57556E-01	0.56393E-01
-2.0	0.55248E-01	0.54122E-01	0.53014E-01	0.51924E-01	0.50851E-01	0.49797E-01	0.48760E-01	0.47741E-01	0.46738E-01	0.45752E-01
-2.1	0.44784E-01	0.43831E-01	0.42895E-01	0.41976E-01	0.41072E-01	0.40184E-01	0.39312E-01	0.38455E-01	0.37613E-01	0.36787E-01
-2.2	0.35975E-01	0.35178E-01	0.34395E-01	0.33627E-01	0.32873E-01	0.32132E-01	0.31406E-01	0.30693E-01	0.29994E-01	0.29307E-01
-2.3	0.28634E-01	0.27974E-01	0.27326E-01	0.26691E-01	0.26068E-01	0.25457E-01	0.24858E-01	0.24271E-01	0.23696E-01	0.23132E-01
-2.4	0.22580E-01	0.22038E-01	0.21508E-01	0.20988E-01	0.20479E-01	0.19980E-01	0.19492E-01	0.19013E-01	0.18545E-01	0.18087E-01
-2.5	0.17638E-01	0.17198E-01	0.16768E-01	0.16348E-01	0.15936E-01	0.15533E-01	0.15139E-01	0.14753E-01	0.14376E-01	0.14007E-01
-2.6	0.13647E-01	0.13294E-01	0.12949E-01	0.12612E-01	0.12282E-01	0.11960E-01	0.11646E-01	0.11338E-01	0.11038E-01	0.10744E-01
-2.7	0.10457E-01	0.10177E-01	0.99035E-02	0.96363E-02	0.93754E-02	0.91207E-02	0.88721E-02	0.86294E-02	0.83925E-02	0.81613E-02
-2.8	0.79357E-02	0.77156E-02	0.75009E-02	0.72914E-02	0.70871E-02	0.68878E-02	0.66935E-02	0.65040E-02	0.63193E-02	0.61392E-02
-2.9	0.59637E-02	0.57926E-02	0.56258E-02	0.54634E-02	0.53050E-02	0.51508E-02	0.50006E-02	0.48543E-02	0.47117E-02	0.45730E-02
-3.0	0.44378E-02	0.43063E-02	0.41782E-02	0.40535E-02	0.39322E-02	0.38141E-02	0.36992E-02	0.35874E-02	0.34787E-02	0.33729E-02
-3.1	0.32700E-02	0.31699E-02	0.30726E-02	0.29780E-02	0.28860E-02	0.27965E-02	0.27096E-02	0.26251E-02	0.25430E-02	0.24632E-02
-3.2	0.23857E-02	0.23104E-02	0.22373E-02	0.21662E-02	0.20972E-02	0.20302E-02	0.19652E-02	0.19020E-02	0.18407E-02	0.17812E-02
-3.3	0.17234E-02	0.16673E-02	0.16130E-02	0.15602E-02	0.15090E-02	0.14593E-02	0.14112E-02	0.13644E-02	0.13191E-02	0.12752E-02
-3.4	0.12326E-02	0.11914E-02	0.11513E-02	0.11126E-02	0.10750E-02	0.10386E-02	0.10033E-02	0.96911E-03	0.93601E-03	0.90394E-03
-3.5	0.87289E-03	0.84281E-03	0.81370E-03	0.78551E-03	0.75822E-03	0.73181E-03	0.70624E-03	0.68150E-03	0.65756E-03	0.63440E-03
-3.6	0.61200E-03	0.59033E-03	0.56936E-03	0.54909E-03	0.52949E-03	0.51053E-03	0.49221E-03	0.47449E-03	0.45737E-03	0.44082E-03
-3.7	0.42483E-03	0.40937E-03	0.39444E-03	0.38002E-03	0.36608E-03	0.35263E-03	0.33963E-03	0.32708E-03	0.31496E-03	0.30326E-03
-3.8	0.29197E-03	0.28107E-03	0.27055E-03	0.26039E-03	0.25059E-03	0.24114E-03	0.23202E-03	0.22322E-03	0.21474E-03	0.20656E-03
-3.9	0.19867E-03	0.19106E-03	0.18372E-03	0.17665E-03	0.16983E-03	0.16326E-03	0.15693E-03	0.15083E-03	0.14495E-03	0.13929E-03

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TABLE 5-3

STANDARD *s*-NORMAL (GAUSSIAN) FAILURE RATE (HAZARD RATE) (Continued)

Z	0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
-4.0	0.13383E-03	0.12858E-03	0.12352E-03	0.11865E-03	0.11395E-03	0.10944E-03	0.10509E-03	0.10090E-03	0.96872E-04	0.92995E-04
-4.1	0.89263E-04	0.85673E-04	0.82219E-04	0.78897E-04	0.75701E-04	0.72627E-04	0.69671E-04	0.66829E-04	0.64096E-04	0.61469E-04
-4.2	0.58944E-04	0.56517E-04	0.54184E-04	0.51942E-04	0.49788E-04	0.47719E-04	0.45731E-04	0.43822E-04	0.41988E-04	0.40227E-04
-4.3	0.38536E-04	0.36912E-04	0.35353E-04	0.33856E-04	0.32420E-04	0.31042E-04	0.29719E-04	0.28449E-04	0.27232E-04	0.26063E-04
-4.4	0.24943E-04	0.23868E-04	0.22837E-04	0.21848E-04	0.20901E-04	0.19992E-04	0.19121E-04	0.18286E-04	0.17486E-04	0.16719E-04
-4.5	0.15984E-04	0.15280E-04	0.14605E-04	0.13959E-04	0.13340E-04	0.12747E-04	0.12180E-04	0.11636E-04	0.11116E-04	0.10618E-04
-4.6	0.10141E-04	0.96845E-05	0.92477E-05	0.88297E-05	0.84298E-05	0.80472E-05	0.76812E-05	0.73311E-05	0.69962E-05	0.66760E-05
-4.7	0.63698E-05	0.60771E-05	0.57972E-05	0.55296E-05	0.52739E-05	0.50295E-05	0.47960E-05	0.45728E-05	0.43596E-05	0.41559E-05
-4.8	0.39613E-05	0.37755E-05	0.35980E-05	0.34285E-05	0.32667E-05	0.31122E-05	0.29647E-05	0.28239E-05	0.26895E-05	0.25613E-05
-4.9	0.24390E-05	0.23222E-05	0.22108E-05	0.21046E-05	0.20033E-05	0.19066E-05	0.18144E-05	0.17265E-05	0.16428E-05	0.15629E-05
-5.0	0.14867E-05	0.14141E-05	0.13450E-05	0.12791E-05	0.12162E-05	0.11564E-05	0.10994E-05	0.10451E-05	0.99339E-06	0.94414E-06
-5.1	0.89724E-06	0.85259E-06	0.81008E-06	0.76961E-06	0.73108E-06	0.69442E-06	0.65953E-06	0.62633E-06	0.59474E-06	0.56469E-06
-5.2	0.53610E-06	0.50891E-06	0.48305E-06	0.45846E-06	0.43508E-06	0.41285E-06	0.39171E-06	0.37162E-06	0.35253E-06	0.33438E-06
-5.3	0.31713E-06	0.30075E-06	0.28518E-06	0.27039E-06	0.25635E-06	0.24300E-06	0.23033E-06	0.21830E-06	0.20688E-06	0.19603E-06
-5.4	0.18574E-06	0.17596E-06	0.16669E-06	0.15789E-06	0.14953E-06	0.14161E-06	0.13409E-06	0.12696E-06	0.12020E-06	0.11378E-06
-5.5	0.10770E-06	0.10193E-06	0.96460E-07	0.91275E-07	0.86360E-07	0.81702E-07	0.77287E-07	0.73104E-07	0.69140E-07	0.65384E-07
-5.6	0.61826E-07	0.58456E-07	0.55264E-07	0.52242E-07	0.49379E-07	0.46669E-07	0.44105E-07	0.41674E-07	0.39375E-07	0.37199E-07
-5.7	0.35140E-07	0.33191E-07	0.31347E-07	0.29603E-07	0.27953E-07	0.26392E-07	0.24916E-07	0.23521E-07	0.22201E-07	0.20953E-07
-5.8	0.19773E-07	0.18658E-07	0.17604E-07	0.16608E-07	0.15667E-07	0.14777E-07	0.13937E-07	0.13143E-07	0.12393E-07	0.11685E-07
-5.9	0.11016E-07	0.10384E-07	0.97877E-08	0.92246E-08	0.86931E-08	0.81913E-08	0.77178E-08	0.72709E-08	0.68492E-08	0.64513E-08
-6.0	0.60759E-08	0.57218E-08	0.53877E-08	0.50727E-08	0.47756E-08	0.44955E-08	0.42314E-08	0.39824E-08	0.37476E-08	0.35264E-08
-6.1	0.33179E-08	0.31214E-08	0.29362E-08	0.27618E-08	0.25974E-08	0.24426E-08	0.22968E-08	0.21595E-08	0.20302E-08	0.19084E-08
-6.2	0.17938E-08	0.16859E-08	0.15843E-08	0.14887E-08	0.13987E-08	0.13140E-08	0.12343E-08	0.11594E-08	0.10889E-08	0.10225E-08
-6.3	0.96014E-09	0.90148E-09	0.84631E-09	0.79444E-09	0.74567E-09	0.69983E-09	0.65674E-09	0.61624E-09	0.57818E-09	0.54242E-09
-6.4	0.50881E-09	0.47725E-09	0.44759E-09	0.41974E-09	0.39358E-09	0.36901E-09	0.34595E-09	0.32429E-09	0.30396E-09	0.28487E-09
-6.5	0.26696E-09	0.25014E-09	0.23437E-09	0.21956E-09	0.20567E-09	0.19264E-09	0.18042E-09	0.16896E-09	0.15820E-09	0.14812E-09
-6.6	0.13867E-09	0.12980E-09	0.12150E-09	0.11371E-09	0.10641E-09	0.99567E-10	0.93157E-10	0.87150E-10	0.81523E-10	0.76251E-10
-6.7	0.71313E-10	0.66689E-10	0.62357E-10	0.58302E-10	0.54504E-10	0.50949E-10	0.47621E-10	0.44506E-10	0.41591E-10	0.38863E-10
-6.8	0.36310E-10	0.33921E-10	0.31686E-10	0.29596E-10	0.27641E-10	0.25812E-10	0.24102E-10	0.22503E-10	0.21008E-10	0.19610E-10
-6.9	0.18303E-10	0.17082E-10	0.15941E-10	0.14874E-10	0.13878E-10	0.12947E-10	0.12077E-10	0.11264E-10	0.10505E-10	0.97966E-11
-7.0	0.91347E-11	0.85157E-11	0.79398E-11	0.74011E-11	0.68984E-11	0.64291E-11	0.59911E-11	0.55825E-11	0.52012E-11	0.48454E-11
-7.1	0.45135E-11	0.42040E-11	0.39153E-11	0.36460E-11	0.33949E-11	0.31608E-11	0.29426E-11	0.27391E-11	0.25495E-11	0.23727E-11
-7.2	0.22080E-11	0.20545E-11	0.19115E-11	0.17783E-11	0.16541E-11	0.15385E-11	0.14309E-11	0.13306E-11	0.12372E-11	0.11503E-11
-7.3	0.10694E-11	0.99405E-12	0.92393E-12	0.85867E-12	0.79794E-12	0.74144E-12	0.68886E-12	0.63995E-12	0.59445E-12	0.55213E-12
-7.4	0.51278E-12	0.47618E-12	0.44214E-12	0.41050E-12	0.38109E-12	0.35375E-12	0.32834E-12	0.30472E-12	0.28277E-12	0.26238E-12
-7.5	0.24343E-12	0.22583E-12	0.20948E-12	0.19430E-12	0.18019E-12	0.16710E-12	0.15494E-12	0.14365E-12	0.13317E-12	0.12344E-12
-7.6	0.11442E-12	0.10604E-12	0.98262E-13	0.91048E-13	0.84355E-13	0.78147E-13	0.72388E-13	0.67047E-13	0.62093E-13	0.57500E-13
-7.7	0.53241E-13	0.49293E-13	0.45633E-13	0.42241E-13	0.39097E-13	0.36183E-13	0.33483E-13	0.30981E-13	0.28664E-13	0.26517E-13
-7.8	0.24529E-13	0.22687E-13	0.20981E-13	0.19402E-13	0.17940E-13	0.16586E-13	0.15333E-13	0.14174E-13	0.13100E-13	0.12107E-13
-7.9	0.11188E-13	0.10338E-13	0.95509E-14	0.88232E-14	0.81502E-14	0.75277E-14	0.69521E-14	0.64198E-14	0.59277E-14	0.54728E-14

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5-4 PARAMETER ESTIMATION, UNCENSORED SAMPLES

Estimation for uncensored (complete) samples is very easy and straightforward; if the samples are censored, more complicated techniques must be used.

Conventional wisdom uses the estimators (for a sample of size N):

$\hat{\mu}$ = sample mean = $\bar{x}$

$\hat{\sigma}$ = (sample standard deviation) $\times \sqrt{\frac{N}{N-1}}$ = s statistic.

$\hat{\mu}$ is unbiased and maximum likelihood. s^2 is an unbiased estimator for σ^2 but s is a biased estimator for σ.

The maximum likelihood estimator for σ is

$\hat{\sigma}_{ML}$ = sample standard deviation.

Other useful estimators are the sample median for μ and various measures involving the sample range for σ. These other estimators will not be discussed further here but can be found in most textbooks on statistical quality control or in Ref. 1.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{N}} \qquad (5\text{-}5a)$$

has the Student's t-distribution with $\nu = N - 1$ degrees of freedom. This fact can be used to set s-confidence limits on $\hat{\mu}$:

$$\mathrm{Conf}\{\mu \le \bar{x} - ts/\sqrt{N}\} = stufc(t; N-1) \qquad (5\text{-}5b)$$

$$(\chi^2/\nu) = s^2/\sigma^2 \qquad (5\text{-}6a)$$

has the chi-square/nu (χ^2/ν) distribution with $\nu = N - 1$ degrees of freedom. This fact can be used to set s-confidence limits on σ:

$$\mathrm{Conf}\{\sigma^2 \le s^2/(\chi^2/\nu)\} = csnfc[(\chi^2/\nu); (N-1)] . \qquad (5\text{-}6b)$$

The subscripts L, U are used to denote Lower and Upper s-confidence limits, respectively.

Joint s-confidence limits on μ and σ are not feasible. The cases where either μ or σ (but not both) is known are simpler to treat but are rarely met in practice.

5-5 EXAMPLES

The following data on strengths of a plastic bar were taken from 1 lot of bars. They are listed in order of occurrence. All have the same units, which are ignored here. Assume s-normality and estimate μ and σ, along with suitable s-confidence limits.

89.0	85.8
105.2	93.3
105.2	87.5
107.7	92.3
99.5	95.6

$N = 10$

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Procedure	Example
1. Calculate the sample mean $\bar{x}$, the s statistic, and the degrees of freedom for s.	1. $\bar{x} = 96.11$ $s = 7.92$ $\nu = 10 - 1 = 9$
2. Estimate μ and σ.	2. $\hat{\mu} = 96.11$ $\hat{\sigma} = 7.92$
3. Calculate s-confidence limits on μ. Use Eq. 5-5b.	3. For $C = 5\%, 95\%$, and $\nu = 9$ $t_{9,5\%} = -1.833$, $t_{9,95\%} = +1.833$. s-confidence level = 95% − 5% = 90% $\mu_L = 91.52$, $\mu_U = 100.70$ Conf $\{91.5 \leqslant \mu \leqslant 100.7\} = 90\%$
4. Calculate s-confidence limits on s. Use Eq. 5-6b.	4. For $C = 5\%, 95\%$ and $\nu = 9$ $(\chi^2/\nu)_{9,5\%} = 0.3694$, $(\chi^2/\nu)_{9,95\%} = 1.8799$. s-confidence level = 95% − 5% = 90% $\sigma_L = 5.78$, $\sigma_U = 13.03$ Conf $\{5.8 \leqslant \sigma \leqslant 13.0\} = 90\%$

There are no data outside the range of 85.8 to 107.7. Therefore, it is difficult to guess what the population is like out there.

The true mean is not known too well, within about 9% (at 90% s-confidence), and the true standard deviation is only known within a factor of about 2 (at 90% s-confidence). Any estimates more certain than those must come from other knowledge—they cannot come from the data. Be very careful not to use the point estimates and blithely forget all the uncertainty. For example, suppose someone wants to know the value of x such that only 1% of the population lies below it. One way to get an idea about an answer is to take the 2 worst cases, "$\mu = \mu_L$ and $\sigma = \sigma_U$", and "$\mu = \mu_U$ and $\sigma = \sigma_L$" and see what the two 1% values come out to be if the distribution were s-normal. The number of standard deviations corresponding to the lower 1% point is −2.33. Therefore $x_{1\%,L} = 61.2$, $x_{1\%,U} = 87.2$. We don't know any s-confidence level for this range, but it gives us an idea anyway. However, we don't really know that the distribution is s-normal down that low, there might be 7% defectives down at about 30 for all we know. If that $x_{1\%}$ value is important to know, we have to get more knowledge from somewhere, or admit that we're just guessing.

The detail in this discussion has been to show that making the calculations is straightforward, but getting some understanding is difficult.

In this example, the data were selected randomly from a s-normal distribution with mean 100 and standard deviation 10; the lower 1% point was 76.7.

5-6 PARAMETER ESTIMATION, CENSORED SAMPLES

Analytic estimation of the parameters is difficult when the samples are censored. The method of maximum likelihood often is used,

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in such cases, but the equations are very complicated, especially for the covariance matrix of the estimators. Analytic techniques that do not provide a measure of the uncertainty can be very misleading because the uncertainties are usually much greater than for uncensored samples. Graphical estimation is a reasonably good method and can give some idea of the uncertainty.

A statistician ought to be consulted for analytic techniques and the meaning of their results. Ref. 4 might be of some help.

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CHAPTER 6

PROBABILITY DISTRIBUTIONS DERIVED FROM THE GAUSSIAN DISTRIBUTION

6-0 LIST OF SYMBOLS

C = *s*-Confidence

C, L, U = subscripts that imply a *s*-confidence level; C is general, L is lower, U is upper.

Cdf = Cumulative distribution function

$CM_i\{\ \}$ = ith central moment

Conf{ } = *s*-Confidence level

csn = base name for chi-square/nu distribution

csq = base name for chi-square distribution

CV{ } = coefficient of variation: StDv{ }/E{ }

$E\{\ \}$ = *s*-Expected value

$\ldots f$ = suffix on base name, implies the *Cdf*

F = F statistic

$\ldots fc$ = suffix on base name, implies the complement of the *Cdf* (i.e., the *Sf*)

fis = base name for Fisher-Snedecor distribution

gau = base name for standard *s*-normal distribution

$M_i\{\ \}$ = ith moment about the origin

$NCM_i\{\ \}$ = normalized ith central moment: $CM_i\{\ \}/[\text{StDv}\{\ \}]^i$

pdf = probability density function

pmf = probability mass function

$Pr\{\ \}$ = Probability

PrD = Probability distribution

R = *s*-Reliability

s = s statistic

s- = denotes statistical definition

Sf = Survivor function

StDv{ } = standard deviation

stu = base name for Student's t distribution

t = t statistic

Var{ } = variance

z = standard *s*-normal variate

ν = degrees of freedom (also used with subscripts)

σ = scale parameter for *s*-normal distribution

$\{\cdot;\cdot\},(\cdot;\cdot)$ = the fixed parameters are listed to the right of the semicolon,

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the random variable is listed to the left of the semicolon

$\bar{}$ = the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

6-1 INTRODUCTION

These *PrD*'s are rarely if ever fitted to experimental data. They are useful because some estimators for other *PrD*'s have these *PrD*'s. The χ^2 and χ^2/ν distributions are related to the Poisson and exponential distributions, in addition to the Gaussian distribution. Failure rates are virtually never used with these distributions; so they are not given. They could be readily (although tediously) calculated.

For many reliability purposes, the χ^2/ν distribution is more useful than the χ^2 distribution.

6-2 CHI-SQUARE (χ^2) DISTRIBUTION

The base name *csq* is given to the χ^2 distribution (for chi-square). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

The sum of the squares of ν *s*-independent standard *s*-normal variates is a χ^2 variate with ν degrees of freedom. In reliability work, the χ^2 distribution itself is rarely needed; it is virtually always the χ^2/ν distribution that is desired.

6-2.1 FORMULAS

ν = degrees of freedom, $\nu > 0$

χ^2 = random variable, $\chi^2 \geq 0$

$$pdf\{\chi^2;\nu\} = c(\chi^2)^{\nu/2-1}\exp(-\chi^2/2) \quad (6\text{-}1)$$

$$c \equiv 2^{\nu/2}[\Gamma(\nu/2)]^{-1}$$

$$Cdf\{\chi^2;\nu\} = csqf(\chi^2;\nu) \quad (6\text{-}2)$$

$$Sf\{\chi^2;\nu\} = csqfc(\chi^2;\nu) \quad (6\text{-}3)$$

$$E\{\chi^2;\nu\} = \nu$$

$$StDv\{\chi^2;\nu\} = \sqrt{2\nu}$$

$$CV\{\chi^2;\nu\} = \sqrt{2/\nu}$$

$$CM_3\{\chi^2;\nu\} = 8\nu$$

$$NCM_3\{\chi^2;\nu\} = \sqrt{8/\nu}$$

$$\text{mode}\{\chi^2;\nu\} = \begin{cases} \nu - 2, & \text{for } \nu > 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{median}\{\chi^2;\nu\} \approx \nu - 0.6$$

Fig. 6-1 shows some curves of the *pdf*.

It is convenient occasionally to define the *Cdf* and *Sf* for $\nu = 0$.

$$csqf(\chi^2;0) \equiv 1, \text{ for } \chi^2 > 0 \quad (6\text{-}4a)$$

$$csqfc(\chi^2;0) \equiv 0, \text{ for } \chi^2 > 0 \quad (6\text{-}4b)$$

$$csqf(0;0) \equiv 0, \; csqfc(0;0) \equiv 1 \quad (6\text{-}4c)$$

Some approximations for χ^2 in terms of the standard *s*-normal variate *z* are

$$\text{for } \nu \to \infty, \; z_Q \approx (\chi^2_{Q,\nu} - \nu)/\sqrt{2\nu} \quad (6\text{-}5)$$

$$\text{for } \nu > 100, \; z_Q \approx \sqrt{2\chi^2_{Q,\nu}} - \sqrt{2\nu - 1} \quad (6\text{-}6)$$

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for $\nu > 20$, $z_Q \approx \left[(\chi^2_{Q,\nu}/\nu)^{1/3} - \left(1 - \frac{2}{9\nu}\right)\right] \Big/ \sqrt{\frac{2}{9\nu}}$ (6-7)

where $gaufc(z_Q) = Q$, $csqfc(\chi^2_{Q,\nu}; \nu) = Q$

6-2.2 TABLES

Calculating the *pdf* is straightforward but tedious. It is rarely used and rarely tabulated. The *pdf* is shown in Fig. 6-1. Table 6-1 gives the percentiles of the chi-square *Cdf*. Other good references for the *Cdf* are Ref. 1 (Tables 7, 8, and Chap. 3) and Ref. 2 (Tables 26.7, 26.8, and Sec. 26.4). Many statistical/quality-control texts give partial tables of the chi-square distribution.

Eq. 6-8 is quite good, even for small values of ν; it is the inverse of Eq. 6-7.

$$\chi^2_{Q,\nu} \approx \chi^{2*}_{Q,\nu} \equiv \nu\left[1 - \frac{2}{9\nu} + z_Q\left(\frac{2}{9\nu}\right)^{1/2}\right]^3 \tag{6-8}$$

where

$$gaufc(z_Q) = Q, \quad csqfc(\chi^2_{Q,\nu}; \nu) = Q$$

Eqs. 6-7 and 6-8 reproduce the $\{Cdf\ \chi^2; \nu\}$ quite well for values of the *Cdf* as low as $1/\nu^2$. Very roughly, the relative error of a tail area of $1/\nu^2$ is less than $1/\nu^2$. For $\nu = 5$, Eq. 6-8 gives the following results:

Q	$csqfc(\chi^{2*}_{Q,\nu}; \nu)$	relative error in tail area
0.001	0.00092	−0.08
0.01	0.00990	−0.01
1 − 0.1	1 − 0.1000	0.0
1 − 0.05	1 − 0.0487	−0.026
1 − 0.01	1 − 0.008	−0.20

6-3 CHI-SQUARE/NU (χ^2/ν) DISTRIBUTION

The base name *csn* is given to the χ^2/ν distribution (for chi-square/nu). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

The average time-to-failure in a sample with r failures from the exponential distribution has the χ^2/ν distribution with $\nu = 2r$; see Chapter 7. The sum of the *pmf*'s (rth term to ∞) from a Poisson distribution has a χ^2/ν distribution with $\nu = 2r$; see Chapter 4. The ratio s^2/σ^2 has a χ^2/ν distribution; see Chapter 5.

6-3.1 FORMULAS

ν = degrees of freedom, $\nu > 0$

χ^2/ν = random variable, $(\chi^2/\nu) \geq 0$

$$pdf\{\chi^2/\nu; \nu\} = c(\chi^2/\nu)^{\nu/2-1}\exp\left[-\frac{\nu}{2}(\chi^2/\nu)\right]$$
$$c \equiv (\nu/2)^{\nu/2}[\Gamma(\nu/2)]^{-1} \tag{6-8}$$

$$Cdf\{\chi^2/\nu; \nu\} = csnf(\chi^2/\nu; \nu) \tag{6-9}$$

$$Sf\{\chi^2/\nu; \nu\} = csnfc(\chi^2/\nu; \nu) \tag{6-10}$$

$$E\{\chi^2/\nu; \nu\} = 1$$

$$\text{StDv}\{\chi^2/\nu; \nu\} = \sqrt{2/\nu}$$

$$\text{CV}\{\chi^2/\nu; \nu\} = \sqrt{2/\nu}$$

$$CM_3\{\chi^2/\nu; \nu\} = 8/\nu^2$$

$$NCM_3\{\chi^2/\nu; \nu\} = \sqrt{8/\nu}$$

$$\text{mode}\{\chi^2/\nu; \nu\} = \begin{cases} 1 - 2/\nu, & \text{for } \nu > 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{median}\{\chi^2/\nu; \nu\} \approx 1 - 0.6/\nu$$

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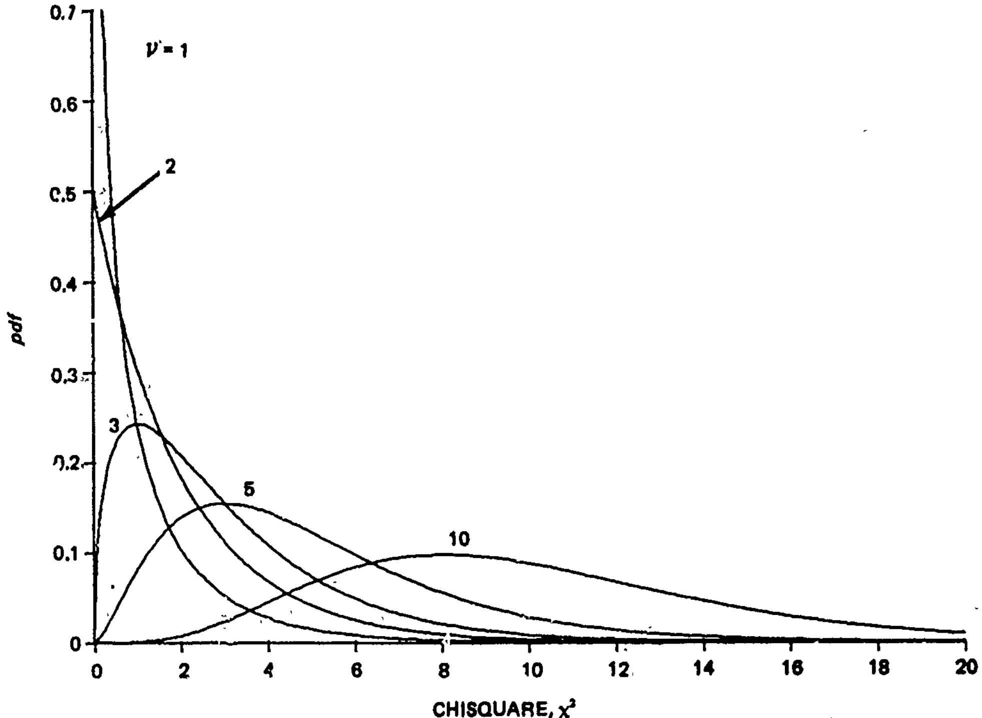
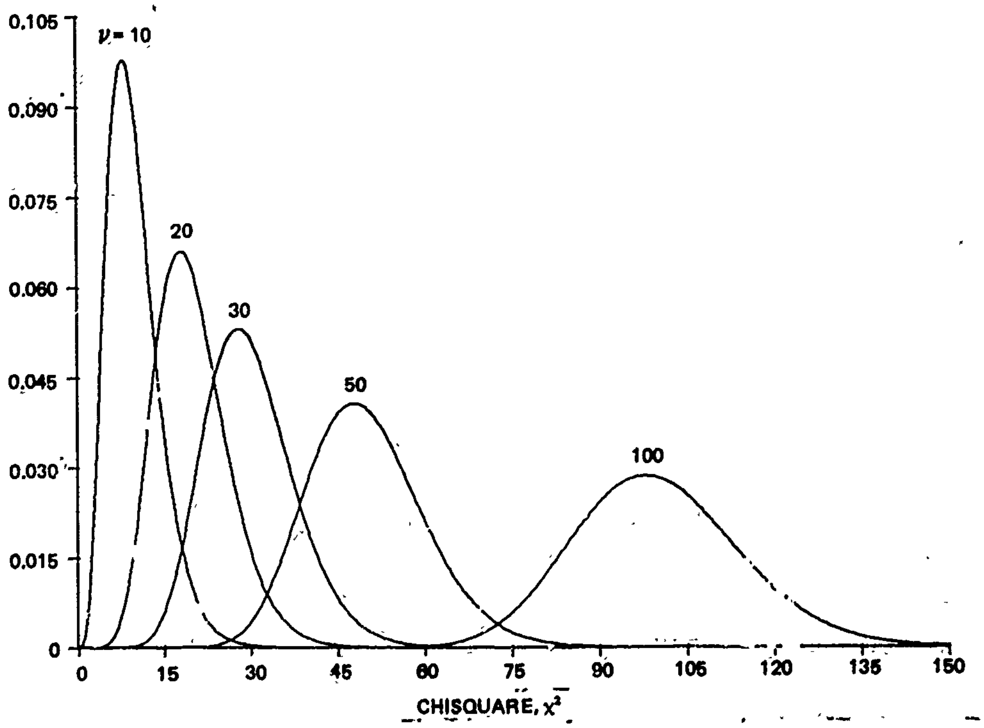


(A) FOR DEGREES-OF-FREEDOM $\nu = 1, 2, 3, 5, 10$ (FOR $\nu = 1$, THE *pdf* $\to \infty$ AS $\chi^2 \to 0^+$.)



(B) FOR DEGREES-OF-FREEDOM $\nu = 10, 20, 30, 50, 100$ (FOR LARGE ν, THE *pdf* IS REASONABLY SYMMETRICAL ABOUT $\chi^2 = \nu - 2$.)

Figure 6-1. Chi-square Distribution, pdf

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TABLE 6-1

PERCENTILES OF THE CHI-SQUARE (χ^2) DISTRIBUTION (ADAPTED FROM Ref. 3)

$csqf\,(\chi^2;\nu) \equiv Cdf\ \ \chi^2;\nu$

The body of the table gives the values of $\chi^2_{P,\nu}$

such that $csqf\,(\chi^2_{P,\nu};\nu) = P$.

ν \ P	Probability in per cent									
	0·05	0·1	0·5	1·0	2·5	5·0	10·0	20·0	30·0	40·0
1	$\cdot0^{6}393$	$\cdot0^{5}157$	$\cdot0^{4}393$	$\cdot0^{3}157$	$\cdot0^{3}982$	$\cdot0^{2}393$	·0158	·0642	·148	·275
2	$\cdot0^{3}100$	$\cdot0^{2}200$	·0100	·0201	·0506	·103	·211	·446	·713	1·02
3	·0153	·0243	·0717	·115	·216	·352	·584	1·00	1·42	1·87
4	·0639	·0908	·207	·297	·484	·711	1·06	1·65	2·19	2·75
5	·158	·210	·412	·554	·831	1·15	1·61	2·34	3·00	3·66
6	·299	·381	·676	·872	1·24	1·64	2·20	3·07	3·83	4·57
7	·485	·598	·989	1·24	1·69	2·17	2·83	3·82	4·67	5·49
8	·710	·857	1·34	1·65	2·18	2·73	3·49	4·59	5·53	6·42
9	·972	1·15	1·73	2·09	2·70	3·33	4·17	5·38	6·39	7·36
10	1·26	1·48	2·16	2·56	3·25	3·94	4·87	6·18	7·27	8·30
11	1·59	1·83	2·60	3·05	3·82	4·57	5·58	6·99	8·15	9·24
12	1·93	2·21	3·07	3·57	4·40	5·23	6·30	7·81	9·03	10·2
13	2·31	2·62	3·57	4·11	5·01	5·89	7·04	8·63	9·93	11·1
14	2·70	3·04	4·07	4·66	5·63	6·57	7·79	9·47	10·8	12·1
15	3·11	3·48	4·60	5·23	6·26	7·26	8·55	10·3	11·7	13·0
16	3·54	3·94	5·14	5·81	6·91	7·96	9·31	11·2	12·6	14·0
17	3·98	4·42	5·70	6·41	7·56	8·67	10·1	12·0	13·5	14·9
18	4·44	4·90	6·26	7·01	8·23	9·39	10·9	12·9	14·4	15·9
19	4·91	5·41	6·84	7·63	8·91	10·1	11·7	13·7	15·4	16·9
20	5·40	5·92	7·43	8·26	9·59	10·9	12·4	14·6	16·3	17·8
21	5·90	6·45	8·03	8·90	10·3	11·6	13·2	15·4	17·2	18·8
22	6·40	6·98	8·64	9·54	11·0	12·3	14·0	16·3	18·1	19·7
23	6·92	7·53	9·26	10·2	11·7	13·1	14·8	17·2	19·0	20·7
24	7·45	8·08	9·89	10·9	12·4	13·8	15·7	18·1	19·9	21·7
25	7·99	8·65	10·5	11·5	13·1	14·6	16·5	18·9	20·9	22·6
26	8·54	9·22	11·2	12·2	13·8	15·4	17·3	19·8	21·8	23·6
27	9·09	9·80	11·8	12·9	14·6	16·2	18·1	20·7	22·7	24·5
28	9·66	10·4	12·5	13·6	15·3	16·9	18·9	21·6	23·6	25·5
29	10·2	11·0	13·1	14·3	16·0	17·7	19·8	22·5	24·6	26·5
30	10·8	11·6	13·8	15·0	16·8	18·5	20·6	23·4	25·5	27·4
31	11·4	12·2	14·5	15·7	17·5	19·3	21·4	24·3	26·4	28·4
32	12·0	12·8	15·1	16·4	18·3	20·1	22·3	25·1	27·4	29·4
33	12·6	13·4	15·8	17·1	19·0	20·9	23·1	26·0	28·3	30·3
34	13·2	14·1	16·5	17·8	19·8	21·7	24·0	26·9	29·2	31·3
35	13·8	14·7	17·2	18·5	20·6	22·5	24·8	27·8	30·2	32·3
36	14·4	15·3	17·9	19·2	21·3	23·3	25·6	28·7	31·1	33·3
37	15·0	16·0	18·6	20·0	22·1	24·1	26·5	29·6	32·1	34·2
38	15·6	16·6	19·3	20·7	22·9	24·9	27·3	30·5	33·0	35·2
39	16·3	17·3	20·0	21·4	23·7	25·7	28·2	31·4	33·9	36·2
40	16·9	17·9	20·7	22·2	24·4	26·5	29·1	32·3	34·9	37·1
41	17·5	18·6	21·4	22·9	25·2	27·3	29·9	33·3	35·8	38·1
42	18·2	19·2	22·1	23·7	26·0	28·1	30·8	34·2	36·8	39·1
43	18·8	19·9	22·9	24·4	26·8	29·0	31·6	35·1	37·7	40·0
44	19·5	20·6	23·6	25·1	27·6	29·8	32·5	36·0	38·6	41·0
45	20·1	21·3	24·3	25·9	28·4	30·6	33·4	36·9	39·6	42·0
46	20·8	21·9	25·0	26·7	29·2	31·4	34·2	37·8	40·5	43·0
47	21·5	22·6	25·8	27·4	30·0	32·3	35·1	38·7	41·5	43·9
48	22·1	23·3	26·5	28·2	30·8	33·1	35·9	39·6	42·4	44·9
49	22·8	24·0	27·2	28·9	31·6	33·9	36·8	40·5	43·4	45·9
50	23·5	24·7	28·0	29·7	32·4	34·8	37·7	41·4	44·3	46·9

Example: $csqf\ (4.40;\,12) = 2.5\%$

Approximate formula: $$\chi^2_{Q,\nu} \approx \nu\left[1 - \frac{2}{9\nu} + Z_Q\left(\frac{2}{9\nu}\right)^{1/2}\right]^3$$

where: $gaufc\,(z_Q) = Q;\ csqfc\,(\chi^2_{Q,z};\nu) = Q$

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TABLE 6-1 (Continued)

Probability in per cent											P/ν
50·0	60·0	70·0	80·0	90·0	95·0	97·5	99·0	99·5	99·9	99·95	
·455	·708	1·07	1·64	2·71	3·84	5·02	6·63	7·88	10·8	12·1	1
1·39	1·83	2·41	3·22	4·61	5·99	7·38	9·21	10·6	13·8	15·2	2
2·37	2·95	3·67	4·64	6·25	7·81	9·35	11·3	12·8	16·3	17·7	3
3·36	4·04	4·88	5·99	7·78	9·49	11·1	13·3	14·9	18·5	20·0	4
4·35	5·13	6·06	7·29	9·24	11·1	12·8	15·1	16·7	20·5	22·1	5
5·35	6·21	7·23	8·56	10·6	12·6	14·4	16·8	18·5	22·5	24·1	6
6·35	7·28	8·38	9·80	12·0	14·1	16·0	18·5	20·3	24·3	26·0	7
7·34	8·35	9·52	11·0	13·4	15·5	17·5	20·1	22·0	26·1	27·9	8
8·34	9·41	10·7	12·2	14·7	16·9	19·0	21·7	23·6	27·9	29·7	9
9·34	10·5	11·8	13·4	16·0	18·3	20·5	23·2	25·2	29·6	31·4	10
10·3	11·5	12·9	14·6	17·3	19·7	21·9	24·7	26·8	31·3	33·1	11
11·3	12·6	14·0	15·8	18·5	21·0	23·3	26·2	28·3	32·9	34·8	12
12·3	13·6	15·1	17·0	19·8	22·4	24·7	27·7	29·8	34·5	36·5	13
13·3	14·7	16·2	18·2	21·1	23·7	26·1	29·1	31·3	36·1	38·1	14
14·3	15·7	17·3	19·3	22·3	25·0	27·5	30·6	32·8	37·7	39·7	15
15·3	16·8	18·4	20·5	23·5	26·3	28·8	32·0	34·3	39·3	41·3	16
16·3	17·8	19·5	21·6	24·8	27·6	30·2	33·4	35·7	40·8	42·9	17
17·3	18·9	20·6	22·8	26·0	28·9	31·5	34·8	37·2	42·3	44·4	18
18·3	19·9	21·7	23·9	27·2	30·1	32·9	36·2	38·6	43·8	46·0	19
19·3	21·0	22·8	25·0	28·4	31·4	34·2	37·6	40·0	45·3	47·5	20
20·3	22·0	23·9	26·2	29·6	32·7	35·5	38·9	41·4	46·8	49·0	21
21·3	23·0	24·9	27·3	30·8	33·9	36·8	40·3	42·8	48·3	50·5	22
22·3	24·1	26·0	28·4	32·0	35·2	38·1	41·6	44·2	49·7	52·0	23
23·3	25·1	27·1	29·6	33·2	36·4	39·4	43·0	45·6	51·2	53·5	24
24·3	26·1	28·2	30·7	34·4	37·7	40·6	44·3	46·9	52·6	54·9	25
25·3	27·2	29·2	31·8	35·6	38·9	41·9	45·6	48·3	54·1	56·4	26
26·3	28·2	30·3	32·9	36·7	40·1	43·2	47·0	49·6	55·5	57·9	27
27·3	29·2	31·4	34·0	37·9	41·3	44·5	48·3	51·0	56·9	59·3	28
28·3	30·3	32·5	35·1	39·1	42·6	45·7	49·6	52·3	58·3	60·7	29
29·3	31·3	33·5	36·3	40·3	43·8	47·0	50·9	53·7	59·7	62·2	30
30·3	32·3	34·6	37·4	41·4	45·0	48·2	52·2	55·0	61·1	63·6	31
31·3	33·4	35·7	38·5	42·6	46·2	49·5	53·5	56·3	62·5	65·0	32
32·3	34·4	36·7	39·6	43·7	47·4	50·7	54·8	57·6	63·9	66·4	33
33·3	35·4	37·8	40·7	44·9	48·6	52·0	56·1	59·0	65·2	67·8	34
34·3	36·5	38·9	41·8	46·1	49·8	53·2	57·3	60·3	66·6	69·2	35
35·3	37·5	39·9	42·9	47·2	51·0	54·4	58·6	61·6	68·0	70·6	36
36·3	38·5	41·0	44·0	48·4	52·2	55·7	59·9	62·9	69·3	72·0	37
37·3	39·6	42·0	45·[illegible]	49·5	53·4	56·9	61·2	64·2	70·7	73·4	38
38·3	40·6	43·1	46·2	50·7	54·6	58·1	62·4	65·5	72·1	74·7	39
39·3	41·6	44·2	47·3	51·8	55·8	59·3	63·7	66·8	73·4	76·1	40
40·3	42·7	45·2	48·4	52·9	56·9	60·6	65·0	68·1	74·7	77·5	41
41·3	43·7	46·3	49·5	54·1	58·1	61·8	66·2	69·3	76·1	78·8	42
42·3	44·7	47·3	50·5	55·2	59·3	63·0	67·5	70·6	77·4	80·2	43
43·3	45·7	48·4	51·6	56·4	60·5	64·2	68·7	71·9	78·7	81·5	44
44·3	46·8	49·5	52·7	57·5	61·7	65·4	70·0	73·2	80·1	82·9	45
45·3	47·8	50·5	53·8	58·6	62·8	66·6	71·2	74·4	81·4	84·2	46
46·3	48·8	51·6	54·9	59·8	64·0	67·8	72·4	75·7	82·7	85·6	47
47·3	49·8	52·6	56·0	60·9	65·2	69·0	73·7	77·0	84·0	86·9	48
48·3	50·9	5[illegible]·7	57·1	62·0	66·3	70·2	74·9	78·2	85·4	88·2	49
49·3	51·9	54·7	58·2	63·2	67·5	71·4	76·2	79·5	86·7	89·6	50

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Fig. 6-2 shows some curves of the *pdf*.

6-3.2 TABLES

Calculating the *pdf* is straightforward but tedious. It is rarely used and almost never tabulated. It is shown in Fig. 6-2.

Table 6-2 gives some percentiles of the χ^2/ν *Cdf*. Tables of the χ^2/ν distribution are handy, but uncommon.

An approximation for χ^2/ν in terms of the standard *s*-normal variate *z* is

$$\text{For } \nu > 20,\ z_Q \approx \left[(\chi^2/\nu)^{1/3} - \left(1 - \frac{2}{9\nu}\right)\right] \Big/ \sqrt{\frac{2}{9\nu}} \quad (6\text{-}7)$$

Eq. 6-11 is the inverse of Eq. 6-7 and is quite good. See Eq. 6-8 et seq.

$$(\chi^2/\nu)_{Q,\nu} \approx \left[1 - \frac{2}{9\nu} + z_Q\left(\frac{2}{9\nu}\right)^{1/2}\right]^3 \quad (6\text{-}11)$$

where

$$gaufc(z_Q) = Q,\ csnfc((\chi^2/\nu)_{Q,\nu};\nu) = Q$$

6-4 STUDENT'S *t*-DISTRIBUTION

The base name *stu* is given to the *t* distribution (for <u>stu</u>dent). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

The ratio of a standard *s*-normal variate to the square-root of a chi-square/nu variate has the *t* distribution. It occurs most frequently with the *s*-normal distribution where both the mean and standard deviation of a *s*-normal distribution are to be estimated from the sample data: <u>Student</u> was used as a pen name by W. S. Gosset in 1908 to publish his derivation of the *t* distribution. See Chapter 5 for examples of the *t* distribution. The distribution is symmetrical about the origin.

6-4.1 FORMULAS

ν = degrees of freedom, $\nu > 0$

t = random variable, it can take any value.

$$pdf\{t;\nu\} = c(1 + t^2/\nu)^{-(\nu+1)/2}$$

$$c \equiv [\sqrt{\pi\nu}\,\Gamma(\nu/2 + 1/2)\Gamma(\nu/2)]^{-1}$$

$$= 2^\nu[\pi\sqrt{\nu}\,\Gamma(\nu)]^{-1} \quad (6\text{-}12)$$

$$Cdf\{t;\nu\} = stuf(t;\nu) \quad (6\text{-}13)$$

$$Sf\{t;\nu\} = stufc(t;\nu) \quad (6\text{-}14)$$

$$E\{t;\nu\} = 0$$

$$\mathrm{StDv}\{t;\nu\} = \begin{cases}\sqrt{\nu/(\nu-2)}, & \text{for } \nu > 2\\ \to\infty, & \text{otherwise}\end{cases}$$

$$CM_3\{t;\nu\} = 0$$

$$NCM_3\{t;\nu\} = 0$$

$$NCM_4\{t;\nu\} = \begin{cases}3 + 6/(\nu-4), & \text{for } \nu > 4\\ \to\infty, & \text{otherwise}\end{cases}$$

$$\mathrm{median}\{t;\nu\} = 0$$

$$\mathrm{mode}\{t;\nu\} = 0$$

Fig. 6-3 shows some curves of the *pdf*. They are quite similar to the Gaussian *pdf*. For $\nu \to \infty$, the *t*-distribution becomes the Gaussian distribution.

6-4.2 TABLES

The *pdf* rarely is used and almost never tabulated. If needed, it can be calculated (tediously) from Eq. 6-12. Table 6-3 gives the percentiles of the *t*-distribution. Tables for the *t*-distribution are quite common; see, for example Ref. 1 (Tables 9, 10, 12, and Chapter 5).

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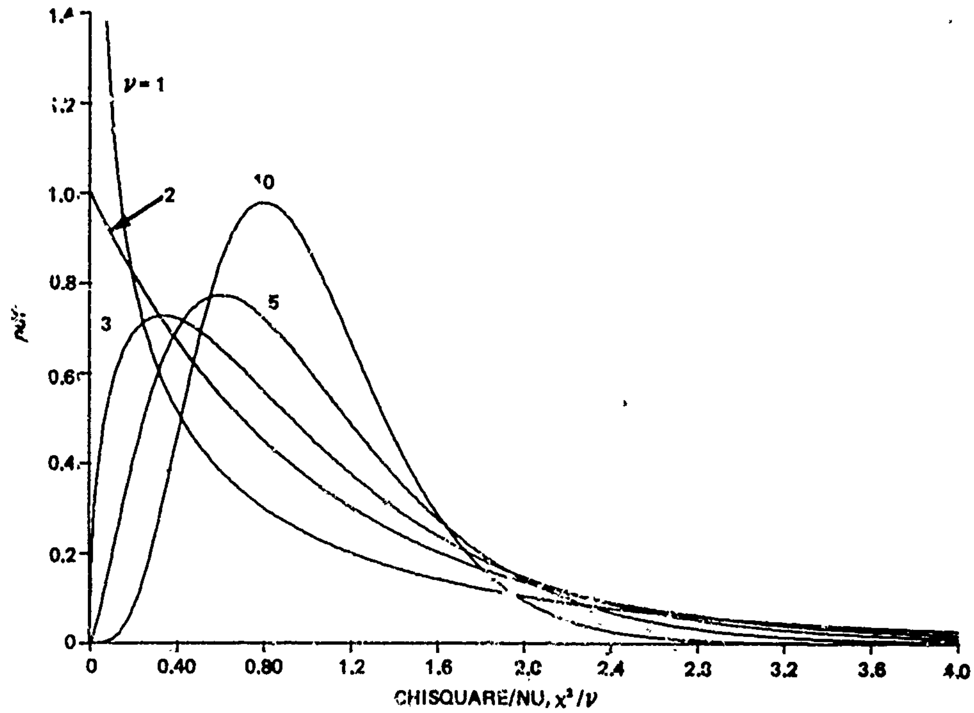
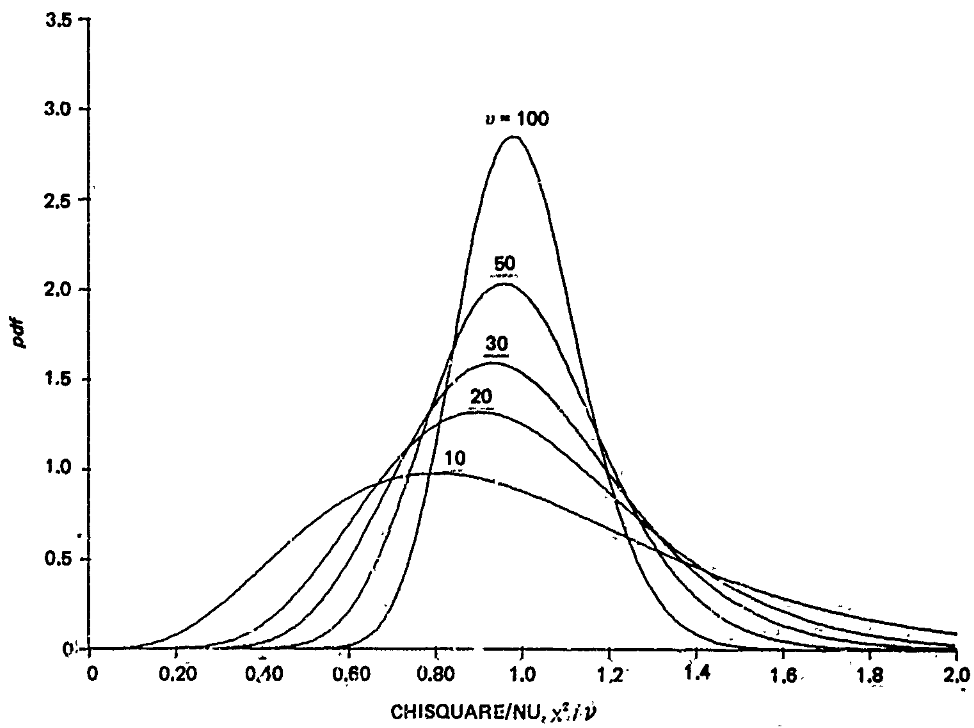


(A) FOR DEGREES-OF-FREEDOM ν = 1, 2, 3, 5, 10 (FOR LARGE ν, THE *pdf* → ∞ AS $\chi^2/\nu \to 0^+$.)



(B) FOR DEGREES-OF-FREEDOM ν = 10, 20, 30, 50, 100 (FOR LARGE ν, THE *pdf* IS REASONABLY SYMMETRICAL ABOUT $\chi^2/\nu = 1-2/\nu$.)

Figure 6-2 Chi-square/Degrees-of-freedom Distribution, pdf

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TABLE 6-2

PERCENTILES OF THE CHI-SQUARE/NU (χ^2/ν) DISTRIBUTION (ADAPTED FROM Ref. 3)

$$csnf\,(\chi^2/\nu;\nu) \equiv Cdf\,\left\{\chi^2/\nu;\nu\right\}$$

The body of the table gives the value of $(\chi^2/\nu)_{P,\nu}$ such that $csnf\,[(\chi^2/\nu)_{P,\nu};\nu] = P$.

ν \ P	Probability in per cent						Probability in per cent					
	0·05	0·1	0·5	1·0	2·5	5·0	95·0	97·5	99·0	99·5	99·9	99·95
1	·0000	·0000	·0000	·0002	·0010	·0039	3·8410	5·0240	6·6350	7·8790	10·8280	12·1160
2	·0005	·0010	·0050	·0100	·0253	·0515	2·9955	3·6890	4·6050	5·2985	6·9080	7·6010
3	·0051	·0081	·0239	·0383	·0720	·1173	2·6050	3·1160	3·7817	4·2793	5·4220	5·9100
4	·0160	·0227	·0518	·0742	·1210	·1778	2·3720	2·7858	3·3192	3·7150	4·6168	4·9995
5	·0316	·0420	·0824	·1108	·1662	·2290	2·2140	2·5664	3·0172	3·3500	4·1030	4·4210
6	·0499	·0635	·1127	·1453	·2062	·2725	2·0987	2·4082	2·8020	3·0913	3·7430	4·0172
7	·0693	·0854	·1413	·1770	·2414	·3096	2·0096	2·2876	2·6393	2·8969	3·4746	3·7169
8	·0888	·1071	·1680	·2058	·2725	·3416	1·9384	2·1919	2·5112	2·7444	3·2656	3·4835
9	·1080	·1281	·1928	·2320	·3000	·3694	1·8799	2·1137	2·4073	2·6210	3·0974	3·2962
10	·1265	·1479	·2156	·2558	·3247	·3940	1·8307	2·0483	2·3209	2·5188	2·9588	3·1419
11	·1443	·1667	·2366	·2775	·3469	·4159	1·7886	1·9927	2·2477	2·4325	2·8422	3·0124
12	·1612	·1845	·2562	·2976	·3670	·4355	1·7522	1·9447	2·1848	2·3583	2·7424	2·9018
13	·1773	·2013	·2742	·3159	·3853	·4532	1·7202	1·9028	2·1298	2·2938	2·6560	2·8060
14	·1926	·2172	·2911	·3329	·4021	·4694	1·6918	1·8656	2·0815	2·2371	2·5802	2·7221
15	·2072	·2322	·3067	·3486	·4175	·4841	1·6664	1·8325	2·0385	2·1867	2·5131	2·6479
16	·2210	·2464	·3214	·3632	·4318	·4976	1·6435	1·8028	2·0000	2·1417	2·4532	2·5818
17	·2341	·2598	·3351	·3769	·4449	·5101	1·6228	1·7759	1·9652	2·1011	2·3994	2·5223
18	·2466	·2725	·3481	·3897	·4573	·5217	1·6038	1·7514	1·9336	2·0542	2·3507	2·4686
19	·2585	·2846	·3602	·4017	·4688	·5325	1·5865	1·7291	1·9048	2·0306	2·3063	2·4196
20	·2699	·2961	·3717	·4130	·4796	·5426	1·5705	1·7085	1·8783	1·9998	2·2658	2·3749
21	·2808	·3070	·3826	·4237	·4897	·5520	1·5558	1·6895	1·8539	1·9715	2·2284	2·3338
22	·2911	·3174	·3929	·4337	·4992	·5608	1·5420	1·6719	1·8313	1·9453	2·1940	2·2960
23	·3010	·3273	·4026	·4433	·5082	·5692	1·5292	1·6555	1·8103	1·9209	2·1621	2·2609
24	·3105	·3369	·4119	·4523	·5167	·5770	1·5173	1·6402	1·7908	1·8982	2·1325	2·2283
25	·3196	·3460	·4208	·4610	·5248	·5844	1·5061	1·6258	1·7726	1·8771	2·1048	2·1979
26	·3284	·3547	·4292	·4692	·5325	·5915	1·4956	1·6124	1·7555	1·8573	2·0789	2·1695
27	·3368	·3631	·4373	·4770	·5397	·5982	1·4857	1·5998	1·7394	1·8387	2·0547	2·1429
28	·3449	·3711	·4450	·4845	·5467	·6046	1·4763	1·5879	1·7242	1·8212	2·0319	2·1179
29	·3527	·3788	·4524	·4916	·5533	·6106	1·4675	1·5766	1·7099	1·8047	2·0104	2·0943
30	·3601	·3863	·4596	·4984	·5597	·6164	1·4591	1·5660	1·6964	1·7891	1·9901	2·0720
31	·3674	·3934	·4664	·5050	·5658	·6220	1·4511	1·5559	1·6836	1·7743	1·9709	2·0510
32	·3743	·4003	·4729	·5113	·5716	·6272	1·4436	1·5462	1·6714	1·7602	1·9527	2·0311
33	·3811	·4070	·4792	·5174	·5772	·6323	1·4364	1·5371	1·6599	1·7469	1·9355	2·0122
34	·3876	·4134	·4853	·5232	·5825	·6372	1·4295	1·5284	1·6489	1·7342	1·9190	1·9942
35	·3939	·4197	·4912	·5288	·5877	·6419	1·4229	1·5201	1·6383	1·7221	1·9034	1·9771
36	·4000	·4257	·4969	·5342	·5927	·6464	1·4166	1·5121	1·6283	1·7106	1·8885	1·9608
37	·4059	·4315	·5023	·5395	·5975	·6507	1·4106	1·5045	1·6187	1·6995	1·8742	1·9452
38	·4117	·4371	·5076	·5445	·6021	·6548	1·4048	1·4972	1·6095	1·6890	1·8606	1·9303
39	·4173	·4426	·5127	·5494	·6065	·6588	1·3993	1·4903	1·6007	1·6789	1·8476	1·9169
40	·4226	·4479	·5177	·5541	·6108	·6627	1·3940	1·4836	1·5923	1·6692	1·8350	1·9024
41	·4279	·4530	·5225	·5587	·6150	·6665	1·3888	1·4771	1·5841	1·6598	1·8230	1·8892
42	·4330	·4580	·5271	·5631	·6190	·6701	1·3839	1·4709	1·5763	1·6509	1·8115	1·8767
43	·4380	·4629	·5316	·5674	·6229	·6736	1·3792	1·4649	1·5688	1·6422	1·8004	1·8646
44	·4428	·4676	·5360	·5715	·6267	·6770	1·3746	1·4591	1·5616	1·6339	1·7898	1·8529
45	·4475	·4722	·5402	·5756	·6304	·6803	1·3701	1·4536	1·5546	1·6259	1·7795	1·8417
46	·4520	·4767	·5444	·5795	·6339	·6835	1·3659	1·4482	1·5478	1·6182	1·7696	1·8309
47	·4565	·4811	·5484	·5833	·6374	·6866	1·3617	1·4430	1·5413	1·6107	1·7600	1·8204
48	·4609	·4853	·5523	·5870	·6407	·6895	1·3577	1·4380	1·5351	1·6035	1·7508	1·8104
49	·4651	·4894	·5561	·5906	·6440	·6924	1·3539	1·4331	1·5290	1·5966	1·7418	1·8006
50	·4692	·4935	·5598	·5941	·6471	·6953	1·3501	1·4284	1·5231	1·5898	1·7332	1·7912

Example: $csnf\,(0.3670; 12) = 2.5\%$

Approximate formula: $(\chi^2/\nu)_{Q,\nu} \approx \left[1 - \frac{2}{9\nu} + z_Q\left(\frac{2}{9\nu}\right)^{1/2}\right]^3$

where: $gaufc\,(z) = Q$; $csnfc\left[(\chi^2/\nu)_{Q,\nu};\nu\right] = Q$ and z_Q can be calculated from Eq. 6-7.

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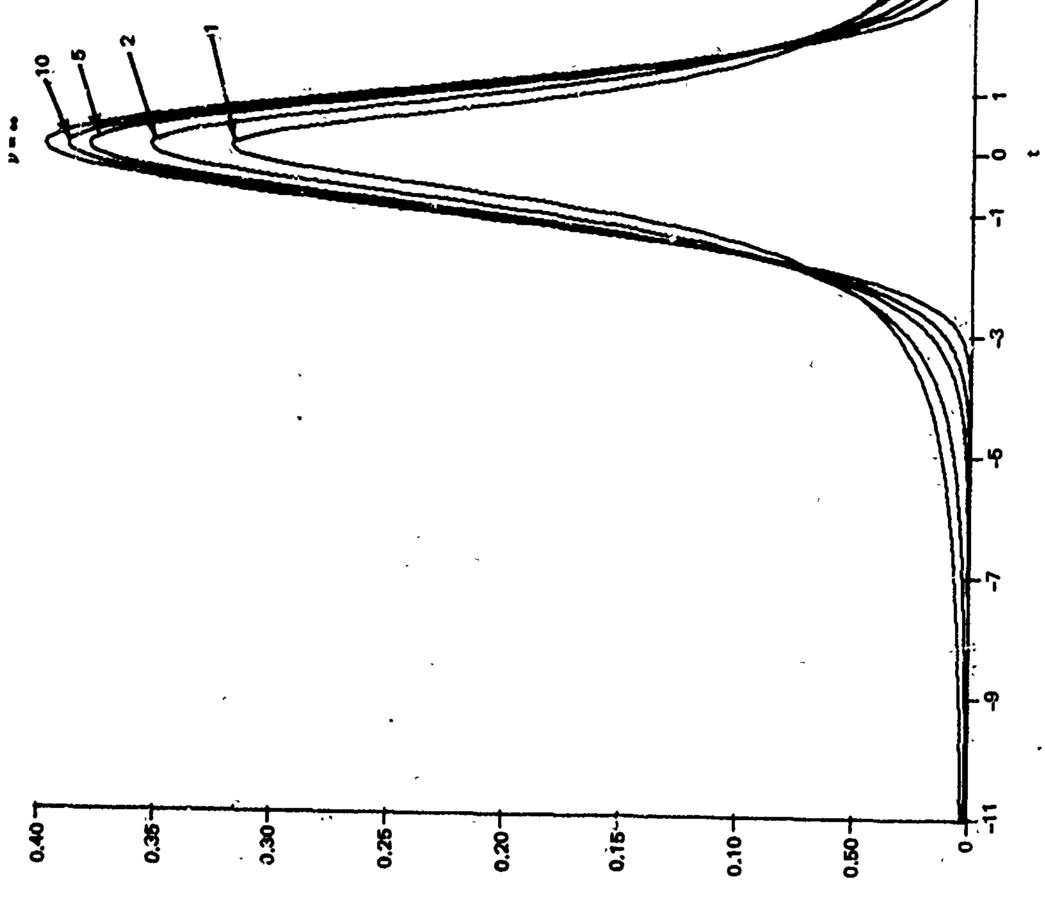

Figure 6-3. Student's t Distribution

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TABLE 6-3

PERCENTILES OF THE t DISTRIBUTION (ADAPTED FROM Ref. 3)

The body of the table gives the value of $t_{P,\nu}$ such that

$$\mathrm{Cdf}\left\{t_{P,\nu};\nu\right\} = \mathrm{stuf}\,(t_{P,\nu};\nu) = P.$$

$$t_{1-P} = -t_P,\ t_{50\%} = 0$$

ν \ P	60	70	80	90	95	97·5	99	99·5	99·9	99·95
	Probability in per cent									
1	·325	·727	1·376	3·078	6·314	12·71	31·82	63·66	318·3	636·6
2	·289	·617	1·061	1·886	2·920	4·303	6·965	9·925	22·33	31·60
3	·277	·584	·978	1·638	2·353	3·182	4·541	5·841	10·22	12·94
4	·271	·569	·941	1·533	2·132	2·776	3·747	4·604	7·173	8·610
5	·267	·559	·920	1·476	2·015	2·571	3·365	4·032	5·893	6·859
6	·265	·553	·906	1·440	1·943	2·447	3·143	3·707	5·208	5·959
7	·263	·549	·896	1·415	1·895	2·365	2·998	3·499	4·785	5·405
8	·262	·546	·889	1·397	1·860	2·306	2·896	3·355	4·501	5·041
9	·261	·543	·883	1·383	1·833	2·262	2·821	3·250	4·297	4·781
10	·260	·542	·879	1·372	1·812	2·228	2·764	3·169	4·144	4·587
11	·260	·540	·876	1·363	1·796	2·201	2·718	3·106	4·025	4·437
12	·259	·539	·873	1·356	1·782	2·179	2·681	3·055	3·930	4·318
13	·259	·538	·870	1·350	1·771	2·160	2·650	3·012	3·852	4·221
14	·258	·537	·868	1·345	1·761	2·145	2·624	2·977	3·787	4·140
15	·258	·536	·866	1·341	1·753	2·131	2·602	2·947	3·733	4·073
16	·258	·535	·865	1·337	1·746	2·120	2·583	2·921	3·686	4·015
17	·257	·534	·863	1·333	1·740	2·110	2·567	2·898	3·646	3·965
18	·257	·534	·862	1·330	1·734	2·101	2·552	2·878	3·611	3·922
19	·257	·533	·861	1·328	1·729	2·093	2·539	2·861	3·579	3·883
20	·257	·533	·860	1·325	1·725	2·086	2·528	2·845	3·552	3·850
21	·257	·532	·859	1·323	1·721	2·080	2·518	2·831	3·527	3·819
22	·256	·532	·858	1·321	1·717	2·074	2·508	2·819	3·505	3·792
23	·256	·532	·858	1·319	1·714	2·069	2·500	2·807	3·485	3·767
24	·256	·531	·857	1·318	1·711	2·064	2·492	2·797	3·467	3·745
25	·256	·531	·856	1·316	1·708	2·060	2·485	2·787	3·450	3·725
26	·256	·531	·856	1·315	1·706	2·056	2·479	2·779	3·435	3·707
27	·256	·531	·855	1·314	1·703	2·052	2·473	2·771	3·421	3·690
28	·256	·530	·855	1·313	1·701	2·048	2·467	2·763	3·408	3·674
29	·256	·530	·854	1·311	1·699	2·045	2·462	2·756	3·396	3·659
30	·256	·530	·854	1·310	1·697	2·042	2·457	2·750	3·385	3·646
40	·255	·529	·851	1·303	1·684	2·021	2·423	2·704	3·307	3·551
50	·255	·528	·849	1·298	1·676	2·009	2·403	2·678	3·262	3·495
60	·254	·527	·848	1·296	1·671	2·000	2·390	2·660	3·232	3·460
80	·254	·527	·846	1·292	1·664	1·990	2·374	2·639	3·195	3·415
100	·254	·526	·845	1·290	1·660	1·984	2·365	2·626	3·174	3·389
200	·254	·525	·843	1·286	1·653	1·972	2·345	2·601	3·131	3·339
500	·253	·525	·842	1·283	1·648	1·965	2·334	2·586	3·106	3·310
∞	·253	·524	·842	1·282	1·645	1·960	2·326	2·576	3·090	3·291
$2(1-P)$	80	60	40	20	10	5	2	1	0·2	0·1

$\mathrm{stuf}\,(2.086;\,20) = 97.5\%$

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Ref. 2 (Form.26.7.8) gives the following approximation for large ν (it is not very good even for moderate ν):

$$z_P = \frac{\left(1 - \frac{1}{4\nu}\right)}{\left(\frac{1}{t^2_{P,\nu}} + \frac{1}{2\nu}\right)^{1/2}}, \ (z_P \text{ has the sign of } t_P \ )$$

(6-15)

where $gauf(z_P) = P$ and $stuf(t_{P,\nu}; \nu) = P$. Eq. 6-15 can be inverted to give

$$t_{P,\nu} = \left[\left(\frac{1 - \frac{1}{4\nu}}{z_P}\right)^2 - \frac{1}{2\nu}\right]^{-1/2}$$

$(t_{P,\nu}$ has the sign of $z_P)$ (6-16)

As an example of the accuracy of Eqs. 6-15 and 6-16, for $\nu = 10$, $t_{1\%,\,10} \to z_{1.1\%}$.

6-5 FISHER-SNEDECOR F DISTRIBUTION

The base name *fis* is given to the F distribution (for Fisher-Snedecor). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

The ratio of the squares of 2 *s*-independent *s* statistics from the same *s*-normal distribution has the F distribution. It is the ratio of any two χ^2/ν variates. Fisher's original distribution used a different function of F. Snedecor introduced the F variable and named it after Fisher. Many authors since then have given both men credit.

The symbol F is not used universally for the random variable, but it is by far the most common symbol.

6-5.1 FORMULAS

ν_1, ν_2 = parameters, degrees of freedom, $\nu_1, \nu_2 > 0$

F = random variable; $F \geqslant 0$

$$pdf\{F; \nu_1, \nu_2\} = c\, F^{\nu_1/2 - 1}$$

(6-17)

$$\times (\nu_2 + \nu_1 F) + \nu_1 F)^{-(\nu_1 + \nu_2)/2}$$

$$c \equiv \left[\frac{\Gamma(\nu_1/2 + \nu_2/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\right] \nu_1^{\nu_1/2}\, \nu_2^{\nu_2/2}$$

$$Cdf\{F; \nu_1, \nu_2\} = fisf(F; \nu_1, \nu_2) \qquad (6\text{-}18)$$

$$Sf\{F; \nu_1, \nu_2\} = fisfc(F; \nu_1, \nu_2) \qquad (6\text{-}19)$$

The first parameter is the degrees-of-freedom of the χ^2/ν variate in the numerator; the second parameter is the degrees-of-freedom of the χ^2/ν variate in the denominator.

The F distribution has some symmetry in its parameters which is often used to shorten tables of the F distribution.

$$fisf(F; \nu_1, \nu_2) = fisfc(1/F; \nu_2, \nu_1) \qquad (6\text{-}20)$$

$$E\{F; \nu_1, \nu_2\} = \begin{cases} \nu_2/(\nu_2 - 2), & \text{for } \nu_2 > 2 \\ \to \infty, & \text{otherwise} \end{cases}$$

$$\text{StDv}\{F; \nu_1, \nu_2\} = \begin{cases} \frac{\nu_2}{\nu_2 - 2}\left[\frac{2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 4)}\right]^{1/2}, & \text{for } \nu_2 > 4 \\ \to \infty, & \text{otherwise} \end{cases}$$

$$\text{CV}\{F; \nu_1, \nu_2\} = \begin{cases} \left[\frac{2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 4)}\right]^{1/2}, & \text{for } \nu_2 > 4 \\ \to \infty, & \text{otherwise} \end{cases}$$

$$CM_3\{F; \nu_1, \nu_2\} = \begin{cases} \left[\frac{\nu_2}{(\nu_2 - 2)}\right]^3 \left[\frac{8(\nu_1 + \nu_2 - 2)(2\nu_1 + \nu_2 - 2)}{\nu_1^2(\nu_2 - 4)(\nu_2 - 6)}\right]; & \nu_2 > 6 \\ \to \infty, & \text{otherwise} \end{cases}$$

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$$\text{mode}\{F; \nu_1, \nu_2\} = \frac{\nu_2(\nu_1 - 2)}{\nu_1(\nu_2 + 2)}$$

$$\text{median}\{F; \nu_1, \nu_2\} \approx \left[\frac{1 - 2/(9\nu_1)}{1 - 2/(9\nu_2)}\right]^3$$

The F distribution is related to other distributions (adapted from Ref. 2).

$$fisfc(t^2; 1, \nu) = 2stufc(t; \nu), \text{ for } t \geq 0 \quad (6\text{-}21)$$

$$fisfc(F; \nu_1, \nu_2) = I_x(\nu_2/2, \nu_1/2) \quad (6\text{-}22)$$

$$x \equiv \nu_2/(\nu_2 + \nu_1 F)$$

where I is the Beta distribution (Chapter 10), also called Incomplete Beta Function, Ref. 2 (Sec. 26.5).

$$fisf(F; \nu, \infty) = csnf(F; \nu) \quad (6\text{-}23a)$$

$$fisf(F; \infty, \nu) = csnfc(1/F; \nu) \quad (6\text{-}23b)$$

6-5.2 TABLES

The *pdf* is neither tabulated nor of engineering interest. Table 6-4 gives percentiles of the F distribution (right-hand tail area only). Because there are 2 parameters, the F distribution is difficult to tabulate extensively. Other tables are in Ref. 1 (Tables 18, 19, and Sec. 9) and Ref. 3 (Table VII, called the v^2 distribution). Abbreviated tables are in most statistics and quality control books..

An approximation is given in Ref. 2 (Formula 26.6.15)

$$z_Q \approx z_Q^* \left[\frac{F_{Q,\nu_1,\nu_2}^{1/3}(1 - c_2) - (1 - c_1)}{(c_1 + c_2 F_{Q,\nu_1,\nu_2}^{2/3})^{1/2}}\right]$$

where

$$\overline{gaufc}(z_Q) = Q$$

$$\overline{fisfc}(F_{Q,\nu_1,\nu_2}; \nu_1, \nu_2) = Q \quad (6\text{-}24)$$

$$c_i \equiv 2/(9\nu_i) \ll 1, \text{ for } i = 1, 2.$$

Eq. 6-24 is reasonably good even for smaller values of ν_1, ν_2, at least in the region where it is usually used (right-hand tail area). Typical results are

ν_1	ν_2	Q, %	$Q(z_Q^*)$, %	relative error in tail area, %
5	10	1.00	1.028	2.8
10	5	1.00	1.18	18
3	5	5.00	5.09	1.8

Eq. 6-24 can be inverted to give Eq. 6-25.

$$F_{Q,\nu_1,\nu_2}$$

$$\approx \left[\left(\frac{1-c_1}{1-c_2}\right) \frac{1 + (1 - U_1U_2)^{1/2}}{U_2}\right]^3, \; z_Q \geq 0$$

$$\approx \left(\frac{1-c_1}{1-c_2}\right)^3, \; z_Q = 0 \; (Q = 50\%)$$

$$\times \left[\left(\frac{1-c_1}{1-c_2}\right) \frac{U_1}{1 + (1 - U_1U_2)^{1/2}}\right]^3, \; z_Q \leq 0$$

(6-25)

where

$$c_i \equiv 2/(9\nu_i) \ll 1$$

$$U_i \equiv 1 - c_i z_Q^2/(1 - c_i)^2 > 0$$

The approximations Eqs. 6-24 and 6-25 reduce to those for χ^2/ν as shown in Eq. 6-23 and par. 6-3.

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TABLE 6-4(A)

F DISTRIBUTIONS (ADAPTED FROM Ref. 4)

fisf $(F;\nu_1,\nu_2)$ = 99%, *fisfc* $(F;\nu_1,\nu_2)$ = 1%
fisf $(1/F;\nu_2,\nu_1)$ = 1%, *fisfc* $(1/F;\nu_2,\nu_1)$ = 99%
Body of the table gives the value of *F*.

Degrees of Freedom in Denominator ν_2	Degrees of Freedom in Numerator ν_1: 1	2	3	4	5	6	7	8	9
1	4052.2	4999.5	5403.3	5624.6	5763.7	5859.0	5928.3	5981.6	6022.5
2	98.503	99.000	99.166	99.249	99.299	99.332	99.356	99.374	99.388
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158
6	13.745	10.925	9.7795	9.1483	8.7459	8.4661	8.2600	8.1016	7.9761
7	12.246	9.5466	8.4513	7.8467	7.4604	7.1914	6.9928	6.8401	6.7188
8	11.259	8.6491	7.5910	7.0060	6.6318	6.3707	6.1776	6.0289	5.9106
9	10.561	8.0215	6.9919	6.4221	6.0569	5.8018	5.6129	5.4671	5.3511
10	10.044	7.5594	6.5523	5.9943	5.6363	5.3858	5.2001	5.0567	4.9424
11	9.6460	7.2057	6.2167	5.6683	5.3160	5.0692	4.8861	4.7445	4.6315
12	9.3302	6.9266	5.9526	5.4119	5.0643	4.8206	4.6395	4.4994	4.3875
13	9.0738	6.7010	5.7394	5.2053	4.8616	4.6204	4.4410	4.3021	4.1911
14	8.8616	6.5149	5.5639	5.0354	4.6950	4.4558	4.2779	4.1399	4.0297
15	8.6831	6.3589	5.4170	4.8932	4.5556	4.3183	4.1415	4.0045	3.8948
16	8.5310	6.2262	5.2922	4.7726	4.4374	4.2016	4.0259	3.8896	3.7804
17	8.3997	6.1121	5.1850	4.6690	4.3359	4.1015	3.9267	3.7910	3.6822
18	8.2854	6.0129	5.0919	4.5790	4.2479	4.0146	3.8406	3.7054	3.5971
19	8.1850	5.9259	5.0103	4.5003	4.1708	3.9386	3.7653	3.6305	3.5225
20	8.0960	5.8489	4.9382	4.4307	4.1027	3.8714	3.6987	3.5644	3.4567
21	8.0166	5.7804	4.8740	4.3688	4.0421	3.8117	3.6396	3.5056	3.3961
22	7.9454	5.7190	4.8166	4.3134	3.9880	3.7583	3.5867	3.4530	3.3458
23	7.8811	5.6637	4.7649	4.2635	3.9392	3.7102	3.5290	3.4057	3.2986
24	7.8229	5.6136	4.7181	4.2184	3.8951	3.6667	3.4959	3.3629	3.2560
25	7.7698	5.5680	4.6755	4.1774	3.8550	3.6272	3.4568	3.3239	3.2172
26	7.7213	5.5263	4.6366	4.1400	3.8183	3.5911	3.4210	3.2884	3.1818
27	7.6767	5.4881	4.0009	4.1056	3.7848	3.5580	3.3882	3.2558	3.1494
28	7.6356	5.4529	4.5681	4.0740	3.7539	3.5276	3.3581	3.2259	3.1195
29	7.5976	5.4205	4.5378	4.0449	3.7254	3.4995	3.3302	3.1982	3.0920
30	7.5625	5.3904	4.5097	4.0179	3.6990	3.4735	3.3045	3.1726	3.0665
40	7.3141	5.1785	4.3126	3.8283	3.5138	3.2910	3.1238	2.9930	2.8876
60	7.0771	4.9774	4.1259	3.6491	3.3389	3.1187	2.9530	2.8233	2.7185
120	6.8510	4.7865	3.9493	3.4796	3.1735	2.9559	2.7918	2.6629	2.5586
∞	6.6349	4.6052	3.7816	3.3192	3.0173	2.8020	2.6393	2.5113	2.4073

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TABLE 6-4(A) (Continued)

F DISTRIBUTIONS (ADAPTED FROM Ref. 4)

fisf $(F;\nu_1,\nu_2)$ = 99%, *fisfc* $(F;\nu_1,\nu_2)$ = 1%
fisf $(1/F;\nu_2,\nu_1)$ = 1%, *fisfc* $(1/F;\nu_2,\nu_1)$ = 99%
Body of the table gives the value of F.

Degrees of Freedom in Denominator ν_2	Degrees of Freedom in Numerator ν_1: 10	12	15	20	24	30	40	60	120	∞
1	6055.8	6106.3	6157.3	6208.7	6234.6	6260.7	6286.8	6313.0	6339.4	6366.0
2	99.399	99.416	99.432	99.449	99.458	99.466	99.474	99.483	99.491	99.501
3	27.229	27.052	26.872	26.690	26.598	26.505	26.411	26.316	26.221	26.125
4	14.546	14.374	14.198	14.020	13.929	13.838	13.745	13.652	13.558	13.463
5	10.051	9.8883	9.7222	9.5527	9.4665	9.3793	9.2912	9.2020	9.1118	9.0204
6	7.8741	7.7183	7.5590	7.3958	7.3127	7.2285	7.1432	7.0568	6.9690	6.8801
7	6.6201	6.4691	6.3143	6.1554	6.0743	5.9921	5.9084	5.8326	5.7572	5.6495
8	5.8143	5.6668	5.5151	5.3591	5.2793	5.1981	5.1156	5.0316	4.9460	4.8588
9	5.2565	5.1114	4.9621	4.8080	4.7290	4.6486	4.5667	4.4831	4.3978	4.3105
10	4.8492	4.7059	4.5582	4.4054	4.3269	4.2469	4.1653	4.0819	3.9965	3.9090
11	4.5393	4.3974	4.2509	4.0990	4.0209	3.9411	3.8596	3.7761	3.6904	3.6025
12	4.2961	4.1553	4.0096	3.8584	3.7805	3.7008	3.6192	3.5355	3.4494	3.3608
13	4.1003	3.9603	3.8154	3.6646	3.5868	3.5070	3.4253	3.3413	3.2548	3.1654
14	3.9394	3.8001	3.6557	3.5052	3.4274	3.3476	3.2656	3.1813	3.0942	3.0040
15	3.8049	3.6662	3.5222	3.3719	3.2940	3.2141	3.1319	3.0471	2.9595	2.8684
16	3.6909	3.5527	3.4089	3.2588	3.1808	3.1007	3.0182	2.9330	2.8447	2.7528
17	3.5931	3.4552	3.3117	3.1615	3.0835	3.0032	2.9205	2.8348	2.7459	2.6530
18	3.5082	3.3706	3.2273	3.0771	2.9990	2.9185	2.8354	2.7493	2.6597	2.5660
19	3.4338	3.2965	3.1533	3.0031	2.9249	2.8442	2.7608	2.6742	2.5839	2.4893
20	3.3682	3.2311	3.0880	2.9377	2.8594	2.7785	2.6947	2.6077	2.5168	2.4212
21	3.3098	3.1729	3.0299	2.8796	2.8011	2.7200	2.6359	2.5484	2.4568	2.3603
22	3.2576	3.1209	2.9780	2.8274	2.7488	2.6675	2.5831	2.4951	2.4029	2.3055
23	3.2106	3.0740	2.9311	2.7805	2.7017	2.6202	2.5355	2.4471	2.3542	2.2559
24	3.1681	3.0316	2.8887	2.7380	2.6591	2.5773	2.4923	2.4035	2.3099	2.2107
25	3.1294	2.9931	2.8502	2.6993	2.6203	2.5383	2.4530	2.3637	2.2695	2.1694
26	3.0941	2.9579	2.8150	2.6640	2.5848	2.5026	2.4170	2.3273	2.2325	2.1315
27	3.0618	2.9256	2.7827	2.6316	2.5522	2.4699	2.3840	2.2938	2.1984	2.0965
28	3.0320	2.8959	2.7530	2.6017	2.5223	2.4397	2.3535	2.2629	2.1670	2.0642
29	3.0045	2.8685	2.7256	2.5742	2.4946	2.4118	2.3253	2.2344	2.1378	2.0342
30	2.9791	2.8431	2.7002	2.5487	2.4689	2.3860	2.2992	2.2079	2.1107	2.0062
40	2.8005	2.6648	2.5216	2.3689	2.2880	2.2034	2.1142	2.0194	1.9172	1.8047
60	2.6318	2.4961	2.3523	2.1978	2.1154	2.0285	1.9360	1.8363	1.7263	1.6006
120	2.4721	2.3363	2.1915	2.0346	1.9500	1.8600	1.7628	1.6557	1.5330	1.3805
∞	2.3209	2.1848	2.0385	1.8783	1.7908	1.6964	1.5923	1.4730	1.3246	1.000

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TABLE 6-4(B)

F DISTRIBUTION (ADAPTED FROM Ref. 4)

fisf $(F;\nu_1,\nu_2)$ = 97.5%, *fisfc* $(F;\nu_1,\nu_2)$ = 2.5%
fisf $(1/F;\nu_2,\nu_1)$ = 2.5%, *fisfc* $(1/F;\nu_2,\nu_1)$ = 97.5%
Body of the table gives the value of *F*.

Degrees of Freedom in Denominator ν_2	Degrees of Freedom in Numerator ν_1								
	1	2	3	4	5	6	7	8	9
1	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28
2	38.506	39.000	39.165	39.248	39.298	39.331	39.355	39.373	39.387
3	17.443	16.044	15.439	15.101	14.885	14.735	14.624	14.540	14.473
4	12.218	10.649	9.9792	9.6045	9.3645	9.1973	9.0741	8.9796	8.9047
5	10.007	8.4336	7.7636	7.3879	7.1464	6.9777	6.8531	6.7572	6.6810
6	8.8131	7.2598	6.5988	6.2272	5.9876	5.8197	5.6955	5.5996	5.5234
7	8.0727	6.5415	5.8898	5.5226	5.2852	5.1186	4.9949	4.8994	4.8232
8	7.5709	6.0595	5.4160	5.0526	4.8173	4.6517	4.5286	4.4332	4.3572
9	7.2093	5.7147	5.0781	4.7181	4.4844	4.3197	4.1971	4.1020	4.0260
10	6.9367	5.4564	4.8256	4.4683	4.2361	4.0721	3.9498	3.8549	3.7790
11	6.7241	5.2559	4.6300	4.2751	4.0440	3.8807	3.7586	3.6638	3.5879
12	6.5538	5.0959	4.4742	4.1212	3.8911	3.7283	3.6065	3.5118	3.4358
13	6.4143	4.9653	4.3472	3.9959	3.7667	3.6043	3.4827	3.3880	3.3120
14	6.2979	4.8567	4.2417	3.8919	3.6634	3.5014	3.3799	3.2853	3.2093
15	6.1995	4.7650	4.1528	3.8043	3.5764	3.4147	3.2934	3.1987	3.1227
16	6.1151	4.6867	4.0768	3.7294	3.5021	3.3406	3.2194	3.1248	3.0488
17	6.0420	4.6189	4.0112	3.6648	3.4379	3.2767	3.1556	3.0610	2.9849
18	5.9781	4.5597	3.9539	3.6083	3.3820	3.2209	3.0999	3.0053	2.9291
19	5.9216	4.5075	3.9034	3.5587	3.3327	3.1718	3.0509	2.9563	2.8800
20	5.8715	4.4613	3.8587	3.5147	3.2891	3.1283	3.0074	2.9128	2.8365
21	5.8266	4.4199	3.8188	3.4754	3.2501	3.0895	2.9686	2.8740	2.7977
22	5.7863	4.3828	3.7829	3.4401	3.2151	3.0546	2.9338	2.8392	2.7628
23	5.7498	4.3492	3.7505	3.4083	3.1835	3.0232	2.9024	2.8077	2.7313
24	5.7167	4.3187	3.7211	3.3794	3.1548	2.9946	2.8738	2.7791	2.7027
25	5.6864	4.2909	3.6943	3.3530	3.1287	2.9685	2.8478	2.7531	2.6766
26	5.6586	4.2655	3.6697	3.3289	3.1048	2.9447	2.8240	2.7293	2.6528
27	5.6331	4.2421	3.6472	3.3067	3.0828	2.9228	2.8021	2.7074	2.6309
28	5.6096	4.2205	3.6264	3.2863	3.0625	2.9027	2.7820	2.6872	2.6106
29	5.5878	4.2006	3.6072	3.2674	3.0438	2.8840	2.7633	2.6686	2.5919
30	5.5675	4.1821	3.5894	3.2499	3.0265	2.8667	2.7460	2.6513	2.5746
40	5.4239	4.0510	3.4633	3.1261	2.9037	2.7444	2.6238	2.5289	2.4519
60	5.2857	3.9253	3.3425	3.0077	2.7863	2.6274	2.5068	2.4117	2.3344
120	5.1524	3.8046	3.2270	2.8943	2.6740	2.5154	2.3948	2.2994	2.2217
∞	5.0239	3.6889	3.1161	2.7858	2.5665	2.4082	2.2875	2.1918	2.1136

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TABLE 6-4(B) (Continued)

F DISTRIBUTION (ADAPTED FROM Ref. 4)

$fisf\,(F;\nu_1,\nu_2) = 97.5\%$, $fisfc\,(F;\nu_1,\nu_2) = 2.5\%$
$fisf\,(1/F;\nu_2,\nu_1) = 2.5\%$, $fisfc\,(1/F;\nu_2,\nu_1) = 97.5\%$
Body of the table gives the value of F.

Degrees of Freedom in Denominator ν_2	Degrees of Freedom in Numerator ν_1: 10	12	15	20	24	30	40	60	120	∞
1	968.63	976.71	984.87	993.10	997.25	1001.4	1005.6	1009.8	1014.0	1018.3
2	39.398	39.415	39.431	39.448	39.456	39.465	39.473	39.481	39.490	39.498
3	14.419	14.337	14.253	14.167	14.124	14.081	14.037	13.992	13.947	13.902
4	8.8439	8.7512	8.6565	8.5599	8.5109	8.4613	8.4111	8.3604	8.3092	8.2573
5	6.6192	6.5246	6.4277	6.3285	6.2780	6.2269	6.1751	6.1225	6.0693	6.0153
6	5.4613	5.3662	5.2687	5.1684	5.1172	5.0652	5.0125	4.9589	4.9045	4.8491
7	4.7611	4.6658	4.5678	4.4667	4.4150	4.3624	4.3089	4.2544	4.1989	4.1423
8	4.2951	4.1997	4.1012	3.9995	3.9472	3.8940	3.8398	3.7844	3.7279	3.6702
9	3.9639	3.8682	3.7694	3.6669	3.6142	3.5604	3.5055	3.4493	3.3918	3.3329
10	3.7168	3.6209	3.5217	3.4186	3.3654	3.3110	3.2554	3.1984	3.1399	3.0798
11	3.5257	3.4296	3.3299	3.2261	3.1725	3.1176	3.0613	3.0035	2.9441	2.8828
12	3.3736	3.2773	3.1772	3.0728	3.0187	2.9633	2.9063	2.8478	2.7874	2.7249
13	3.2497	3.1532	3.0527	2.9477	2.8932	2.8373	2.7797	2.7204	2.6590	2.5955
14	3.1469	3.0501	2.9493	2.8437	2.7888	2.7324	2.6742	2.6142	2.5519	2.4872
15	3.0602	2.9633	2.8621	2.7559	2.7006	2.6437	2.5850	2.5242	2.4611	2.3953
16	2.9862	2.8890	2.7875	2.6808	2.6252	2.5678	2.5085	2.4471	2.3831	2.3163
17	2.9222	2.8249	2.7230	2.6158	2.5598	2.5021	2.4422	2.3801	2.3153	2.2474
18	2.8664	2.7689	2.6667	2.5590	2.5027	2.4445	2.3842	2.3214	2.2558	2.1869
19	2.8173	2.7196	2.6171	2.5089	2.4523	2.3937	2.3329	2.2695	2.2032	2.1333
20	2.7737	2.6758	2.5731	2.4645	2.4076	2.3486	2.2873	2.2234	2.1562	2.0853
21	2.7348	2.6368	2.5338	2.4247	2.3675	2.3082	2.2465	2.1819	2.1141	2.0422
22	2.6998	2.6017	2.4984	2.3890	2.3315	2.2718	2.2097	2.1446	2.0760	2.0032
23	2.6682	2.5699	2.4665	2.3567	2.2989	2.2389	2.1763	2.1107	2.0415	1.9677
24	2.6396	2.5412	2.4374	2.3273	2.2693	2.2090	2.1460	2.0799	2.0099	1.9353
25	2.6135	2.5149	2.4110	2.3005	2.2422	2.1816	2.1183	2.0517	1.9811	1.9055
26	2.5895	2.4909	2.3867	2.2759	2.2174	2.1565	2.0928	2.0257	1.9545	1.8781
27	2.5676	2.4688	2.3644	2.2533	2.1946	2.1334	2.0693	2.0018	1.9299	1.8527
28	2.5473	2.4484	2.3438	2.2324	2.1735	2.1121	2.0477	1.9796	1.9072	1.8291
29	2.5286	2.4295	2.3248	2.2131	2.1540	2.0923	2.0276	1.9591	1.8861	1.8072
30	2.5112	2.4120	2.3072	2.1952	2.1359	2.0739	2.0089	1.9400	1.8664	1.7867
40	2.3882	2.2882	2.1819	2.0677	2.0069	1.9429	1.8752	1.8028	1.7242	1.6371
60	2.2702	2.1692	2.0613	1.9445	1.8817	1.8152	1.7440	1.6668	1.5810	1.4822
120	2.1570	2.0548	1.9450	1.8249	1.7597	1.6899	1.6141	1.5299	1.4327	1.3104
∞	2.0483	1.9447	1.8326	1.7085	1.6402	1.5660	1.4835	1.3883	1.2684	1.0000

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TABLE 6-4 (C)

F DISTRIBUTION (ADAPTED FROM Ref. 4)

$fisf\,(F;\nu_1,\nu_2) = 95\%$, $fisfc\,(F;\nu_1,\nu_2) = 5\%$
$fisf\,(1/F;\nu_2,\nu_1) = 5\%$, $fisfc\,(1/F;\nu_2,\nu_1) = 95\%$
Body of the table gives the value of F.

Degrees of Freedom in Denominator, ν_2	Degrees of Freedom in Numerator, ν_1: 1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385
3	10.128	9.5521	9.2766	9.1172	9.0135	8.9406	8.8868	8.8452	8.8123
4	7.7086	6.9443	6.5914	6.3883	6.2560	6.1631	6.0942	6.0410	5.9988
5	6.6079	5.7861	5.4095	5.1922	5.0503	4.9503	4.8759	4.8183	4.7725
6	5.9874	5.1433	4.7571	4.5337	4.3874	4.2839	4.2066	4.1463	4.0990
7	5.5914	4.7374	4.3468	4.1203	3.9715	3.8660	3.7870	3.7257	3.6767
8	5.3177	4.4590	4.0662	3.8378	3.6875	3.5806	3.5005	3.4381	3.3881
9	5.1174	4.2565	3.8626	3.6331	3.4817	3.3738	3.2927	3.2296	3.1789
10	4.9646	4.1028	3.7083	3.4780	3.3258	3.2172	3.1355	3.0717	3.0204
11	4.8443	3.9823	3.5874	3.3567	3.2039	3.0946	3.0123	2.9480	2.8962
12	4.7472	3.8853	3.4903	3.2592	3.1059	2.9961	2.9134	2.8486	2.7964
13	4.6672	3.8056	3.4105	3.1791	3.0254	2.9153	2.8321	2.7669	2.7144
14	4.6001	3.7389	3.3439	3.1122	2.9582	2.8477	2.7642	2.6987	2.6458
15	4.5431	3.6823	3.2874	3.0556	2.9013	2.7905	2.7066	2.6408	2.5876
16	4.4940	3.6337	3.2389	3.0069	2.8524	2.7413	2.6572	2.5911	2.5377
17	4.4513	3.5915	3.1968	2.9647	2.8100	2.6987	2.6143	2.5480	2.4943
18	4.4139	3.5546	3.1599	2.9277	2.7729	2.6613	2.5767	2.5102	2.4563
19	4.3803	3.5219	3.1274	2.8951	2.7401	2.6283	2.5435	2.4768	2.4227
20	4.3513	3.4928	3.0984	2.8661	2.7109	2.5990	2.5140	2.4471	2.3928
21	4.3248	3.4668	3.0725	2.8401	2.6848	2.5727	2.4876	2.4205	2.3661
22	4.3009	3.4434	3.0491	2.8187	2.6613	2.5491	2.4638	2.3965	2.3419
23	4.2793	3.4221	3.0280	2.7955	2.6400	2.5277	2.4422	2.3748	2.3201
24	4.2597	3.4028	3.0088	2.7763	2.6207	2.5082	2.4226	2.3551	2.3002
25	4.2417	3.3852	2.9912	2.7587	2.6030	2.4904	2.4047	2.3371	2.2821
26	4.2252	3.3690	2.9751	2.7426	2.5868	2.4741	2.3883	2.3205	2.2655
27	4.2100	3.3541	2.9604	2.7278	2.5719	2.4591	2.3732	2.3053	2.2501
28	4.1960	3.3404	2.9467	2.7141	2.5581	2.4453	2.3593	2.2913	2.2360
29	4.1830	3.3277	2.9340	2.7014	2.5454	2.4324	2.3463	2.2782	2.2229
30	4.1709	3.3158	2.9223	2.6896	2.5336	2.4205	2.3343	2.2662	2.2107
40	4.0848	3.2317	2.8387	2.6060	2.4495	2.3359	2.2490	2.1802	2.1240
60	4.0012	3.1504	2.7581	2.5252	2.3683	2.2540	2.1665	2.0970	2.0401
120	3.9201	3.0718	2.6802	2.4472	2.2900	2.1750	2.0867	2.0164	1.9588
∞	3.8415	2.9957	2.6049	2.3719	2.2141	2.0986	2.0096	1.9384	1.8799

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TABLE 6-4(C) (Continued)

F DISTRIBUTION (ADAPTED FROM Ref. 4)

$fisf\,(F;\nu_1,\nu_2) = 95\%,\ fisfc\,(F;\nu_1,\nu_2) = 5\%$
$fisf\,(1/F;\nu_2,\nu_1) = 5\%,\ fisfc\,(1/F;\nu_2,\nu_1) = 95\%$
Body of the table gives the value of *F*.

Degrees of Freedom in Denominator, ν_2	Degrees of Freedom in Numerator, ν_1									
	10	12	15	20	24	30	40	60	120	∞
1	241.88	243.91	245.95	248.01	249.05	250.09	251.14	252.20	253.25	254.32
2	19.396	19.413	19.429	19.446	19.454	19.462	19.471	19.479	19.487	19.496
3	8.7855	8.7446	8.7029	8.6602	8.6385	8.6166	8.5944	8.5720	8.5494	8.5265
4	5.9644	5.9117	5.8578	5.8025	5.7744	5.7459	5.7170	5.6878	5.6581	5.6281
5	4.7351	4.6777	4.6188	4.5581	4.5272	4.4957	4.4638	4.4314	4.3984	4.3650
6	4.0600	3.9999	3.9381	3.8742	3.8415	3.8082	3.7743	3.7398	3.7047	3.6688
7	3.6365	3.5747	3.5108	3.4445	3.4105	3.3758	3.3404	3.3043	3.2674	3.2298
8	3.3472	3.2840	3.2184	3.1503	3.1152	3.0794	3.0428	3.0053	2.9669	2.9276
9	3.1373	3.0729	3.0061	2.9365	2.9005	2.8637	2.8259	2.7872	2.7473	2.7067
10	2.9782	2.9130	2.8450	2.7740	2.7372	2.6996	2.6609	2.6211	2.5801	2.5379
11	2.8536	2.7876	2.7186	2.6464	2.6090	2.5705	2.5309	2.4901	2.4480	2.4045
12	2.7534	2.6866	2.6169	2.5436	2.5055	2.4663	2.4259	2.3842	2.3410	2.2962
13	2.6710	2.6037	2.5331	2.4589	2.4202	2.3803	2.3392	2.2966	2.2524	2.2064
14	2.6021	2.5342	2.4630	2.3879	2.3487	2.3082	2.2664	2.2230	2.1778	2.1307
15	2.5437	2.4753	2.4035	2.3275	2.2878	2.2468	2.2043	2.1601	2.1141	2.0658
16	2.4935	2.4247	2.3522	2.2756	2.2354	2.1938	2.1507	2.1058	2.0589	2.0096
17	2.4499	2.3807	2.3077	2.2304	2.1898	2.1477	2.1040	2.0584	2.0107	1.9604
18	2.4117	2.3421	2.2686	2.1906	2.1497	2.1071	2.0629	2.0166	1.9681	1.9168
19	2.3779	2.3080	2.2341	2.1555	2.1141	2.0712	2.0264	1.9796	1.9302	1.8780
20	2.3479	2.2776	2.2033	2.1242	2.0825	2.0391	1.9938	1.9464	1.8963	1.8432
21	2.3210	2.2504	2.1757	2.0960	2.0540	2.0102	1.9645	1.9165	1.8657	1.8117
22	2.2967	2.2258	2.1508	2.0707	2.0283	1.9842	1.9380	1.8895	1.8380	1.7831
23	2.2747	2.2036	2.1282	2.0476	2.0050	1.9605	1.9190	1.8649	1.8128	1.7570
24	2.2547	2.1834	2.1077	2.0267	1.9838	1.9390	1.8920	1.8424	1.7897	1.7331
25	2.2365	2.1649	2.0889	2.0075	1.9643	1.9192	1.8718	1.8217	1.7684	1.7110
26	2.2197	2.1479	2.0716	1.9898	1.9464	1.9010	1.8533	1.8027	1.7488	1.6906
27	2.2043	2.1323	2.0558	1.9736	1.9299	1.8842	1.8361	1.7851	1.7307	1.6717
28	2.1900	2.1179	2.0411	1.9586	1.9147	1.8687	1.8203	1.7689	1.7138	1.6541
29	2.1768	2.1045	2.0275	1.9446	1.9005	1.8543	1.8055	1.7537	1.6981	1.6377
30	2.1646	2.0921	2.0148	1.9317	1.8874	1.8409	1.7918	1.7396	1.6835	1.6223
40	2.0772	2.0035	1.9245	1.8389	1.7929	1.7444	1.6928	1.6373	1.5766	1.5089
60	1.9926	1.9174	1.8364	1.7480	1.7001	1.6491	1.5943	1.5343	1.4673	1.3893
120	1.9105	1.8337	1.7505	1.6587	1.6084	1.5543	1.4952	1.4290	1.3519	1.2539
∞	1.8307	1.7522	1.6664	1.5705	1.5173	1.4591	1.3940	1.3180	1.2214	1.0000

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TABLE 6-4(D)

F DISTRIBUTION (ADAPTED FROM Ref. 4)

fisf $(F;\nu_1,\nu_2) = 90\%$, *fisfc* $(F;\nu_1,\nu_2) = 10\%$
fisf $(1/F;\nu_2,\nu_1) = 10\%$, *fisfc* $(1/F;\nu_2,\nu_1) = 90\%$
Body of the table gives values of *F*.

Degrees of Freedom in Denominator ν_2	Degrees of Freedom in Numerator ν_1: 1	2	3	4	5	6	7	8	9
1	39.864	49.500	53.593	55.833	57.241	58.204	58.906	59.439	59.858
2	8.5263	9.0000	9.1618	9.2434	9.2926	9.3255	9.3491	9.3668	9.3805
3	5.5383	5.4624	5.3908	5.3427	5.3092	5.2847	5.2662	5.2517	5.2400
4	4.5448	4.3246	4.1908	4.1073	4.0506	4.0098	3.9790	3.9549	3.9357
5	4.0604	3.7797	3.6195	3.5202	3.4530	3.4045	3.3679	3.3393	3.3163
6	3.7760	3.4633	3.2888	3.1808	3.1075	3.0546	3.0145	2.9830	2.9577
7	3.5894	3.2574	3.0741	2.9605	2.8833	2.8274	2.7849	2.7516	2.7247
8	3.4579	3.1131	2.9238	2.8064	2.7265	2.6683	2.6241	2.5893	2.5612
9	3.3603	3.0065	2.8129	2.6927	2.6106	2.5509	2.5053	2.4694	2.4403
10	3.2850	2.9245	2.7277	2.6053	2.5216	2.4606	2.4140	2.3772	2.3473
11	3.2252	2.8595	2.6602	2.5362	2.4512	2.3891	2.3416	2.3040	2.2735
12	3.1765	2.8068	2.6055	2.4801	2.3940	2.3310	2.2828	2.2446	2.2135
13	3.1362	2.7632	2.5603	2.4337	2.3467	2.2830	2.2341	2.1953	2.1638
14	3.1022	2.7265	2.5222	2.3947	2.3069	2.2426	2.1931	2.1539	2.1220
15	3.0732	2.6952	2.4898	2.3614	2.2730	2.2081	2.1582	2.1185	2.0862
16	3.0481	2.6682	2.4618	2.3327	2.2438	2.1783	2.1280	2.0862	2.0553
17	3.0262	2.6446	2.4374	2.3077	2.2183	2.1524	2.1017	2.0613	2.0284
18	3.0070	2.6239	2.4160	2.2858	2.1958	2.1296	2.0785	2.0379	2.0047
19	2.9899	2.6056	2.3970	2.2663	2.1760	2.1094	2.0580	2.0171	1.9836
20	2.9747	2.5893	2.3801	2.2489	2.1582	2.0913	2.0391	1.9985	1.9649
21	2.9609	2.5746	2.3649	2.2333	2.1423	2.0751	2.0232	1.9819	1.9480
22	2.9486	2.5613	2.3512	2.2193	2.1279	2.0605	2.0084	1.9668	1.9327
23	2.9374	2.5493	2.3387	2.2065	2.1149	2.0472	1.9949	1.9531	1.9189
24	2.9271	2.5383	2.3274	2.1949	2.1030	2.0351	1.9826	1.9407	1.9063
25	2.9177	2.5283	2.3170	2.1843	2.0922	2.0241	1.9714	1.9292	1.8947
26	2.9091	2.5191	2.3075	2.1745	2.0822	2.0139	1.9610	1.9188	1.8841
27	2.9012	2.5106	2.2987	2.1655	2.0730	2.0045	1.9515	1.9091	1.8743
28	2.8939	2.5028	2.2906	2.1571	2.0045	1.9959	1.9427	1.9001	1.8652
29	2.8871	2.4955	2.2831	2.1494	2.0566	1.9878	1.9345	1.8918	1.8568
30	2.8807	2.4887	2.2761	2.1422	2.0492	1.9803	1.9269	1.8841	1.8490
40	2.8354	2.4404	2.2261	2.0909	1.9968	1.9269	1.8725	1.8289	1.7929
60	2.7914	2.3932	2.1774	2.0410	1.9457	1.8747	1.8194	1.7748	1.7380
120	2.7478	2.3473	2.1300	1.9923	1.8959	1.8238	1.7675	1.7220	1.6843
∞	2.7055	2.3026	2.0838	1.9449	1.8473	1.7741	1.7167	1.6702	1.6315

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TABLE 6-4(D) (Continued)

F DISTRIBUTION (ADAPTED FROM Ref. 4)

$fisf\,(F;\nu_1,\nu_2) = 90\%,\ fisfc\,(F;\nu_1,\nu_2) = 10\%$

$fisf\,(1/F;\nu_2,\nu_1) = 10\%,\ fisfc\,(1/F;\nu_2,\nu_1) = 90\%$

Body of the table gives values of *F*.

Degrees of Freedom in Denominator ν_2	Degrees of Freedom in Numerator ν_1: 10	12	15	20	24	30	40	60	120	∞
1	60.195	60.705	61.220	61.740	62.002	62.265	62.529	62.794	63.061	63.328
2	9.3916	9.4081	9.4247	9.4413	9.4496	9.4579	9.4663	9.4746	9.4829	9.4913
3	5.2304	5.2156	5.2003	5.1845	5.1764	5.1681	5.1597	5.1512	5.1425	5.1337
4	3.9199	3.8955	3.8689	3.8443	3.8310	3.8174	3.8036	3.7896	3.7753	3.7607
5	3.2974	3.2682	3.2380	3.2067	3.1905	3.1741	3.1573	3.1402	3.1228	3.1050
6	2.9369	2.9047	2.8712	2.8363	2.8183	2.8000	2.7812	2.7620	2.7423	2.7222
7	2.7025	2.6681	2.6322	2.5947	2.5753	2.5555	2.5351	2.5142	2.4928	2.4708
8	2.5380	2.5020	2.4642	2.4246	2.4041	2.3830	2.3614	2.3391	2.3162	2.2926
9	2.4163	2.3789	2.3396	2.2983	2.2768	2.2547	2.2320	2.2085	2.1843	2.1592
10	2.3226	2.2841	2.2435	2.2007	2.1784	2.1554	2.1317	2.1072	2.0818	2.0554
11	2.2482	2.2087	2.1671	2.1230	2.1000	2.0762	2.0516	2.0261	1.9997	1.9721
12	2.1878	2.1474	2.1049	2.0597	2.0360	2.0115	1.9861	1.9597	1.9323	1.9036
13	2.1376	2.0966	2.0532	2.0070	1.9827	1.9576	1.9315	1.9043	1.8759	1.8462
14	2.0954	2.0537	2.0095	1.9625	1.9377	1.9119	1.8852	1.8572	1.8280	1.7973
15	2.0593	2.0171	1.9722	1.9243	1.8990	1.8728	1.8454	1.8168	1.7867	1.7551
16	2.0281	1.9854	1.9399	1.8913	1.8656	1.8388	1.8108	1.7816	1.7507	1.7182
17	2.0009	1.9577	1.9117	1.8624	1.8362	1.8090	1.7805	1.7506	1.7191	1.6856
18	1.9770	1.9333	1.8868	1.8368	1.8103	1.7827	1.7537	1.7232	1.6910	1.6567
19	1.9557	1.9117	1.8647	1.8142	1.7873	1.7592	1.7298	1.6988	1.6659	1.6308
20	1.9367	1.8924	1.8449	1.7938	1.7667	1.7382	1.7083	1.6768	1.6433	1.6074
21	1.9197	1.8750	1.8272	1.7756	1.7481	1.7193	1.6890	1.6569	1.6209	1.5862
22	1.9043	1.8593	1.8111	1.7590	1.7312	1.7021	1.6714	1.6389	1.6042	1.5668
23	1.8903	1.8450	1.7964	1.7439	1.7159	1.6864	1.6554	1.6224	1.5871	1.5490
24	1.8775	1.8319	1.7831	1.7302	1.7019	1.6721	1.6407	1.6073	1.5715	1.5327
25	1.8658	1.8200	1.7708	1.7175	1.6890	1.6589	1.6272	1.5934	1.5570	1.5176
26	1.8550	1.8090	1.7596	1.7059	1.6771	1.6468	1.6147	1.5805	1.5437	1.5036
27	1.8451	1.7989	1.7492	1.6951	1.6662	1.6356	1.6032	1.5686	1.5313	1.4906
28	1.8359	1.7895	1.7395	1.6852	1.6560	1.6252	1.5925	1.5575	1.5198	1.4784
29	1.8274	1.7808	1.7306	1.6759	1.6465	1.6155	1.5825	1.5472	1.5090	1.4670
30	1.8195	1.7727	1.7223	1.6673	1.6377	1.6065	1.5732	1.5376	1.4989	1.4564
40	1.7627	1.7146	1.6624	1.6052	1.5741	1.5411	1.5056	1.4672	1.4248	1.3769
60	1.7070	1.6574	1.6034	1.5435	1.5107	1.4755	1.4373	1.3952	1.3476	1.2915
120	1.6524	1.6012	1.5450	1.4821	1.4472	1.4094	1.3676	1.3203	1.2646	1.1926
∞	1.5987	1.5458	1.4871	1.4206	1.3832	1.3419	1.2951	1.2400	1.1686	1.0000

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CHAPTER 7

EXPONENTIAL DISTRIBUTION

7-0 LIST OF SYMBOLS

C	= *s*-Confidence
Cdf	= Cumulative distribution function
C, L, U	= subscripts that imply a *s*-confidence level; C is general, L is lower, U is upper
Conf{ }	= *s*-Confidence level
$CM_i\{\ \}$	= ith central moment
CV{ }	= coefficient of variation: StDv{ }/E{ }
$E\{\ \}$	= *s*-Expected value
exp	= base name for exponential distribution
exp*f*	= *Cdf* for exponential distribution
exp*fc*	= *Sf* for exponential distribution
$M_i\{\ \}$	= ith moment about the origin
$NCM_i\{\ \}$	= normalized ith central moment; $CM_i\{\ \}/[\mathrm{StDv}\{\ \}]^i$
pdf	= probability density function
pmf	= probability mass function
$Pr\{\ \}$	= Probability
PrD	= Probability distribution
R	= *s*-Reliability
s-	= denotes statistical definition
Sf	= Survivor function
StDv{ }	= standard deviation
Var{ }	= variance
θ	= scale parameter
λ	= rate parameter
ν	= degrees of freedom
τ	= random variable
$\{\cdot;\cdot\}$, $(\cdot;\cdot)$	= the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon
$\bar{\ }$	= the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

7-1 INTRODUCTION

This is the most commonly used *PrD* for life (e.g., time to failure). It is closely related to the Poisson process. The base name exp is given to the exponential distribution (for exponential). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

7-2 FORMULAS

λ = rate parameter, $\lambda > 0$, $(\lambda \equiv 1/\theta)$

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θ = scale parameter, $\theta > 0$, $(\theta \equiv 1/\lambda)$

τ = random variable, $\tau \geq 0$

$$pdf\{\tau;\lambda\} = \lambda \exp(-\lambda\tau) \quad (7\text{-}1a)$$

$$pdf\{\tau;\theta\} = (1/\theta)\exp(-\tau/\theta) \quad (7\text{-}1b)$$

$$Cdf\{\tau;\lambda\} = \exp f(\lambda\tau) = 1 - \exp(-\lambda\tau) \quad (7\text{-}2a)$$

$$Cdf\{\tau;\theta\} = \exp f(\tau/\theta) = 1 - \exp(-\tau/\theta) \quad (7\text{-}2b)$$

$$Sf\{\tau;\lambda\} = \exp fc(\lambda\tau) = \exp(-\lambda\tau) \quad (7\text{-}3a)$$

$$Sf\{\tau;\theta\} = \exp fc(\tau/\theta) = \exp(-\tau/\theta) \quad (7\text{-}3b)$$

$$\text{failure rate}\{\tau;\lambda\} = \lambda \quad (7\text{-}4a)$$

$$\text{failure rate}\{\tau;\theta\} = 1/\theta \quad (7\text{-}4b)$$

$$E\{\tau;\theta\} = \theta$$

$$\text{StDv}\{\tau;\theta\} = \theta$$

$$\text{CV}\{\tau;\theta\} = 1$$

$$CM_3\{\tau;\theta\} = 2\theta^3$$

$$NCM_3\{\tau;\theta\} = 2$$

$$\text{mode}\{\tau;\theta\} = 0$$

$$\text{median}\{\tau;\theta\} = \theta \ln 2 \approx 0.7\theta$$

Fig. 7-1 shows some curves of the *pdf*. The failure rate λ is constant; so no graphs of it are shown.

It is possible to substitute $(\tau - \tau_0)$ for τ, where τ_0 often is called the "guarantee period". Ref. 6 (Chapter 5) discusses this case thoroughly.

7-3 TABLES

In the 1950's several tables were generated for the exponential distribution in reliability, e.g., Ref. 2. Since the exponential function has been common in mathematics for hundreds of years, several extensive tables exist in their own right, e.g., Ref. 1 (Tables 4.4 and 4.5). The electronic calculator with engineering functions often contains the exp and ℓn functions, thus tables are virtually unnecessary.

Table 7-1 is an abbreviated set of tables. It uses the fact that $\exp(x_1 + \cdots + x_5) = \exp(x_1) \cdots \exp(x_5)$.

In reliability work, more accuracy usually is required for small values of the argument than for large ones, because very low reliability (large values of the argument) is bad anyway and the degree of badness need not be known to many significant figures. Eq. 7-5 is good for all values of the argument, but is easiest to use for small values. The error is always less than the next unused term. (It is the usual power series: $x^n/n!$.)

$$\exp f(x) = 1 - e^{-x} = x\left(1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \cdots\right) \quad (7\text{-}5)$$

For example, $\exp f(0.1) = 1 - e^{-0.1}$
$= 0.1(1 - 0.0500 + 0.0017) = 0.09517$.

Figure 7-2 is a nomograph for estimating $R = \exp(-\lambda t) = \exp(-t/\theta)$.

7-4 PARAMETER ESTIMATION

An *s*-sufficient statistic for estimating the parameter of the exponential distribution is total-test-time, i.e., the total time to acquire the specified number of failures.

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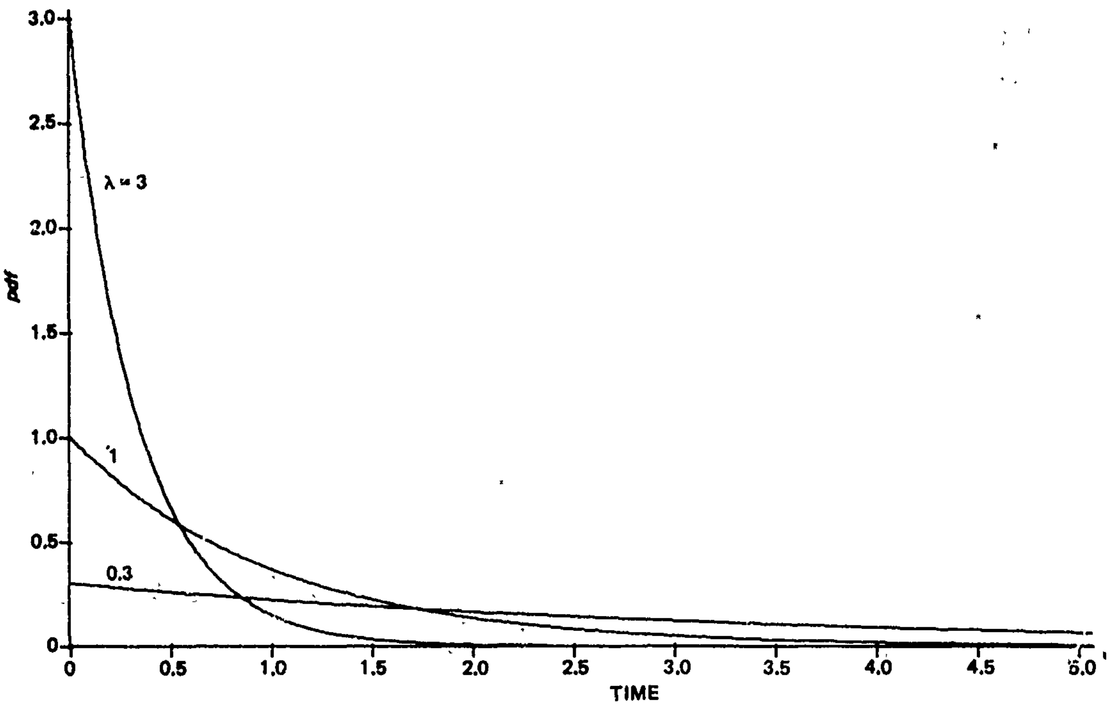

Figure 7-1. Exponential Distribution

TABLE 7-1

TABLES OF e^{-x}

$\text{expf}\ (x) = 1 - e^{-x}$
$\text{expfc}\ (x) = e^{-x}$

$\frac{x}{0}$	$\frac{e^{-x}}{1}$	$\frac{e^{-0.1x}}{1}$	$\frac{e^{-0.01x}}{1}$	$\frac{e^{-0.001x}}{1}$	$\frac{e^{-0.0001x}}{1}$
1	0.367879	0.904837	0.9900498	$0.9^3\,000500$	$0.9^4\,000050$
2	0.135335	0.818731	0.9801987	$0.9^2\,800200$	$0.9^3\,8000200$
3	0.0497871	0.740818	0.9704455	$0.9^2\,700450$	$0.9^3\,7000450$
4	0.0183164	0.670320	0.9607894	$0.9^2\,600799$	$0.9^3\,6000800$
5	$0.0^2\,673795$	0.606531	0.9512294	$0.9^2\,501248$	$0.9^3\,5001250$
6	$0.0^2\,247875$	0.548812	0.9417645	$0.9^2\,401796$	$0.9^3\,4001800$
7	$0.0^3\,911882$	0.496585	0.9323938	$0.9^2\,302444$	$0.9^3\,3002449$
8	$0.0^3\,335463$	0.449329	0.9231163	$0.9^2\,203191$	$0.9^3\,2003199$
9	$0.0^3\,123410$	0.406570	0.9139312	$0.9^2\,104038$	$0.9^3\,1004049$
10	$0.0^4\,453999$	0.367879	0.9048374	$0.9^2\,004983$	$0.9^3\,0004998$

Example $e^{-1.7963} = 0.367879 \times 0.496585 \times 0.9139312 \times 0.99401796 \times 0.9997000450 = 0.165911$

Example $e^{-0.0012} = 1 \times 1 \times 0.999000500 \times 0.9998000200 = 0.99880072$

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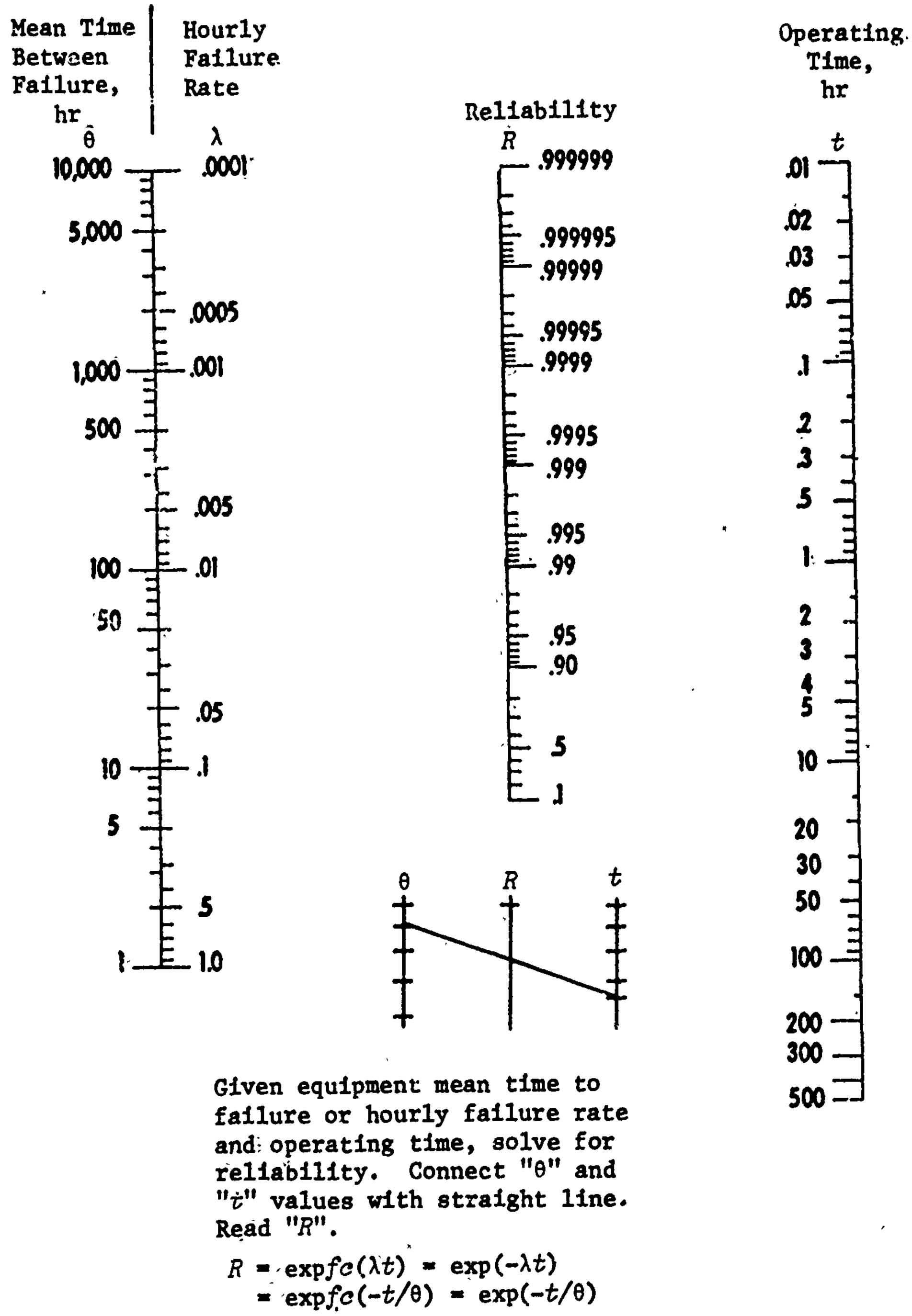

Figure 7-2. Reliability Nomograph for the Exponential Distribution

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Notation:

T = total test time for all units; $T \geqslant 0$. This is the random variable.

r = number of failures; $r > 0$. This is not a random variable; in principle, it is fixed at the beginning of the test.

The items can be tested in any order, at any time, and with or without replacement. The only restriction is that items be removed from test (e.g., end of test) only upon the failure of some item. If this restriction is not fulfilled, then the Poisson distribution in Chapter 4 must be used.

The usual point estimates for θ and λ are

$$\hat{\theta} = T/r \tag{7-6a}$$

$$\hat{\lambda} = r/T \tag{7-6b}$$

$\hat{\theta}$ is unbiased and maximum likelihood (in fact, it has virtually all the desirable properties). $\hat{\lambda}$ is maximum likelihood (but is biased).

$\hat{\theta}/\theta$ has the χ^2/ν distribution with $\nu = 2r$, viz.,

$$Cdf\{\hat{\theta}/\theta; r\} = csnf(\hat{\theta}/\theta; 2r) \tag{7-7a}$$

$\lambda/\hat{\lambda}$ has the χ^2/ν distribution with $\nu = 2r$, i.e.,

$$Cdf\{\lambda/\hat{\lambda}; r\} = csnf(\lambda/\hat{\lambda}; 2r) \tag{7-7b}$$

s-Confidence limits can be set by Eq. 7-8.

$$\mathrm{Conf}\{\lambda \leq \hat{\lambda}(\chi^2/\nu)\} = csnf[(\chi^2/\nu); 2r] \tag{7-8a}$$

$$\mathrm{Conf}\{\theta \leq \hat{\theta}/(\chi^2/\nu)\} = csnfc[(\chi^2/\nu); 2r] \tag{7-8b}$$

Table 7-2 shows the ratio of the upper and lower symmetrical s-confidence limits as a function of the number of failures; the ratio is not a function of anything else. This ratio is very large for any reasonable number of failures; e.g., for 5 failures and only 80% s-confidence, the ratio is 3.3 (from Table 7-2). That means that the true value is uncertain to a factor of over 3. To get an uncertainty of 10% (a ratio of 1.10) at a 95% s-confidence level requires about 1700 failures (from Table 7-2).

Example. Ten items are put on test. The failure/censoring times are as listed in the table. All times are in hours and are ordered.

1. 142
2. 205
3. 249
4. 448 (3 unfailed items were also removed)
5. 1351
6. 2947 (the last item was also removed).

Make estimates for θ, λ, assuming that the times-to-failure are exponentially distributed. All censoring was done at a failure; so this section applies; i.e., the number of failures is not a random variable, the total-test-time is a random variable.

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Procedure	Example
1. Calculate total-test-time T.	1. T = 142 + 205 + 249 + 4 X 448 + 1351 + 2 X 2947 = 9633
2. State the number of failures r.	2. r = 6
3. Calculate $\hat{\lambda}$ and $\hat{\theta}$ from Eq. 7-6.	3. $\hat{\lambda}$ = 0.6229/1000-hr $\hat{\theta}$ = 1605.5 hr
4. Calculate the 5% and 95% *s*-confidence limits using Eq. 7-8 and Table 6-2. (Subscripts L, U imply Lower and Upper *s*-confidence limits.)	4. $(\chi^2/\nu)_{5\%,\,12} = 0.4355$ $(\chi^2/\nu)_{95\%,\,12} = 1.7522$ λ_L = (0.6229/1000-hr) × 0.4355 = 0.2713/1000-hr θ_U = (1605.5 hr)/0.4355 = 3687 hr λ_U = (0.6229/1000-hr) × 1.7522 = 1.091/1000-hr θ_L = (1605.5 hr)/1.7522 = 916 hr
5. Make the *s*-confidence statements.	5. *s*-Confidence level = 95% − 5% = 90%. Conf $\{0.2713/1000\text{-hr} \leq \lambda \leq 1.091/1000\text{-hr}\}$ = 90% Conf $\{916 \text{ hr} \leq \theta \leq 3687 \text{ hr}\}$ = 90%
6. Calculate the ratio of upper to lower *s*-confidence limits. (See also Table 7-2).	6. ratio = 4.0

Note how misleading the point estimates are with all of their apparent precision.

The test did not give as much information about the parameter as we would have liked. This paradox is well known and has led to several suggestions to avoid it; e.g., use Bayesian methods, or use smaller *s*-confidence levels.

The data in this test were generated using a set of 10 pseudorandom numbers with θ = 1000 hr (λ = 1.0/1000-hr). The data are quite unevenly distributed which again shows the wide variability in samples which are not large.

Special tables for making these inferences have been generated; e.g., Refs. 2, 4, and 5. The χ^2/ν (or χ^2) tables are just as easy to use, and one doesn't get lost in someone else's partially explained mathematics.

Often it is desirable to test whether or not the data might reasonably have come from an exponential distribution. For general alternatives, see Chapter 14.

For the specific alternative of a Weibull distribution, see Chapter 8. A *s*-confidence interval usually is generated on the shape parameter. If that interval includes unity, the exponential hypothesis need not be rejected. In most practical situations there are so few data that the exponential hypothesis is not rejected; indeed, it would be difficult to reject many other hypotheses as well.

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TABLE 7-2

RATIO OF UPPER TO LOWER s-CONFIDENCE LIMITS FOR THE EXPONENTIAL PARAMETER (WITH EQUAL SIZE TAILS ON EACH SIDE)

Body of table gives the ratio

s-Confidence Level

number of failures r	60% (20%, 80%)	80% (10%, 90%)	90% (5%, 95%)	95% (2½%, 97½%)	98% (1%, 99%)	99% (½%, 99½%)
1	7.2	22	58	150	460	1100
2	3.6	7.3	13	23	45	72
3	2.8	4.8	7.7	12	19	27
4	2.4	3.8	5.7	8.0	12	16
5	2.2	3.3	4.6	6.3	9.1	12
6	2.0	2.9	4.0	5.3	7.3	9.2
7	1.92	2.7	3.6	4.6	6.2	7.7
8	1.83	2.5	3.3	4.2	5.5	6.7
9	1.77	2.4	3.1	3.8	5.0	5.9
10	1.71	2.3	2.9	3.6	4.6	5.4
12	1.64	2.1	2.6	3.2	3.9	4.6
14	1.57	2.0	2.4	2.9	3.6	4.1
16	1.53	1.91	2.3	2.7	3.3	3.7
18	1.49	1.84	2.2	2.6	3.1	3.4
20	1.46	1.78	2.1	2.4	2.9	3.2
25	1.41	1.68	1.94	2.2	2.6	2.8
30	1.36	1.60	1.83	2.1	2.4	2.6
35	1.33	1.55	1.75	1.95	2.2	2.4
40	1.31	1.50	1.69	1.86	2.1	2.3
45	1.29	1.47	1.64	1.80	2.0	2.2
50	1.27	1.44	1.60	1.75	1.94	2.1
60	1.24	1.39	1.53	1.66	1.83	1.95
70	1.22	1.36	1.48	1.60	1.75	1.86
80	1.21	1.33	1.45	1.55	1.69	1.78
90	1.19	1.31	1.42	1.51	1.64	1.72
100	1.18	1.29	1.39	1.48	1.59	1.68
150	1.15	1.23	1.31	1.38	1.46	1.53
200	1.13	1.20	1.26	1.32	1.39	1.44
250	1.11	1.18	1.23	1.28	1.34	1.39
300	1.10	1.16	1.21	1.25	1.31	1.35
400	1.088	1.14	1.18	1.22	1.26	1.29
500	1.078	1.12	1.16	1.19	1.23	1.26
1000	1.055	1.084	1.11	1.13	1.16	1.18
1500	1.044	1.068	1.089	1.11	1.13	1.14
2000	1.038	1.059	1.076	1.092	1.11	1.12
2500	1.034	1.053	1.068	1.082	1.097	1.11
3000	1.031	1.048	1.062	1.074	1.089	1.099
4000	1.027	1.041	1.053	1.064	1.076	1.085
5000	1.024	1.037	1.048	1.057	1.068	1.076
10000	1.017	1.026	1.033	1.040	1.048	1.053
$r \to \infty$	$1+(1.7/\sqrt{r})$	$1+(2.6/\sqrt{r})$	$1+(3.3/\sqrt{r})$	$1+(3.9/\sqrt{r})$	$1+(4.7/\sqrt{r})$	$1+(5.2/\sqrt{r})$

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CHAPTER 8

WEIBULL DISTRIBUTION

8-0 LIST OF SYMBOLS

ACov{ } = estimated asymptotic covariance

AVar{ } = estimated asymptotic variance

C = *s*-Confidence

Cdf = Cumulative distribution function

C, L, U = subscripts that imply a *s*-confidence level; *C* is general, *L* is lower, *U* is upper

$CM_i\{\ \}$ = *i*th central moment

Conf{ } = *s*-Confidence level

CV{ } = coefficient of variation: StDv{ }/*E*{ }

E{ } = *s*-Expected value

$M_i\{\ \}$ = *i*th moment about the origin

$NCM_i\{\ \}$ = normalized *i*th central moment; $CM_i\{\ \}/[\text{StDv}\{\ \}]^i$

n_f = number of failures

pdf = probability density function

pmf = probability mass function

Pr{ } = Probability

PrD = Probability distribution

R = *s*-Reliability

s- = denotes statistical definition

Sf = Survivor function

StDv{ } = standard deviation

Var{ } = variance

wei = base name for Weibull distribution

weif = *Cdf* for Weibull distribution

weifc = *Sf* for Weibull distribution

α = scale parameter

β = shape parameter

τ = random variable

ψ = $\ell n(-\ell n R)$

{·;·}, (·;·) = the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon

$\overline{\ }$ = the complement, e.g., $\overline{\theta} \equiv 1 - \theta$ where θ is any probability

8-1 INTRODUCTION

This is a very commonly used *PrD* for life (e.g., time-to-failure). It is a rather tractable *PrD* and is reasonably rich in being able to fit various sets of data. The base name *wei* is given to the Weibull distribution (for Weibull). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

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The 2-parameter Weibull distribution always is implied, unless stated otherwise.

8-2 FORMULAS

α = scale parameter, $\alpha > 0$

β = shape parameter, $\beta > 0$

τ = random variable, $\tau \geq 0$

(if $\beta < 1$, then $\tau > 0$ for the *pdf*)

$$pdf\{\tau; \alpha, \beta\} = (\beta/\alpha)\,(\tau/\alpha)^{\beta-1}\exp[-(\tau/\alpha)^{\beta}] \qquad (8\text{-}1)$$

$$Cdf\{\tau; \alpha, \beta\} = weif(\tau/\alpha; \beta) = \exp f[(\tau/\alpha)^{\beta}]$$

$$= 1 - \exp[-(\tau/\alpha)^{\beta}] \qquad (8\text{-}2)$$

$$Sf\{\tau; \alpha, \beta\} = weifc(\tau/\alpha; \beta) = \exp fc[(\tau/\alpha)^{\beta}]$$

$$= \exp[-(\tau/\alpha)^{\beta}] \qquad (8\text{-}3)$$

$$\text{failure rate}\{\tau; \alpha, \beta\} = (\beta/\alpha)\,(\tau/\alpha)^{\beta-1} \qquad (8\text{-}4)$$

$$b_i \equiv \Gamma(1 + i/\beta)$$

$$E\{\tau; \alpha, \beta\} = \alpha b_1$$

$$\text{StDv}\{\tau; \alpha, \beta\} = \alpha(b_2 - b_1^2)^{1/2}$$

$$CV = \left(\frac{b_2}{b_1^2} - 1\right)^{1/2}$$

$$CM_n/(b_1^n \alpha^n) = (-1)^n(1-n) + \sum_{i=2}^{n} \binom{n}{i} (-1)^{n-i} \left(\frac{b_i}{b_1^i}\right),$$

for $n \geq 1$

$(b_i/b_1^i > 1$, for $i > 1)$

$M_n = \alpha^n b_n$ (nth moment about the origin)

$$\text{mode}\{\tau; \alpha, \beta\} = \begin{cases} \alpha(\beta - 1)^{1/\beta}, & \text{for } \beta \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{median}\{\tau; \alpha, \beta\} = \alpha(\ln 2)/\beta \approx 0.7\alpha/\beta$$

Other types of parameters are often used, e.g., ξ in place of $-\beta \ln \alpha$; but the ones used here have the most direct engineering meaning. α often is called the characteristic life; it is useful because the *Sf* for $t = \alpha$ is $1/e \approx 0.36788$, regardless of β.

Fig. 8-1 shows the *pdf*, and Fig. 8-2 shows the hazard rate, both as a function of α and β. When $\beta = 1$, the Weibull distribution reduces to the exponential. Fig. 8-3 is a contour plot of the failure rate.

It is possible to substitute $(\tau - \tau_o)$ for τ, where τ_o is called the "guarantee period". Ref. 1 (Chapter 5) discusses this case thoroughly. Unless there is a strong physical reason why τ_o ought not to be zero, it is wise to set $\tau_o = 0$ and deal only with the 2-parameter Weibull distribution.

8-3 TABLES

The exponential function has been common in mathematics for many years. Explicit tables of the Weibull distribution are rare if they exist at all. Electronic calculators that have engineering functions generally can calculate the desired expressions for the Weibull distribution.

8-4 PARAMETER ESTIMATION

Only the 2-parameter Weibull distribution is discussed here. The 3-parameter Weibull distribution [$(\tau - \tau_o)$ substituted for τ] is not recommended unless there are compelling physical reasons to use it. Ordinarily there are not enough data to estimate τ_o with any certainty at all. Often τ_o is adjusted by an analyst to "straighten out" the graph on Weibull probability paper; this is very poor practice because sample *Cdf*'s of reasonable size are rarely straight when plotted on the proper probability paper. See Ref. 1 (Chapter 5) for parameter estimation of the 3-parameter Weibull distribution.

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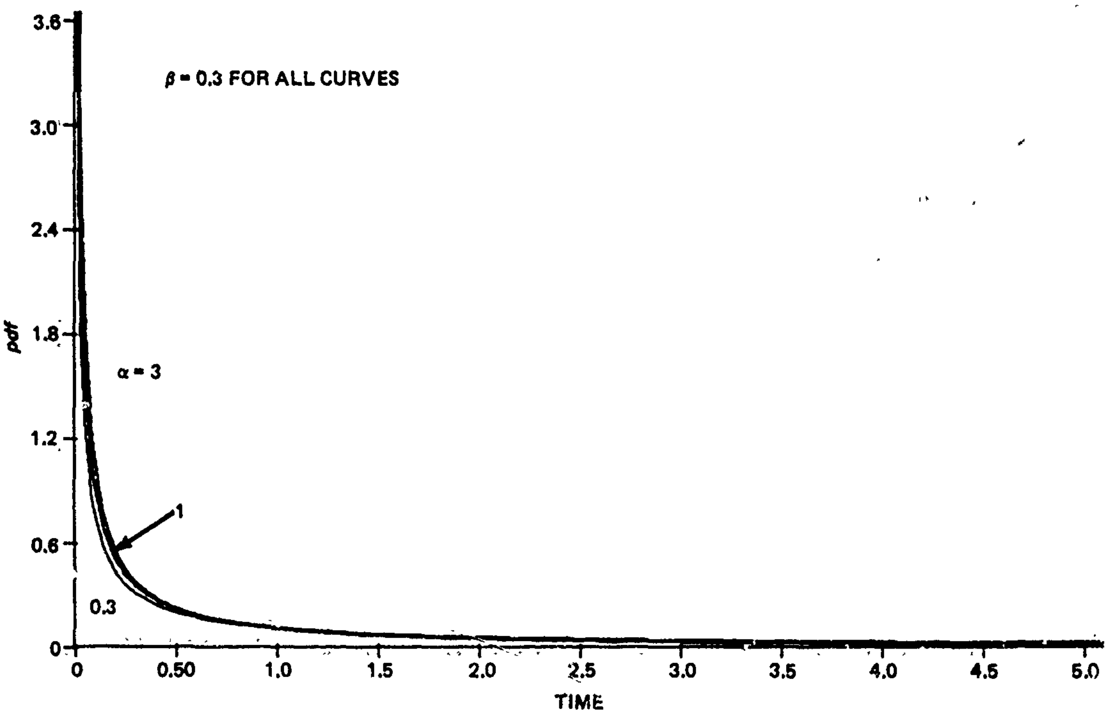
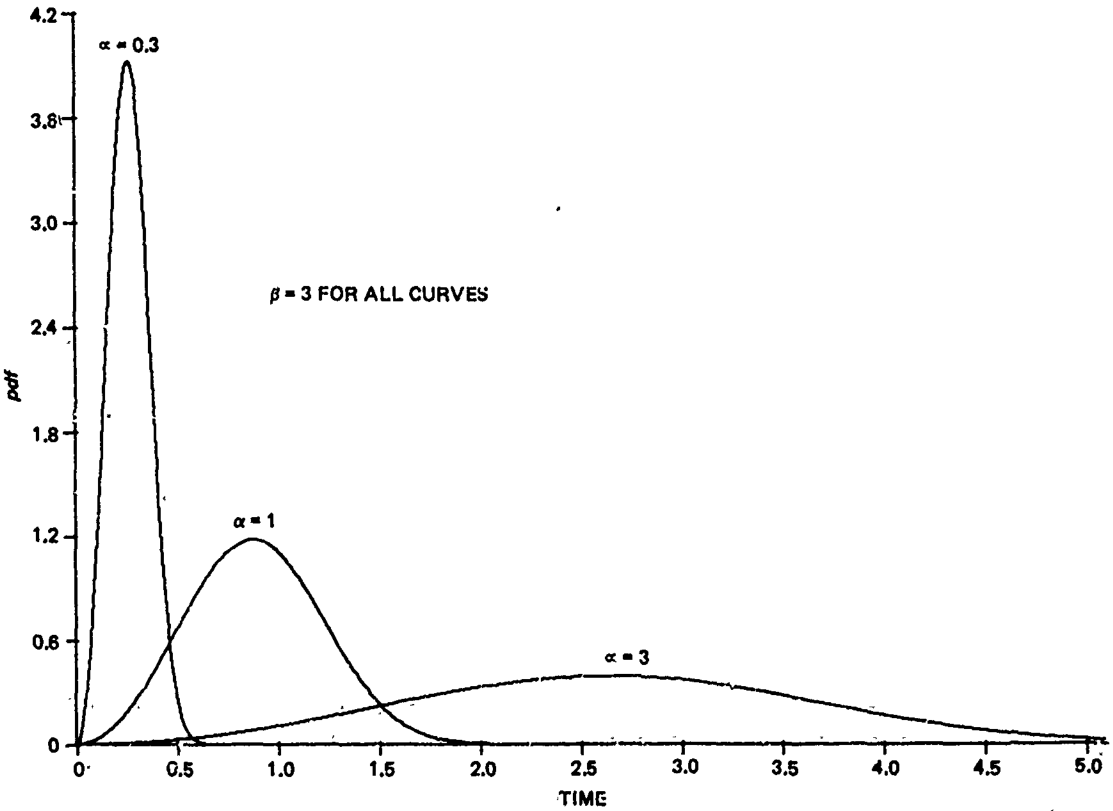


(A) FOR SHAPE PARAMETER $\beta = 0.3$. THIS SET OF CURVES IS TYPICAL FOR $0 < \beta < 1$. (THE *pdf's* $\rightarrow \infty$ AS TIME $\rightarrow 0^+$.)



(B) FOR SHAPE PARAMETER $\beta = 3.0$. THIS IS TYPICAL FOR $\beta > 1$. (FOR $1 < \beta < 2$, THE *pdf*-SLOPE $\rightarrow \infty$ AS TIME $\rightarrow 0^+$.)

Figure 8-1. Weibull Distribution, pdf

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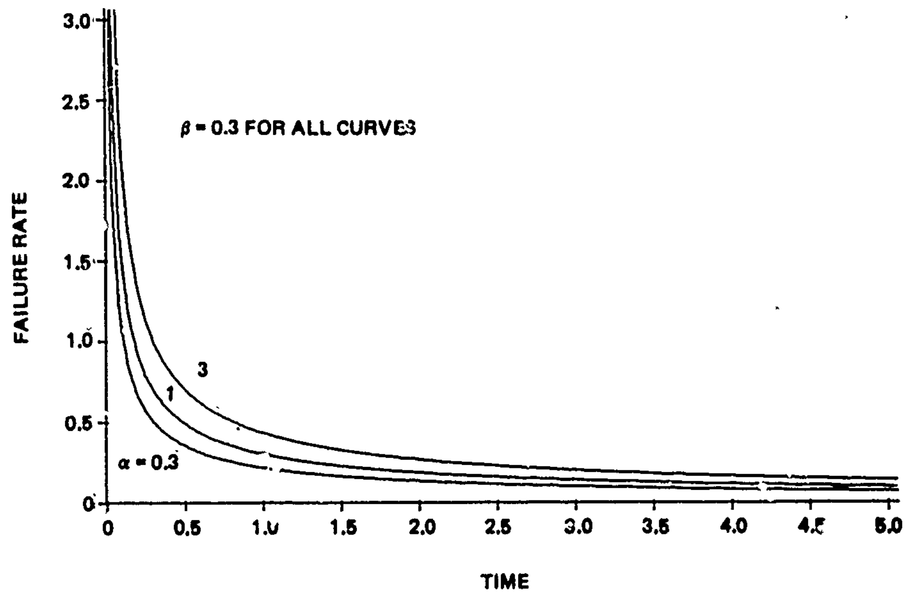
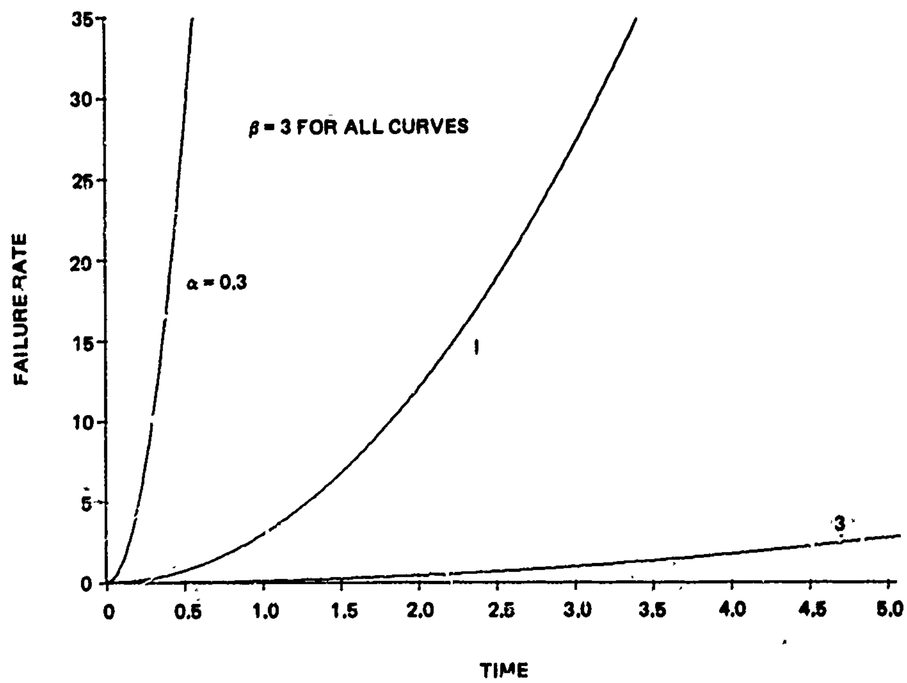


(A) FOR SHAPE PARAMETER $\beta = 0.3$. THIS SET OF CURVES IS TYPICAL FOR $0 < \beta < 1$. (THE FAILURE RATE $\to \infty$ AS TIME $\to 0^+$.)



(B) FOR SHAPE PARAMETER $\beta = 3$, THIS IS TYPICAL FOR $\beta > 1$. (FOR $1 < \beta < 2$, THE FAILURE-RATE SLOPE $\to \infty$ AS TIME $\to 0^+$. AS TIME $\to \infty$, THE FAILURE RATE $\to \infty$.

Figure 8-2. Weibull Distribution, Failure Rate

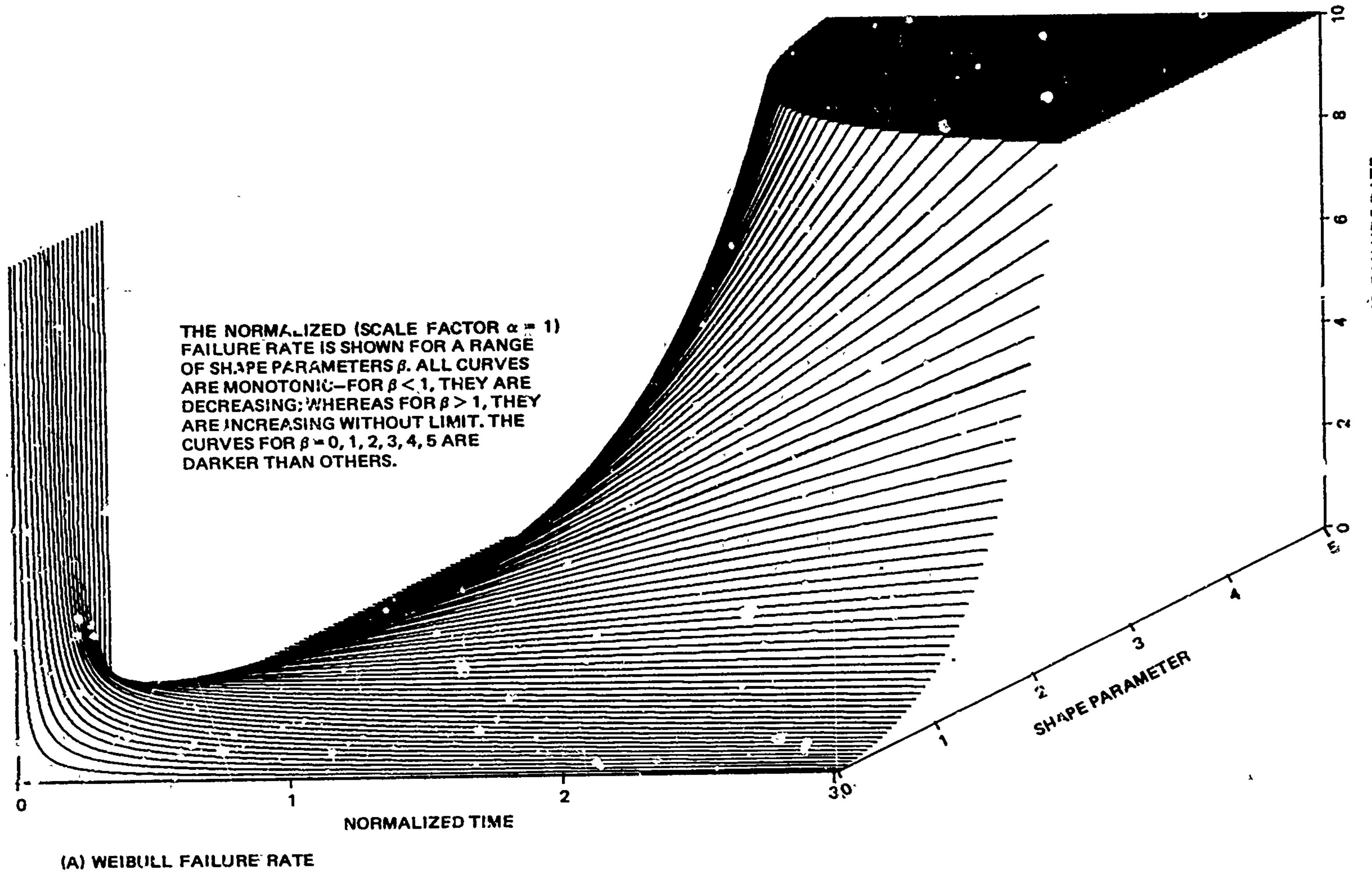


Figure 8-3(A). Weibull Distribution, Contour Plot

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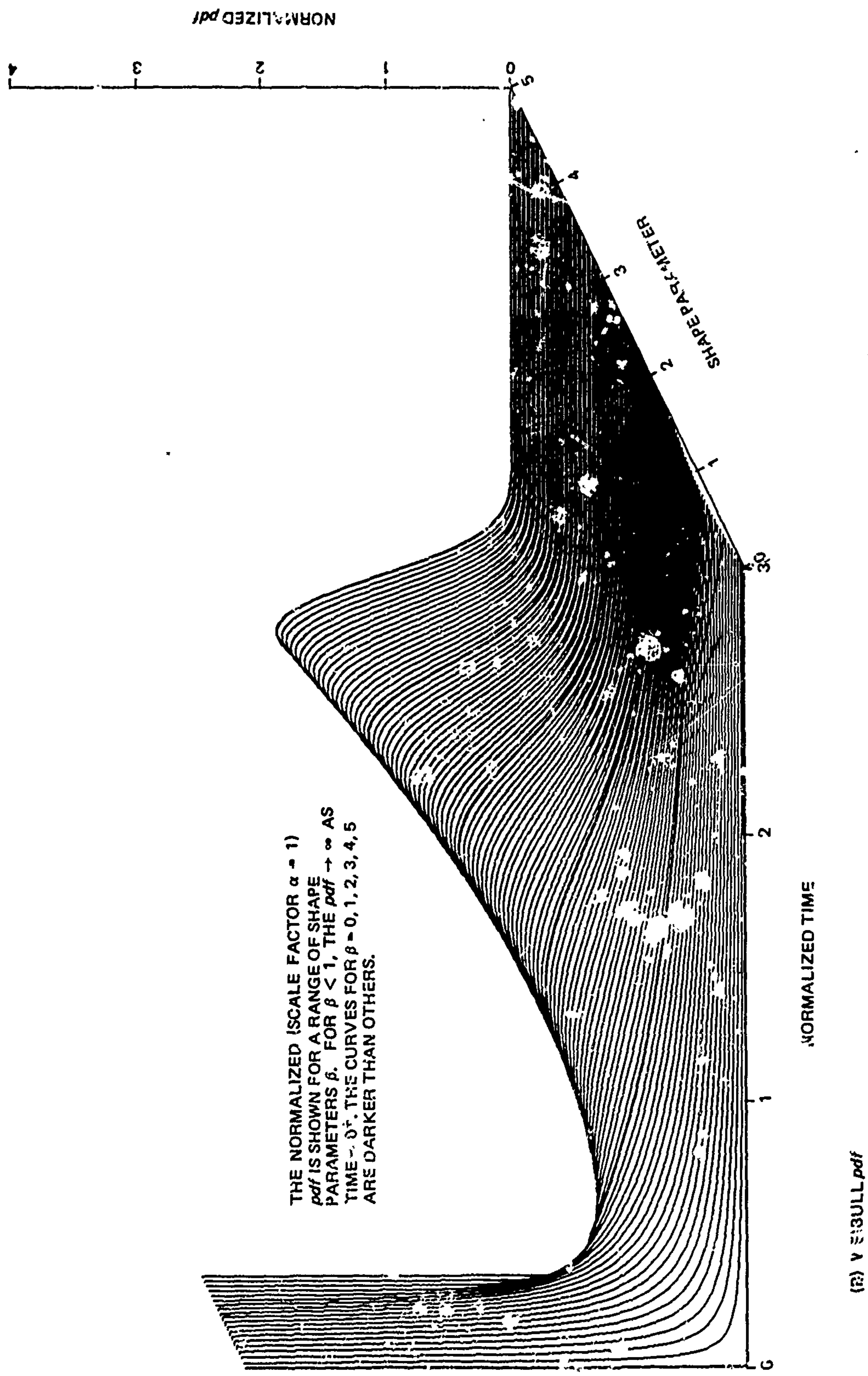

Figure 8-3(B). Weibull Distribution, Contour Plot

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8-4.1 GRAPHICAL METHOD

Graphical estimation is feasible using special Weibull probability paper. One ought always to try to get some idea of the uncertainty involved in the estimates, no matter how roughly the uncertainty is guessed at.

8-4.2 MAXIMUM LIKELIHOOD METHOD

The method of maximum likelihood (ML) is virtually the only analytic technique that is used. Unfortunately, one of the 2 ML equations cannot be solved explicitly, and iterative techniques must be used. The ML method is suitable for any kind of censoring, and always can be used. Even though it is somewhat tedious, it is recommended here because the amateur analyst "can't" go wrong with it.

Estimating the uncertainty in the ML estimates is difficult because one knows only the asymptotic (large sample) behavior. The estimates are asymptotically *s*-normal; this fact is used in making *s*-confidence statements.

Notation:

n_f = number of failures

x_i = failure or censoring time for item i (if more than 1 item is censored at the same time, each is given the value x_i)

$\sum_{\text{fail}}$ = sum only over all failed items

$\sum_{\text{all}}$ = sum over all times (failed and censored)

$$\gamma_i(\beta) = x_i^{\beta} \Big/ \sum_{\text{all}} x_i^{\beta}$$

The equation to be solved iteratively is

$$g(\hat{\beta}) \equiv \frac{1}{\hat{\beta}} + \frac{1}{n_f} \sum_{\text{fail}} \ln x_i - \sum_{\text{all}} \gamma_i(\hat{\beta}) \ln x_i = 0 \tag{8-5}$$

The $\gamma_i(\beta)$ must be calculated each time the value of β is changed. There is only one positive value of β which satisfies the equation; so Newton's method, linear interpolation, or any standard method works quite well. If the data are first graphed on Weibull probability paper, the graphical value of β can be used to begin the iteration. Otherwise, begin with $\hat{\beta} = 1$ (it's about as good as any other).

The equation for $\hat{\alpha}$ is

$$\hat{\alpha} = \left(\frac{1}{n_f} \sum_{\text{all}} x_i^{\hat{\beta}}\right)^{1/\hat{\beta}}. \tag{8-6}$$

The elements of the asymptotic covariance matrix are estimated by the following expressions:

$$\text{AVar}\{\ln \hat{\alpha}\} = \left(\frac{1}{n_f}\right)\left(\frac{1}{\hat{\beta}^2}\right)\left(\frac{\hat{T}_2^2}{\hat{T}_2^2 - \hat{T}_1^2}\right) \tag{8-7}$$

$$\text{AVar}\{\hat{\beta}\} = \left(\frac{1}{n_f}\right)\left(\frac{1}{\hat{T}_2^2 - \hat{T}_1^2}\right) \tag{8-8}$$

$$\text{ACov}\{\ln \hat{\alpha}, \hat{\beta}\} = -\left(\frac{1}{n_f}\right)\left(\frac{1}{\hat{\beta}}\right)\left(\frac{\hat{T}_1}{\hat{T}_2^2 - \hat{T}_1^2}\right) \tag{8-9}$$

where

$$\hat{T}_1 \equiv \sum_{\text{all}} \gamma_i(\hat{\beta}) \ln\left(\frac{x_i}{\hat{\alpha}}\right)$$

$$= \left[\sum_{\text{all}} \gamma_i(\hat{\beta}) \ln x_i\right] - \ln \hat{\alpha}$$

$$\hat{T}_2^2 \equiv \frac{1}{\hat{\beta}^2} + \sum_{\text{all}} \gamma_i(\hat{\beta}) \left[\ln\left(\frac{x_i}{\hat{\alpha}}\right)\right]^2$$

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AVar = estimated asymptotic variance

ACov = estimated asymptotic covariance

$A\rho$ = estimated asymptotic correlation coefficient

AStDv = estimated asymptotic standard deviation

$$\text{AStDv}\{\cdot\} = [\text{AVar}\{\cdot\}]^{1/2} \quad (8\text{-}10a)$$

$$A\rho\{\ln\hat{\alpha}, \hat{\beta}\} = \frac{\text{ACov}\{\ln\hat{\alpha}, \hat{\beta}\}}{[\text{AVar}\{\ln\hat{\alpha}\}\text{AVar}\{\hat{\beta}\}]^{1/2}}$$

Eq. 8-4 can be rewritten as

$$\psi = \beta\ln\tau - \beta\ln\alpha \quad (8\text{-}11)$$

where

$$\psi \equiv \ln(-\ln R).$$

The two usual problems are to estimate ψ (and thus R) given τ, and to estimate τ, given R (and thus ψ). Since the variances are not always small, it is wise to consider $\ell n\tau$ and $\ell n\alpha$ instead of τ and α. The AVar's of $\ell n\tau$ and ψ are

$$\begin{aligned}\text{AVar}\{\psi\} = &(\ln[\tau/\hat{\alpha}])^2\,\text{AVar}\{\hat{\beta}\}\\ &+ \hat{\beta}^2\,\text{AVar}\{\ln\hat{\alpha}\}\\ &- 2\hat{\beta}(\ln[\tau/\hat{\alpha}])\,\text{ACov}\{\ln\hat{\alpha}, \hat{\beta}\}\end{aligned} \quad (8\text{-}12)$$

$$\begin{aligned}\text{AVar}\{\ell n\tau\} = &(\psi^2/\hat{\beta}^4)\,\text{Avar}\{\hat{\beta}\}\\ &+ \text{AVar}\{\ell n\hat{\alpha}\} - (2\psi/\hat{\beta}^2)\\ &\times \text{ACov}\{\ell n\hat{\alpha}, \hat{\beta}\}\end{aligned} \quad (8\text{-}13)$$

Approximate *s*-confidence limits are set by assuming that the parameters are *s*-normally distributed with mean given by the maximum likelihood value and standard deviation given by the square root of the asymptotic variance. For small samples, the answers are very rough, but they do serve the purpose of showing the uncertainty.

Example. Ten items were put on test. The failure/censoring times are as listed in the table. All times are in hours and are ordered.

1. 142
2. 205
3. 249
4. 448 (3 unfailed items were also removed)
5. 1351
6. 2947 (the last item was also removed)

Estimate α, β assuming that the times to failure have the Weibull distribution. (The following calculations were all performed on an HP-45 electronic calculator—a computer is not necessary if there aren't too many data.)

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Procedure	Example
1. Solve Eq. 8-5 by successive approximation. Linear interpolation on the values of g closest to zero is simple; it is quite similar to Newton's method with numerical differentiation. Choose $\hat{\beta} = 1.0$ and 0.8 for the first 2 trials (just guess). Use 2 or 3 point interpolation to estimate further values of $\hat{\beta}$.	1. (see table below) $\hat{\beta} = 0.9069$
2. Solve Eq. 8-6 for $\hat{\alpha}$.	2. $\hat{\alpha} = 1614.77$
3. Use Eqs. 8-7, 8-8, and 8-9 to get the asymptotic covariance matrix. ($n_f = 6$)	3. $\hat{T}_1 = -0.1012$ $\hat{T}_2^2 = 2.0959$ AVar $\{\ell n\hat{\alpha}\} = 0.2036 = 0.451^2$ AVar $\{\hat{\beta}\} = 0.07991 = 0.283^2$ ACov $\{\ell n\hat{\alpha},\hat{\beta}\} = -0.00892 = -0.0699 \times 0.451 \times 0.283$
4. Use Eq. 8-10 to get the asymptotic standard deviations and correlation coefficient.	4. AStDv $\{\ell n\hat{\alpha}\} = 0.451$ AStDv $\{\hat{\beta}\} = 0.283$ $A\rho$ $\{\ell n\hat{\alpha}, \hat{\beta}\} = -0.0699$
5. Use ±1 standard deviation to put approximate 68% symmetrical s-confidence limits on $\ell n\,\alpha$, α, β.	5. Conf $\{6.936 \leq \ell n\,\alpha \leq 7.838\} \approx 68\%$ Conf $\{1029 \text{ hr} \leq \alpha \leq 2535 \text{ hr}\} \approx 68\%$ Conf $\{0.62 \leq \beta \leq 1.19\} \approx 68\%$
6. Use Eq. 8-12 to find AVar $\{\psi\}$ = AVar $\{\ell n(-\ell n R)\}$	6. AVar $\{\psi\} = \left[\ell n \frac{\tau}{1615 \text{ hr}}\right]^2 \times 0.0799 + 0.167 + 0.0162 \left(\ell n \frac{\tau}{1615 \text{ hr}}\right)$
7. From Step 6, evaluate $\psi \pm$ AStDv $\{\psi\}$ and thus the uncertainty in R (for approximately 68% s-confidence) at τ = 100 hr (R = 93.5%).	7. AVar $\{\psi\} = 0.740 = 0.860^2$ $\psi = \hat{\beta}\ell n(\tau/\hat{\alpha}) = -2.705$ Conf $\{-1.845 \leq \psi \leq -3.565\} \approx 68\%$ Conf $\{0.854 \leq R \leq 0.972\} \approx 68\%$
8. Use Eq. 8-13 to find AVar $\{\ell n\tau\}$	8. AVar $\{\ell n\tau\} = 0.118\psi^2 + 0.0217\psi + 0.204$.
9. From Step 8, evaluate $\ell n\tau \pm$ AStDv $\{\ell n\tau\}$ and thus the uncertainty in τ (for approximately 68% s-confidence) at R = 93.5% (τ = 100 hr), ψ = −2.70	9. AVar $\{\ell n\tau\} = 1.006 = 1.003^2$ $\ell n\tau = 4.605$ Conf $\{3.602 \leq \ell n\tau \leq 5.608\} \approx 68\%$ Conf $\{36.7 \text{ hr} \leq \tau \leq 273 \text{ hr}\} \approx 68\%$

Example, step 1:

$\hat{\beta}$	$g(\hat{\beta})$
1.0	−0.1760
0.8	+0.2444
0.916	−0.0188
0.9082	−0.0027
→0.9069	−0.000011

$\hat{\beta} = 0.9069$

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The point of going into the analysis in detail is to show that the calculation of the uncertainties is by far the most important contribution of statistics. Without those calculations of uncertainty, a dangerous delusion of accuracy would prevail. These data are the same as in the example in Chapter 7. They are from an exponential distribution with parameter θ = 1000 hr., i.e., a Weibull distribution with $\beta = 1$, $\alpha = 1000$ hr.

8-4.3 LINEAR ESTIMATION METHODS

There are many linear estimation techniques in the literature. None are given here because they require extensive tabulations. The tabulations are usually only for specific sample sizes. Ref. 1 (Chapter 5) discusses many of these and gives the references. Ref. 2 (Chapter 3) collects several of the tables. It is usually wise to go back to the original paper if the explanation of the use of any reproduced table is not clear and explicit. For example, some techniques are designed for uncensored samples, others can be used only if all censoring is at the end of the test. Many of the methods are very good when they apply; it is wise to consult a statistician about using them and about the methods for estimating uncertainty (e.g., *s*-confidence intervals).

8-4.4 TEST FOR FAILURE RATE: INCREASING, DECREASING, OR CONSTANT

The literature contains tables for testing the hypothesis about whether $\beta > 1$ (increasing failure rate), $\beta = 1$ (constant failure rate), $\beta < 1$ (decreasing failure rate). Some tables are not valid if there is any kind of censoring.

You must remember what such a test really does. It says, "Do the test data virtually force me to reject my hypothesis?" If they do force you to reject the hypothesis (i.e., data as bad as yours rarely would be obtained if the hypothesis were true), then you ordinarily accept the alternate hypothesis. But very often the data do not force you to reject your hypothesis—your data are quite reasonable if the hypothesis is true. Then what you really ought to say is "This is a reasonable hypothesis; there may also be many other reasonable hypotheses."

For the example in par 8-4.2, $\hat{\beta} \approx 0.91$, and $\hat{\sigma}_\beta \approx 0.28$ ($\hat{\sigma}_\beta \equiv \text{StDv}\{\hat{\beta}\}$). If $\beta_{true} = 1$, you'd get a $\hat{\beta}$ as low as (or lower than) 0.91 at least $gaufl[(0.91 - 1.00)/0.28] = 37\%$ of the time. The data do not force you to reject the hypothesis that $\beta = 1$.

Now if $\beta_{true} = 0.8$, you'd get a $\hat{\beta}$ as high as (or higher than) 0.91 at least $gaufc[(0.91 - 0.80)/0.28] = 35\%$ of the time. The data do not force you to reject that hypothesis.

Now if $\beta_{true} = 1.2$, you'd get a $\hat{\beta}$ as low as (or lower than) 0.91 at least $gaufl[(0.91 - 1.2)/0.28] = 15\%$ of the time. The data do not really force you to reject that hypothesis.

Often it is desirable to make as simple an assumption as the data will allow. That usually means to assume $\beta = 1$ (exponential distribution) if the data will allow it.

8-5 COMPARISON WITH LOGNORMAL DISTRIBUTION

For 10 failures or less (or perhaps even 20 failures or less) data sets from a Weibull and a lognormal distribution are virtually indistinguishable from each other. It is not wise to use a goodness-of-fit test to find which is the better fit because neither one ought to fit the sample data very well; there is just too much scatter in small to medium size samples. This is an illustration of why it is not wise to extrapolate very far from the sample data. These 2 distributions will generally have quite different behavior in the right tail region.

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REFERENCES

1. Mann, Schafer, Singpurwalla, *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley & Sons. 1974.
2. W. Yurkowsky, *Nonelectronic Reliability Notebook*, March 1970, RADC-TR-69-458, AD-868 372.

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CHAPTER 9

LOGNORMAL DISTRIBUTION

9-0 LIST OF SYMBOLS

C	=	s-Confidence
Cdf	=	Cumulative distribution function
C, L, U	=	subscripts that imply a s-confidence level; C is general, L is lower, U is upper
$CM_i\{\ \}$	=	ith central moment
Conf$\{\ \}$	=	s-Confidence level
CV$\{\ \}$	=	coefficient of variation: StDv$\{\ \}/E\{\ \}$
$E\{\ \}$	=	s-Expected value
...f	=	suffix for base name, implies the *Cdf*
...fc	=	suffix for base name, implies the *Sf*
gau	=	base name for Gaussian distribution
gaud	=	*pdf* for Gaussian distribution
gauhr	=	hazard rate (failure rate) for Gaussian distribution
lgn	=	base name for lognormal distribution
$NCM_i\{\ \}$	=	normalized ith central moment; $CM_i\{\ \}/[\text{StDv}\{\ \}]^i$
$M_i\{\ \}$	=	ith moment about the origin
pdf	=	probability density function
pmf	=	probability mass function
$Pr\{\ \}$	=	Probability
PrD	=	Probability distribution
R	=	s-Reliability
s-	=	denotes statistical definition
Sf	=	Survivor function
StDv$\{\ \}$	=	standard deviation
Var$\{\ \}$	=	variance
$\bar{x}$	=	sample mean
α	=	scale parameter
β	=	shape parameter
τ	=	random variable
$\{\cdot;\cdot\},(\cdot;\cdot)$	=	the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon
$\bar{\ }$	=	the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

9-1 INTRODUCTION

This distribution is used occasionally for the life of semiconductors and mechanical

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parts. Part of its popularity is traceable to its relation to the s-normal distribution, and part is traceable to the non-negative character of the random variable. The basic name *lgn* is given to the lognormal distribution (for lognormal). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*).

The 2-parameter lognormal distribution is always implied unless otherwise stated.

9-2 FORMULAS

α = scale parameter, $\alpha > 0$

β = shape parameter, $\beta > 0$

τ = random variable, $\tau \geq 0$

$$pdf\{\tau;\alpha,\beta\} = \frac{\beta}{\tau\sqrt{2\pi}} \exp[-1/2[\ln(\tau/\alpha)^\beta]^2] \quad (9\text{-}1a)$$

$$= \frac{1}{\sqrt{2\pi}(1/\beta)}\left(\frac{1}{\tau}\right) \times \exp\left[-1/2\left(\frac{\ln\tau - \ln\alpha}{1/\beta}\right)^2\right] \quad (9\text{-}1b)$$

$$= (\beta/\tau) gaud\left[\beta\ln\left(\frac{\tau}{\alpha}\right)\right] \quad (9\text{-}1c)$$

where *gaud* is the Gaussian *pdf*; see Par. 5-1.

$$Cdf\{\tau;\alpha,\beta\} = lgnf(\tau/\alpha;\beta) \quad (9\text{-}2a)$$

$$= gauf\left(\frac{\ln\tau - \ln\alpha}{1/\beta}\right)$$

$$= gauf\left[\beta\ln\left(\frac{\tau}{\alpha}\right)\right]$$

$$= gauf\left[\ln\left(\frac{\tau}{\alpha}\right)^\beta\right] \quad (9\text{-}2b)$$

$$Sf\{\tau;\alpha,\beta\} = lgnfc(\tau/\alpha;\beta) \quad (9\text{-}3a)$$

$$= gaufc\left(\frac{\ln\tau - \ln\alpha}{1/\beta}\right)$$

$$= gaufc\left[\beta\ln\left(\frac{\tau}{\alpha}\right)\right]$$

$$= gaufc\left[\ln\left(\frac{\tau}{\alpha}\right)^\beta\right] \quad (9\text{-}3b)$$

$$\text{failure rate }\{\tau;\alpha,\beta\} = pdf\{\tau;\alpha,\beta\}/Sf\{\tau;\alpha,\beta\} \quad (9\text{-}4a)$$

$$= \beta\, gauhr\left[\beta\ln\left(\frac{\tau}{\alpha}\right)\right]$$

$$= \beta\, gauhr\left[\ln\left(\frac{\tau}{\alpha}\right)^\beta\right] \quad (9\text{-}4b)$$

where *gauhr* is the Gaussian hazard rate (failure rate); see par. 5.2.

$B \equiv \exp[1/(2\beta^2)] > 1$

$E\{\tau;\alpha,\beta\} = \alpha B > \alpha$

$\mathrm{StDv}\{\tau;\alpha,\beta\} = \alpha B(B^2 - 1)^{1/2}$

$\mathrm{CV}\{\tau;\alpha,\beta\} = (B^2 - 1)^{1/2}$

$CM_3\{\tau;\alpha,\beta\} = \alpha^3 B^3 (B^2 - 1)^2 (B^2 + 2)$

$$\frac{CM_n}{(\alpha B)^n} = (-1)^n\left[1 - n + \sum_{i=2}^{n}(-1)^i\binom{n}{i}B^{i(i-1)}\right]$$

$\mathrm{mode}\{\tau;\alpha,\beta\} = \alpha/B^2 < \alpha$

$\mathrm{median}\{\tau;\alpha,\beta\} = \alpha$

Mode of failure rate occurs at $(\alpha/B^2)e^\gamma$. γ $(0 \leqslant \gamma \leqslant 1)$ is given in Fig. 9-5. The failure rate is zero for $\tau = 0$, then rises to a single maximum, and finally decreases toward zero. Fig. 9-1 shows the *pdf*, and Fig. 9-2 shows the

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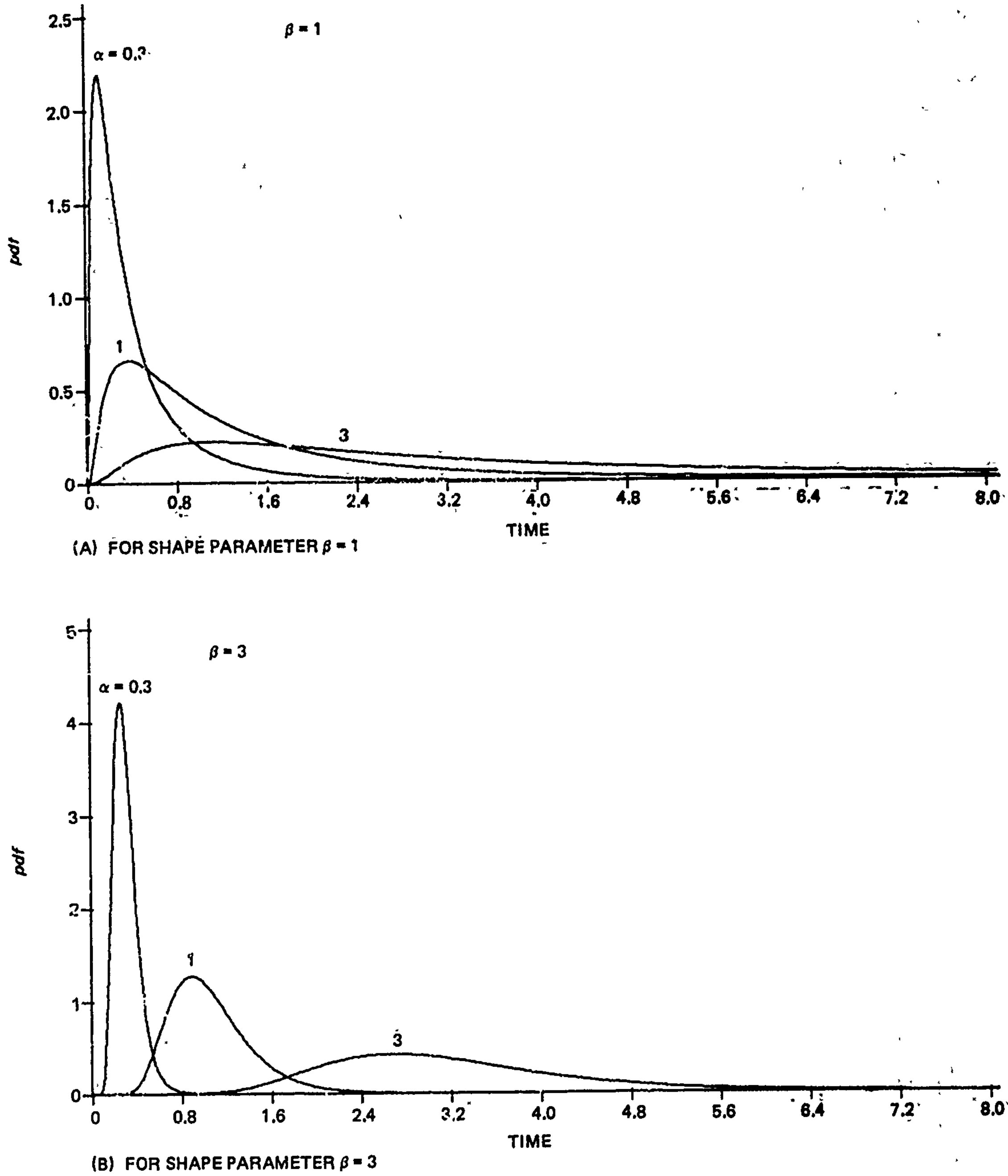

Figure 9-1. Lognormal Distribution, pdf

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failure rate, for several values of α, β. Figs. 9-3 and 9-4 are contour plots of the *pdf* and failure rate.

There are several ways of putting the parameters into the lognormal distribution, but the common ones in the literature which use the symbols μ and σ are confusing because μ and σ do not stand for the mean and standard deviation of τ, but for $\ell n\tau$.

If a change of variable is made from τ to $\ell n\tau$, then $\ell n\tau$ has the *s*-normal distribution. See Chapter 5.

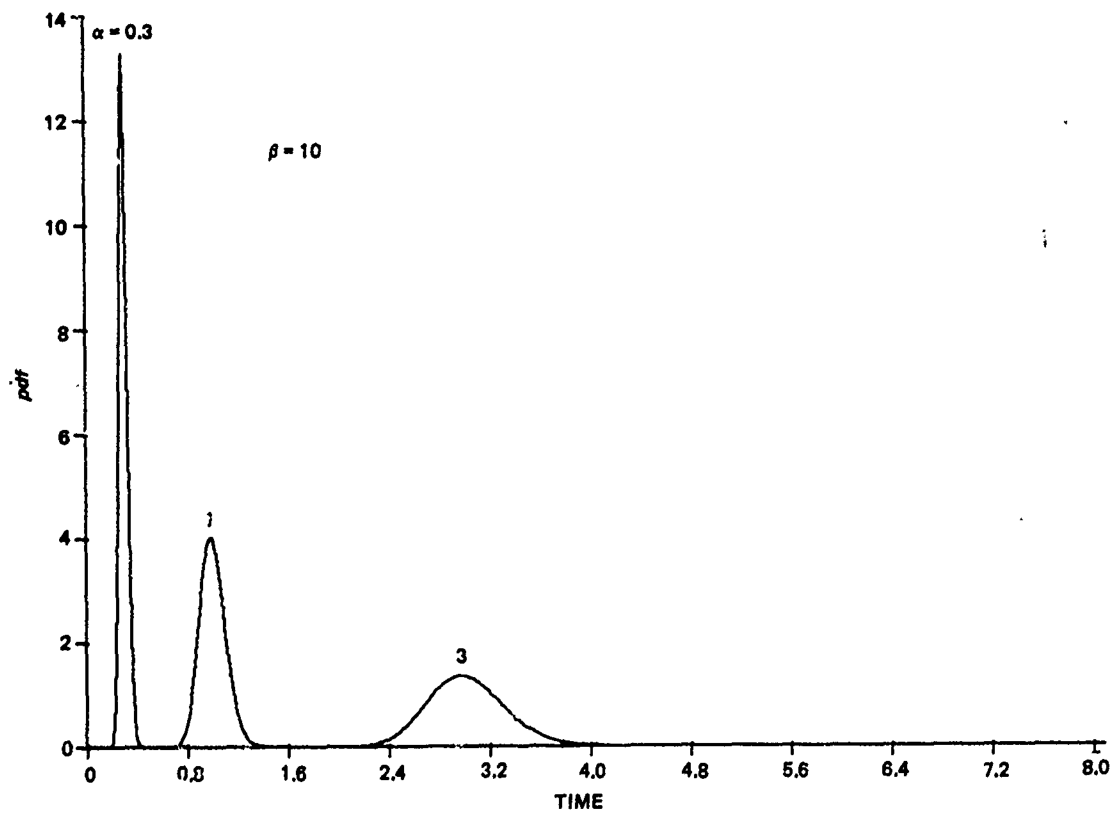
It is possible to substitute $(\tau - \tau_o)$ for τ, where τ_o is called the "guarantee period", Ref. 1 (Chapter 5) discusses this case thoroughly. Unless there is a strong physical reason why τ_o ought not to be zero, it is wise to set $\tau_o = 0$ and deal only with the 2-parameter lognormal distribution.

9-3 TABLES

Tables of the lognormal distribution are virtually nonexistent. The *Cdf* and *Sf* are calculated easily using *gauf* and *gaufc* tables (standard *s*-distribution), see Eqs. 9-2 and 9-3. The *pdf* can be calculated directly, or from the *s*-normal *pdf* using Eq. 9-1.

9-4 PARAMETER ESTIMATION

Only the 2-parameter lognormal distribution is discussed here. The 3-parameter lognormal distribution [$(\tau - \tau_o)$ substituted for τ] is not recommended unless there are compelling physical reasons to use it. Ordinarily, there are not enough data to estimate τ_o



(C) FOR SHAPE PARAMETER β = 10

Figure 9-1. Lognormal Distribution, pdf

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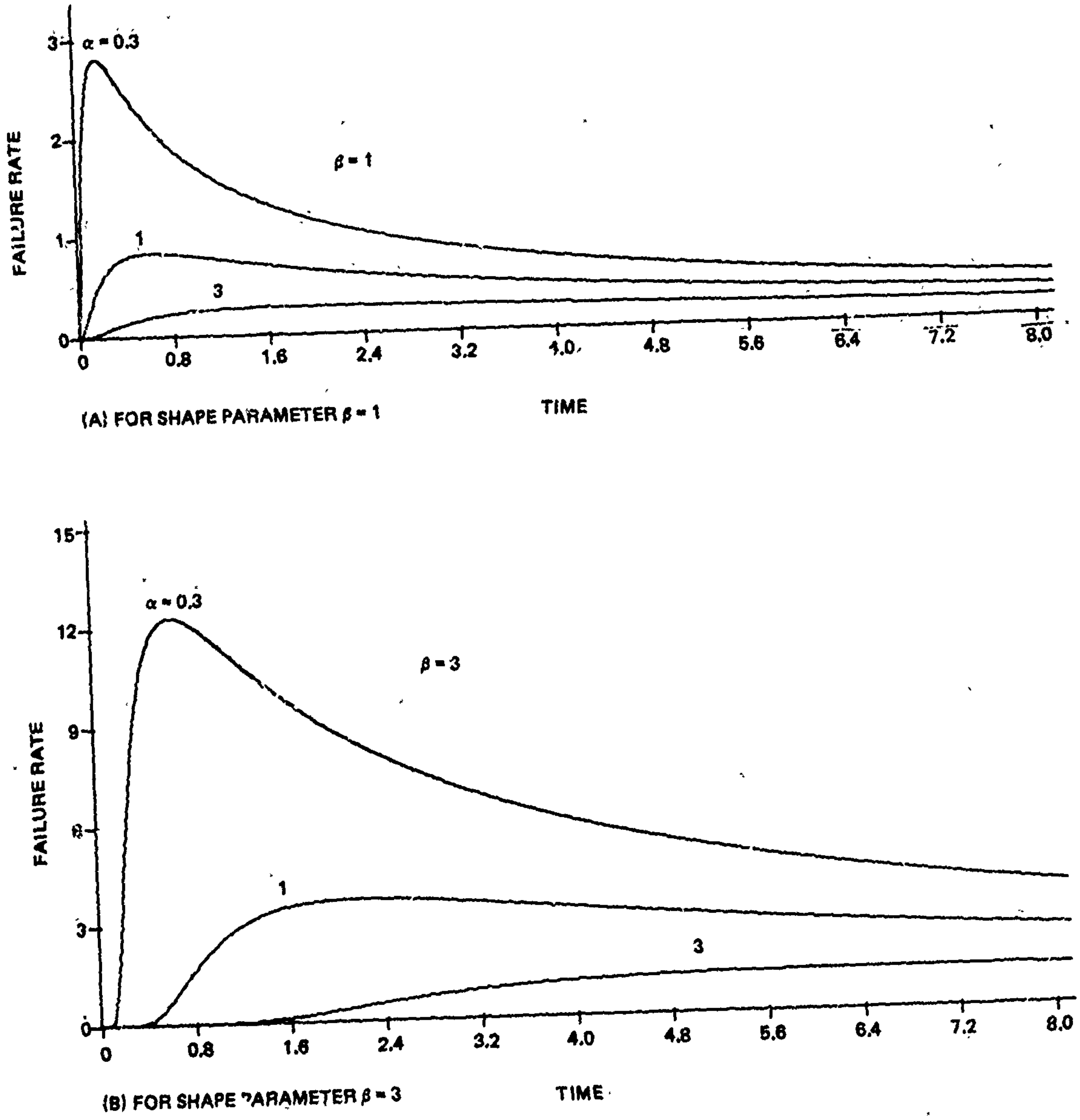

Figure 9-2. Lognormal Distribution, Failure Rate

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with any certainty at all. Often τ_0 is adjusted by an analyst to "straighten out" the graph on lognormal probability paper; this is very poor practice because sample *Cdf*'s of reasonable size are rarely straight when plotted on the proper probability paper. See Ref. 1 (Chapter 5) for parameter estimation of the 3-parameter lognormal distribution.

9-4.1 UNCENSORED DATA

By far the most satisfactory procedure is to use $x \equiv \ln \tau$. Then x is from a Gaussian distribution; see par. 5-3 for parameter estimation.

$$\alpha \equiv e^{\mu} \tag{9-5a}$$

$$\beta \equiv 1/\sigma \tag{9-5b}$$

where μ and σ are the mean and standard deviations of x, respectively. *s*-confidence statements about μ and σ will hold for the corresponding α and β.

Example. The following failure times were observed (all times are in hours). Assume a lognormal distribution. Estimate the parameters.

566	2171
625	2226
1000	2638
1073	2773
1240	3781

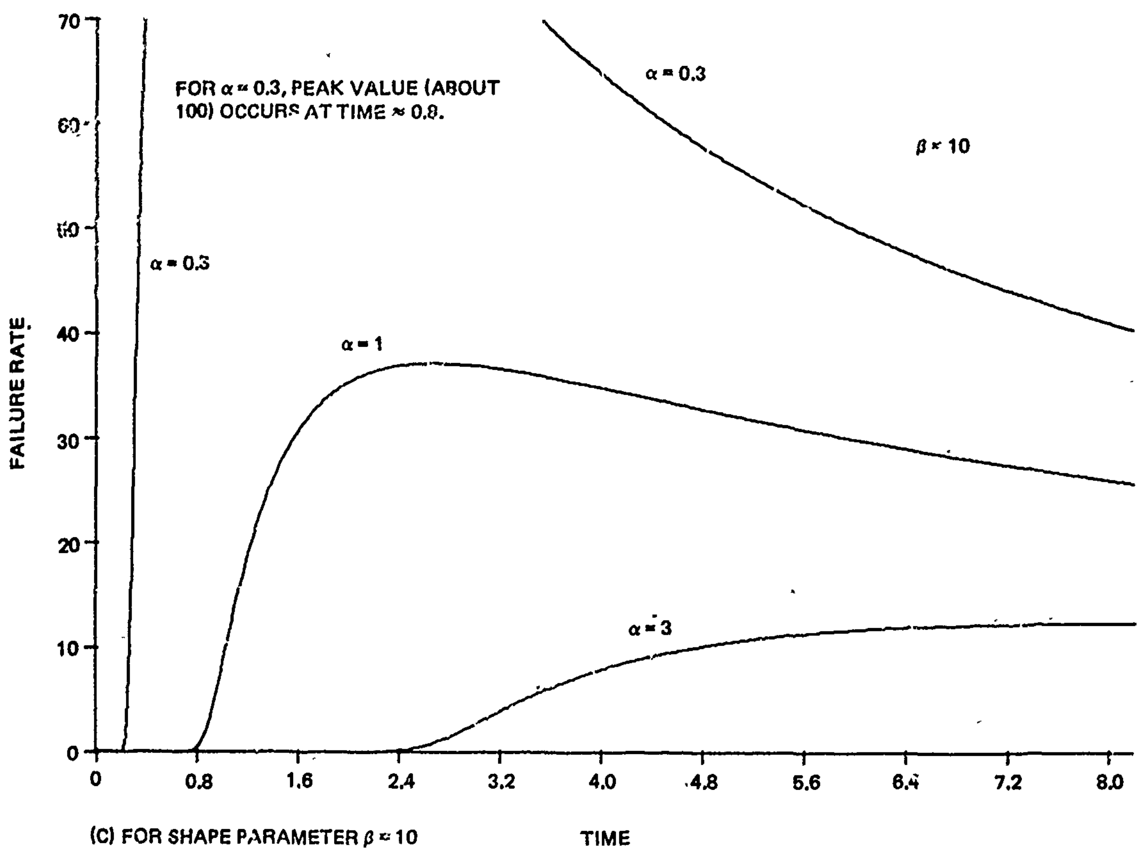

Figure 9-2. Lognormal Distribution, Failure Rate

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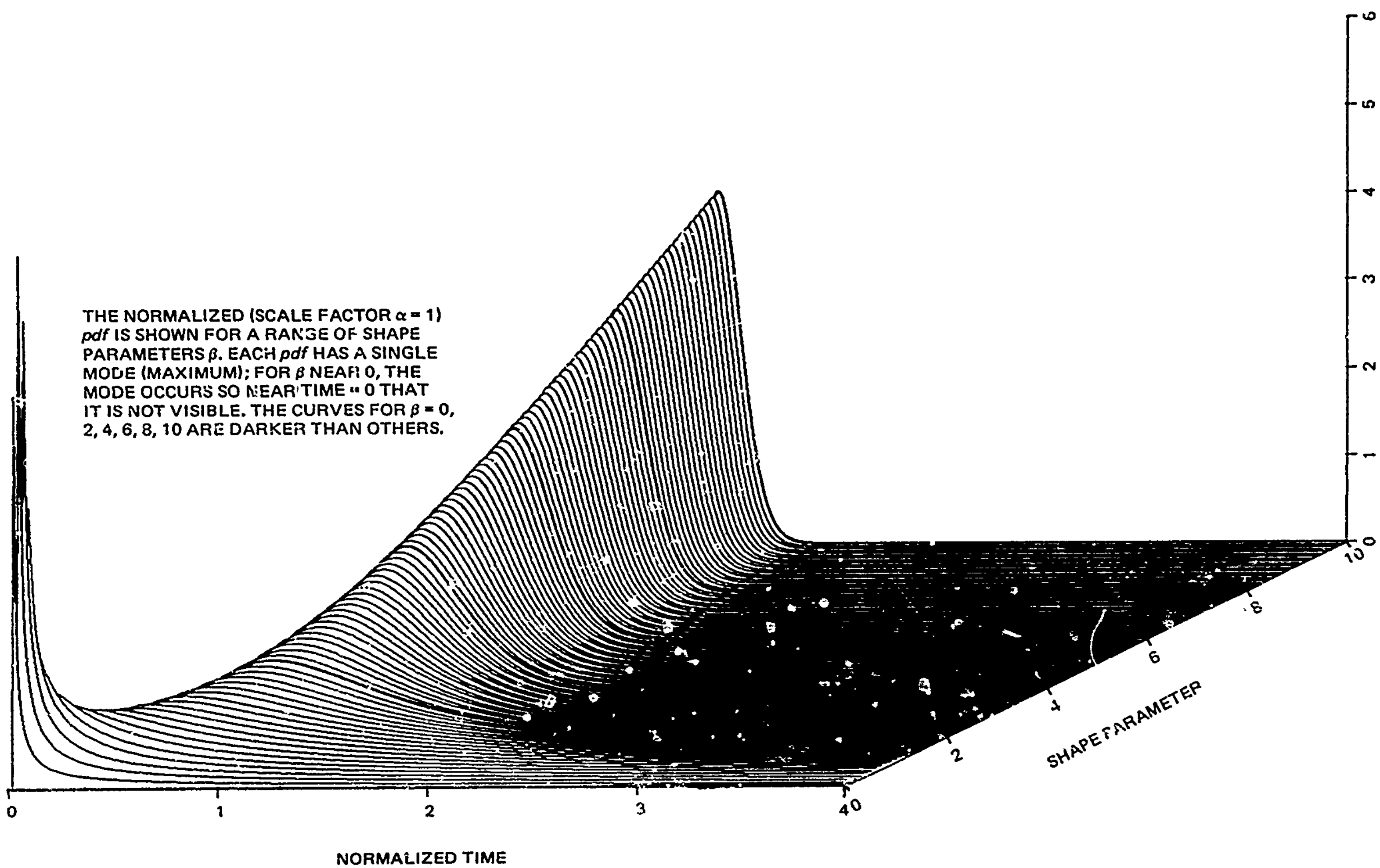

Figure 9-3. Lognormal Distribution, Contour Plot

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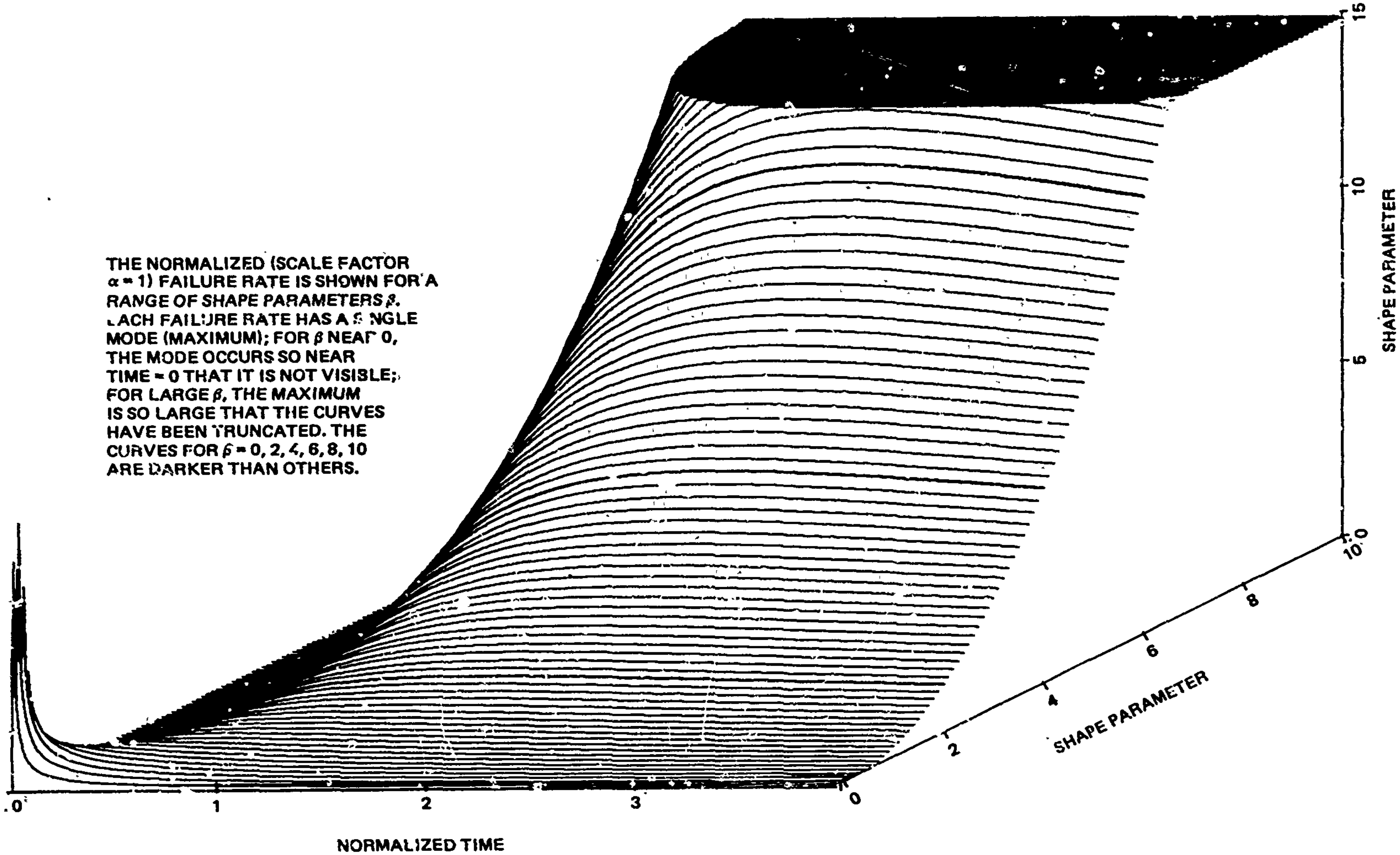

Figure 9-4. Lognormal Failure Rate, Contour Plot

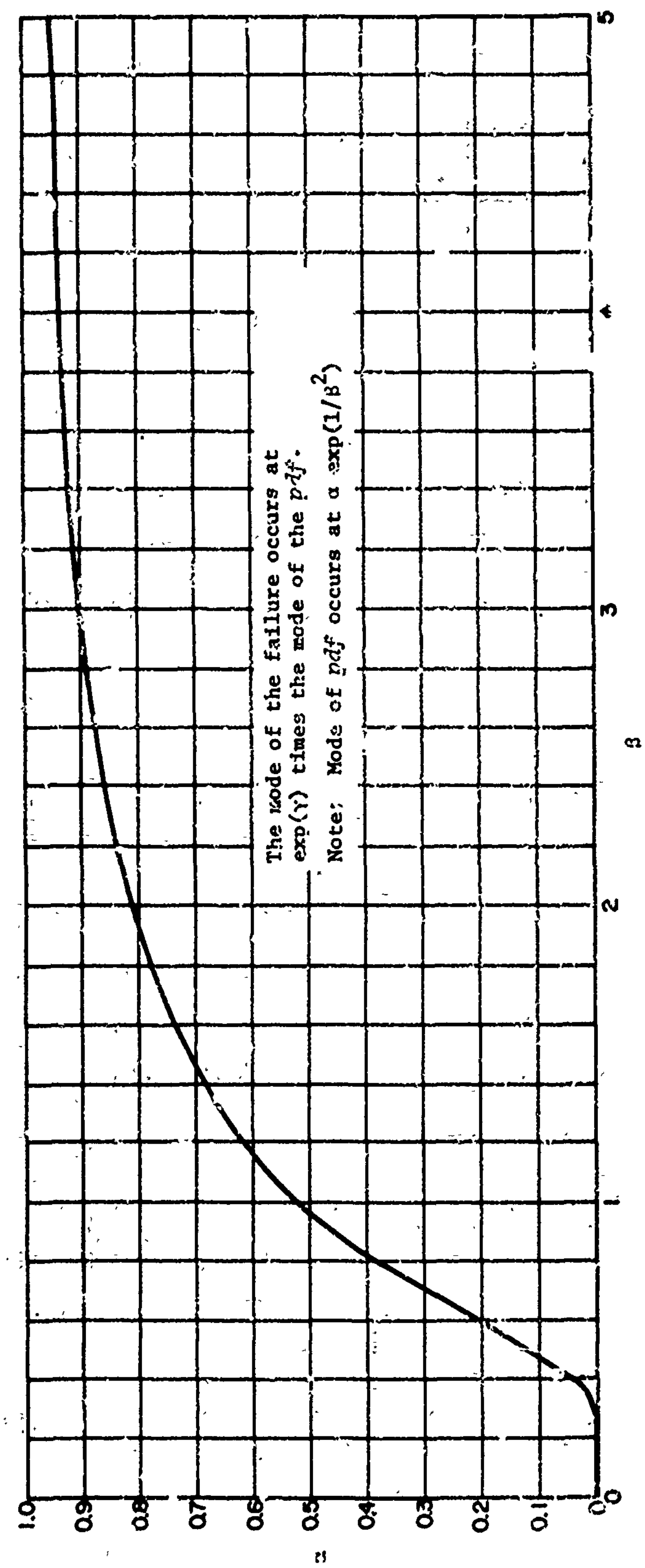

Figure 9-5. Lognormal Distribution: Failure-rate Mode

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Procedure	Example
1. Calculate the logs (natural) of the failure times (any base will give correct answers if it is used consistently) and find the sample mean $\bar{x}$ and s statistic. State sample size and degrees of freedom.	1. $\bar{x} = 7.322$ $s = 0.656$ $N = 10$ $\nu = 9$
2. Calculate point estimates from Eq. 9-5 $\hat{\alpha} = e^{\hat{\mu}} = e^{\bar{x}}$ $\hat{\beta} = 1/\hat{\sigma} = 1/s$	2. $\hat{\alpha} = 1513$ hr $\hat{\beta} = 1.525$
3. Calculate 90% symmetrical s-confidence limits for α. Calculate then for μ first. Use Eq. 5-5b.	3. $t_{.05,9} = -1.833$ $t_{.95,9} = +1.833$ $\mu_L = 7.322 - 0.380 = 6.942$ $\mu_H = 7.322 + 0.380 = 7.702$ $\alpha_L = 1034$ hr $\alpha_H = 2213$ hr Conf $\{1030 \text{ hr} \leqslant \alpha \leqslant 2210 \text{ hr}\} = 90\%$
4. Calculate 90% symmetrical s-confidence limits for β. Calculate them for σ first. Use. Eq. 5-6b.	4. $(\chi^2/\nu)_{0.05,9} = 0.3694 = 0.6078^2$ $(\chi^2/\nu)_{0.95,9} = 1.8799 = 1.3711^2$ $\sigma_H = 0.656/0.6078 = 1.0792$ $\sigma_L = 0.656/1.3711 = 0.4784$ $\beta_H = 2.09$ $\beta_L = 0.927$ Conf $\{0.93 \leqslant \beta \leqslant 2.1\} = 90\%$

The uncertainty in α is more than a factor of 2, and in β is more than a factor of 2. Thus the statistical analysis has shown how little we know about the distribution after taking this sample of 10. The actual data were a random sample from a lognormal distribution with α = 1000 hr and β = 2.

9-4.2 CENSORED DATA

Maximum likelihood is complicated because 2 simultaneous equations must be solved iteratively just as for the s-normal distribution. Ref. 2 shows how order statistics can be used. Ref. 1 also discusses this situation. A statistician ought to be consulted.

REFERENCES

1. Mann, Schafer, Singpurwalla, *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley & Sons, 1974.
2. Sarhan and Greenberg, *Contributions to Order Statistics*, John Wiley & Sons, NY, 1962.

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CHAPTER 10

BETA DISTRIBUTION

10-0 LIST OF SYMBOLS

B = beta function

bet = base name for beta distribution

betf = *Cdf* for beta distribution

betfc = *Sf* for beta distribution

C = *s*-Confidence

Cdf = Cumulative distribution function

C, L, U = subscripts that imply a *s*-confidence level; C is general, L is lower, U is upper

$CM_i\{\ \}$ = ith central moment

Conf$\{\ \}$ = *s*-Confidence level

csqfc = *Sf* for the chi-square distribution

CV$\{\ \}$ = coefficient of variation: StDv$\{\ \}/E\{\ \}$

$E\{\ \}$ = *s*-Expected value

fisfc = *Sf* of the Fisher-Snedecor F distribution

gauf = *Cdf* for Gaussian distribution

I_p = incomplete beta function

$M_i\{\ \}$ = ith moment about the origin

$NCM_i\{\ \}$ = normalized ith central moment; $CM_i\{\ \}/[\text{StDv}\{\ \}]^i$

pdf = probability density function

pmf = probability mass function

$Pr\{\ \}$ = Probability

PrD = Probability distribution

R = *s*-Reliability

r, n = parameters

s- = denotes statistical definition

Sf = Survivor function

StDv$\{\ \}$ = standard deviation

stufc = *Sf* of the Student's *t*-distribution

Var$\{\ \}$ = variance

x = random variable

α, β = parameters

ν_1, ν_2 = degrees of freedom

$\{\cdot;\cdot\}, (\cdot;\cdot)$ = the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon

$\bar{\ }$ = the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

$*$ = implies use of the (r, n) parameter set

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10-1 INTRODUCTION

This is sometimes used as a *PrD* for *s*-reliability since the random variable has the range 0 – 1. It also finds some use, in principle, as a prior *PrD* for Bayesian analysis of the binomial parameter. The base name *bet* is given to the beta distribution (for beta). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*). Most of the formulas were obtained from Refs. 1, 2, and 3 (many formulas appear all 3 places). The beta distribution is also called the incomplete beta function.

10-2 FORMULAS

α, β = parameters, $\alpha \geq 0, \beta \geq 0$

x = random variable, $0 \leq x \leq 1$ (for some values of α, β the *pdf* is not defined at the end points)

r, n alternate parameters, $0 \leq r \leq n$ (the restriction on n is more stringent than mathematically necessary)

$r \equiv \alpha$, $n \equiv \alpha + \beta - 1$ (r and n are usually restricted to non-negative integers)

n sometimes is called a "scale" parameter and r a shape parameter

$B(\alpha, \beta) \equiv \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$ is called the beta function.

$$pdf\{x; \alpha, \beta\} = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha, \beta) \quad \text{for } \alpha, \beta \neq 0 \qquad (10\text{-}1a)$$

$$\left.\begin{array}{l} pmf\{0; 0, \beta\} = 1 \\ pdf\{x; 0, \beta\} = 0, \text{ for } x \neq 0 \end{array}\right\} \text{for } \beta \neq 0$$

$$\left.\begin{array}{l} pmf\{1; \alpha, 0\} = 1 \\ pdf\{x; \alpha, 0\} = 0 \text{ for } x \neq 1 \end{array}\right\} \text{for } \alpha \neq 0$$

$$pmf\{0; 0, 0\} = pmf\{1; 0, 0\} = 1/2$$

$$pdf\{x; 0, 0\} = 0, \text{ for } x \neq 0$$

$$pdf^*\{x; r, n\} = x^{r-1}(1-x)^{n-r} r\binom{n}{r}, \quad \text{for } r \geq 1 \qquad (10\text{-}1b)$$

$$pmf^*\{0; 0, n\} = 1$$

$$pdf^*\{x; 0, n\} = 0 \text{ for } x \neq 0$$

$$Cdf\{x; \alpha, \beta\} = I_x(\alpha, \beta) = 1 - I_{1-x}(\beta, \alpha) \qquad (10\text{-}2a)$$

$$\begin{aligned} Cdf^*\{x; r, n\} &= I_x(r, n-r+1) \\ &= 1 - I_{1-x}(n-r+1, r) \\ &= betf^*(x; r, n) \qquad (10\text{-}2b) \end{aligned}$$

$$Sf\{x; \alpha, \beta\} = 1 - I_x(\alpha, \beta) = I_{1-x}(\beta, \alpha) \qquad (10\text{-}3a)$$

$$\begin{aligned} Sf^*\{x; r, n\} &= 1 - I_x(r, n-r+1) \\ &= I_{1-x}(n-r+1, r) \\ &= betfc^*(x; r, n) \qquad (10\text{-}3b) \end{aligned}$$

$$E\{x; \alpha, \beta\} = \alpha/(\alpha + \beta)$$

$$E^*\{x; r, n\} = r/(n+1)$$

$$\text{StDv}\{x; \alpha, \beta\} = \frac{1}{\alpha + \beta}\left[\frac{\alpha\beta}{\alpha + \beta + 1}\right]^{1/2}$$

$$\text{StDv}^*\{x; r, n\} = \frac{1}{n+1}\left[\frac{r(n+1-r)}{n+2}\right]^{1/2}$$

$$\text{CV}\{x; \alpha, \beta\} = \left[\frac{\beta}{\alpha(\alpha + \beta + 1)}\right]^{1/2}$$

$$\text{CV}^*\{x; r, n\} = \left[\frac{n+1-r}{r(n+2)}\right]^{1/2}$$

$$CM_3\{x; \alpha, \beta\} = \frac{2\alpha\beta(\beta - \alpha)}{(\alpha+\beta)^3(\alpha+\beta+1)(\alpha+\beta+2)}$$

$$CM_3^*\{x; r, n\} = \frac{2r(n+1-r)(n+1-2r)}{(n+1)^3(n+2)(n+3)}$$

$$NCM_3\{x; \alpha, \beta\} = \frac{2(\beta - \alpha)}{(\alpha + \beta + 2)}\left[\frac{\alpha + \beta + 1}{\alpha\beta}\right]^{1/2}$$

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$$NCM_3^*\{x; r, n\} = \frac{2(n+1-2r)}{n+3} \times \left[\frac{n+2}{r(n+1-r)}\right]^{1/2}$$

$$\text{mode}\{x; \alpha, \beta\} = (\alpha-1)/(\alpha+\beta-2)$$

$$\text{mode}^*\{x; r, n\} = (r-1)/(n-1)$$

$$\text{median}\{x; \alpha, \beta\} \approx (\alpha-0.3)/(\alpha+\beta-0.6)$$

$$\text{median}^*\{x; r, n\} \approx (r-0.3)/(n+0.4)$$

The beta distribution is related to other distributions as follows:

$$\sum_{i=r}^{n} \binom{n}{r} p^i \bar{p}^{n-i} = I_p(r, n-r+1) = betf^*(x; r, n) \qquad (10\text{-}4)$$

$$|stufc(t; \nu)| = I_x(\nu/2, 1/2), \quad x \equiv \nu/(\nu+t^2) \qquad (10\text{-}5)$$

$$fisfc(F; \nu_1, \nu_2) = I_x(\nu_2/2, \nu_1/2), \quad x \equiv \frac{\nu_2}{\nu_2+\nu_1 F} \qquad (10\text{-}6)$$

Fig. 10-1 shows some graphs of Eq. 10-1a. Fig. 10-2 shows some graphs of Eq. 10-1b.

10-3 TABLES

Ref. 4 is an extensive set of tables of Eq. 10-4. Other tables are given in Refs. 5 and 6. Ref. 2 has tables, a chart, and a discussion of the distribution.

Ref. 1 (formulas 26.5.20 and 26.5.21) has two approximations for calculating I_x.

For $(\alpha+\beta-1)(1-x) \leqslant 0.8$,

$$I_x(\alpha, \beta) = csqfc(\chi^2; 2\beta) + \epsilon$$

$$|\epsilon| < 0.005, \text{ if } \alpha+\beta > 6$$

$$\chi^2 \equiv (\alpha+\beta-1)(1-x)(3-x) - (1-x)(\beta-1) \qquad (10\text{-}7)$$

$$I_x(\alpha, \beta) = gauf(y) + \epsilon$$

$$|\epsilon| < 0.005 \text{ if } \alpha+\beta > 6$$

$$y \equiv \frac{3\left[w_1\left(1-\frac{1}{9\beta}\right) - w_2\left(1-\frac{1}{9\alpha}\right)\right]}{\left[\frac{w_1^2}{\beta} + \frac{w_2^2}{\alpha}\right]^{1/2}}$$

$$w_1 \equiv (\beta x)^{1/3}, \quad w_2 \equiv [\alpha(1-x)]^{1/3}. \qquad (10\text{-}8)$$

Eq. 10-8 is related to Eq. 6-24. No tables are included in this Handbook series. Eq. 10-6 and Table 6-4 can be used.

10-4 PARAMETER ESTIMATION

There are no simple good estimation procedures. The method of moments is reasonably straightforward but does not readily allow an estimation of the uncertainties in the parameter estimates.

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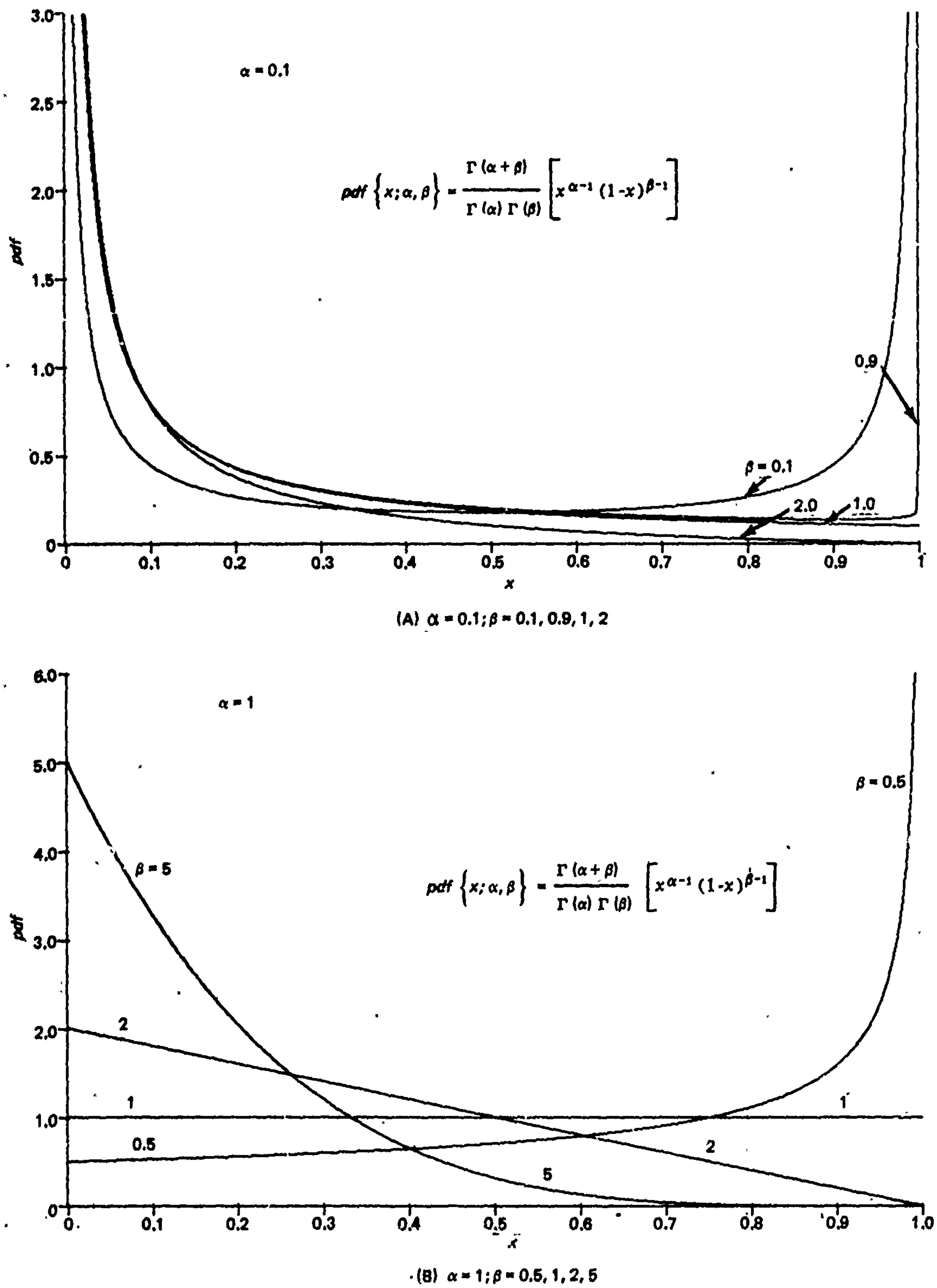

Figure 10-1. Beta Distribution

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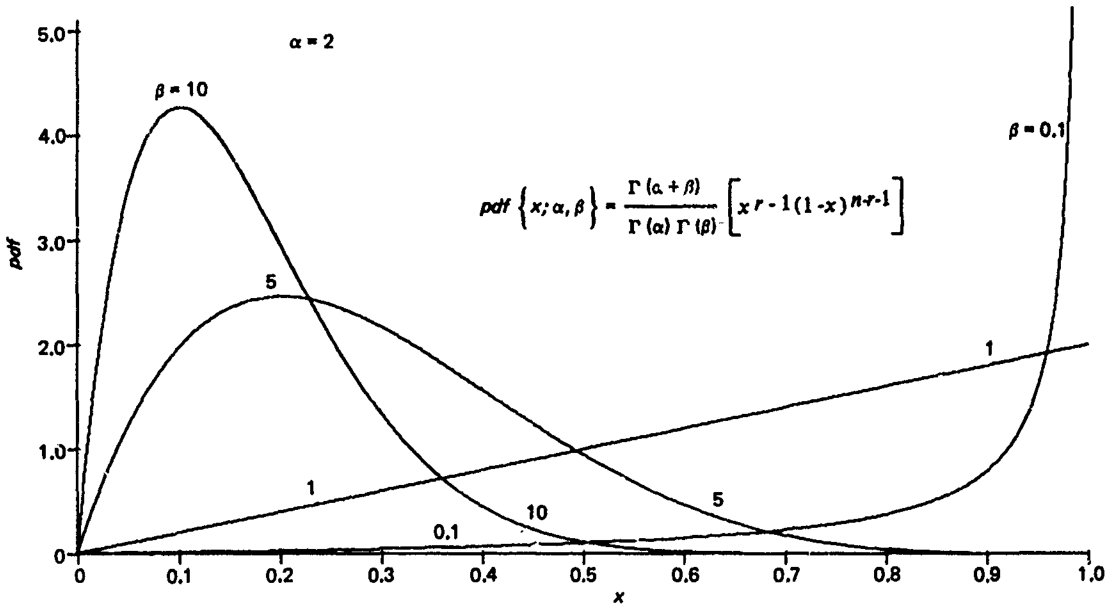
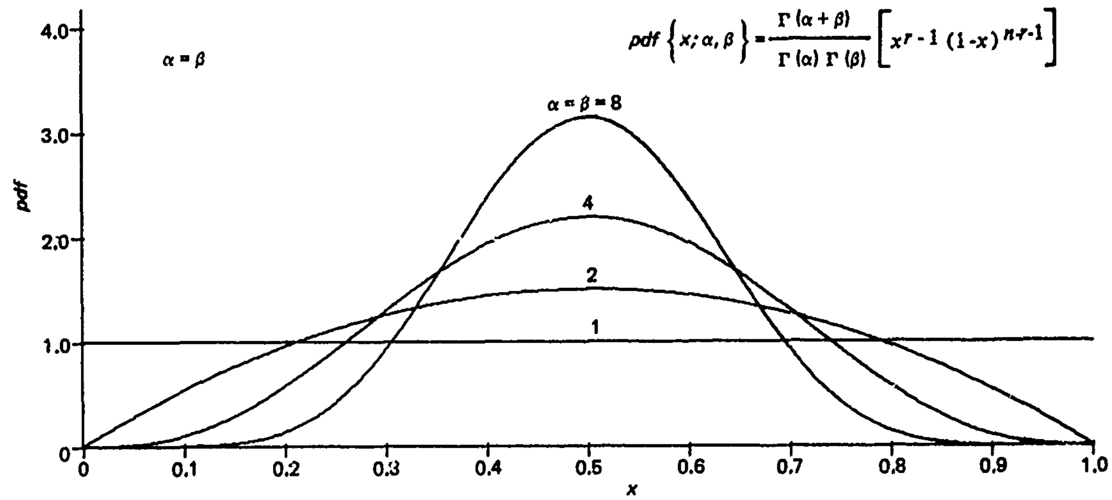


(C) α = 2; β = 0.1, 1, 5, 10



(D) α = β = 1, 2, 4, 8

Figure 10-1. Beta Distribution (Continued)

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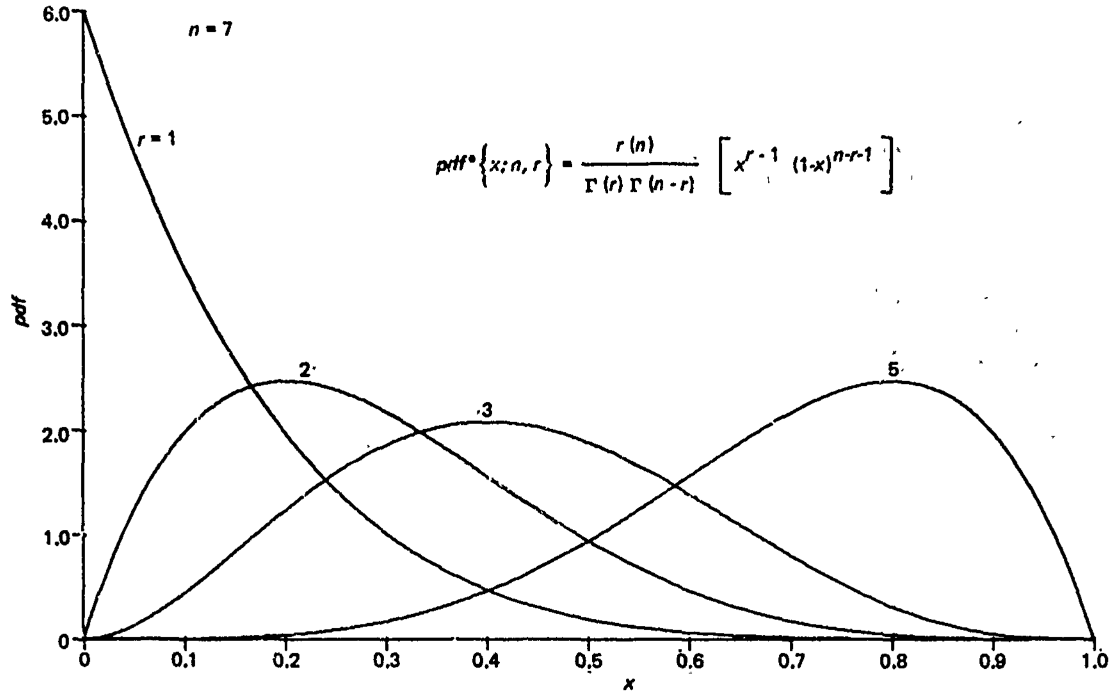
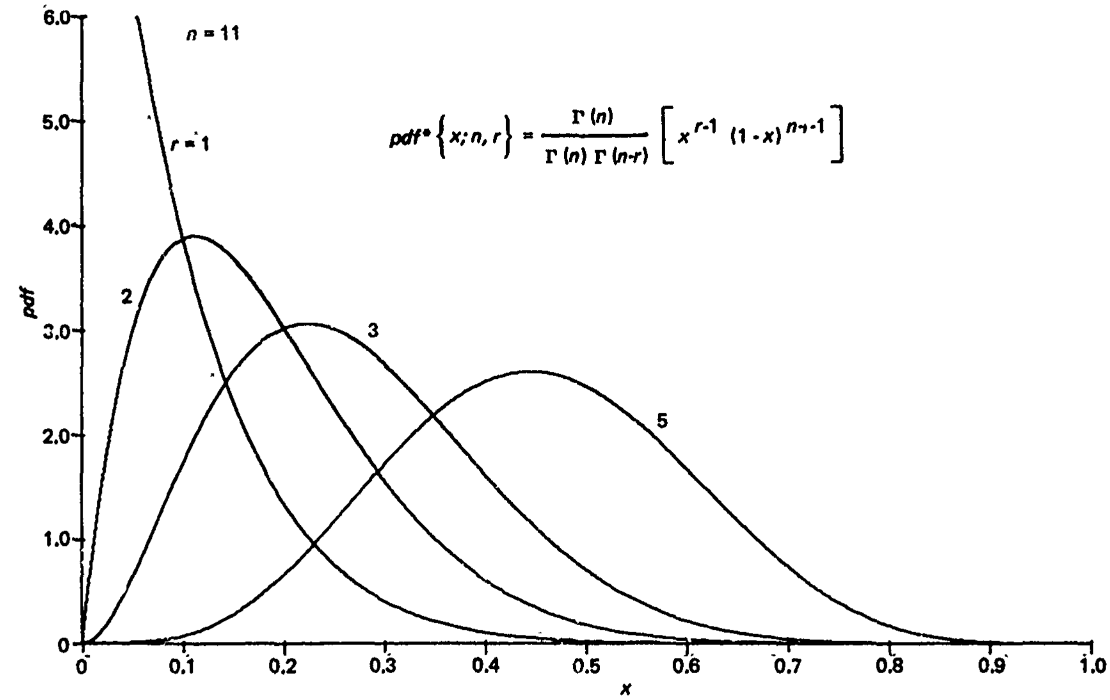


(A) $n = 7; r = 1, 2, 3, 5$



(B) $n = 11; r = 1, 2, 3, 5$

Figure 10-2. Beta Distribution (alternate parameters)

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REFERENCES

1. Abramowitz and Stegun, Eds., *Handbook of Mathematical Functions,* AM555, NBS, USGPO, June 1964 with subsequent corrected printings.

2. E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians,* Vol. I, Cambridge Univ. Press, 1956.

3. W. G. Ireson, Ed., *Reliability Handbook,* McGraw-Hill, NY, 1966.

4. AMCP 706-109, Engineering Design Handbook, *Tables of the Cumulative Binomial Probabilities.*

5. K. Pearson, *Tables of the Incomplete Beta Function,* University Press, Cambridge (England), 1968 (first ed., 1934).

6. H. L. Harter, *New Tables of the Incomplete Gamma Function Ratio and of Percentage Points of the Chi-squcre and Beta Distributions,* USGPO, 1964.

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CHAPTER 11

GAMMA DISTRIBUTION

11-0 LIST OF SYMBOLS

C	=	*s*-Confidence
Cdf	=	Cumulative distribution function
C, L, U	=	subscripts that imply a *s*-confidence level; C is general, L is lower, U is upper
Conf{ }	=	*s*-Confidence level
$CM_i\{\ \}$	=	ith central moment
csqf	=	*Cdf* for chi-square distribution
CV{ }	=	coefficient of variation: StDv{ }/E{ }
$E\{\ \}$	=	*s*-Expected value
gam	=	base name for gamma distribution
gamf	=	*Cdf* for gamma distribution
gamfc	=	*Sf* for gamma distribution
gaufc	=	*Sf* for Gaussian distribution
$M_i\{\ \}$	=	ith moment about the origin
$NCM_i\{\ \}$	=	normalized ith central moment; $CM_i\{\ \}/[\mathrm{StDv}\{\ \}]^i$
pdf	=	probability density function
pmf	=	probability mass function
$Pr\{\ \}$	=	Probability
PrD	=	Probability distribution
R	=	*s*-Reliability
s-	=	denotes statistical definition
Sf	=	Survivor function
StDv{ }	=	standard deviation
Var{ }	=	variance
α	=	scale parameter
β	=	shape parameter
Γ	=	gamma function
τ	=	random variable
$\{\cdot;\cdot\}, (\cdot;\cdot)$	=	the fixed parameters are listed to the right of the semicolon, the random variable is listed to the left of the semicolon
$\overline{}$	=	the complement, e.g., $\bar{\phi} \equiv 1 - \phi$ where ϕ is any probability

11-1 INTRODUCTION

This is the distribution actually used in some of the exponential distribution examples when there is more than 1 failure. It is related closely to the chi-square distribution. The base name *gam* is given to the gamma distribution (for gamma). The suffix *f* implies the *Cdf*, and the suffix *fc* implies the *Sf* (complement of the *Cdf*). The gamma distribution is also called the incomplete gamma function.

11-2 FORMULAS

α = scale parameter, $\alpha > 0$

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β = shape parameter, $\beta > 0$

τ = random variable, $\tau \geq 0$ (if $\beta < 1$, then $\tau > 0$ for the *pdf*)

$$pdf\{\tau;\alpha,\beta\} = (1/\alpha)e^{-(\tau/\alpha)}(\tau/\alpha)^{\beta-1}/\Gamma(\beta) \quad (11\text{-}1)$$

$$Cdf\{\tau;\alpha,\beta\} = gamf(\tau/\alpha;\beta) \quad (11\text{-}2)$$

$$Sf\{\tau;\alpha,\beta\} = gamfc(\tau/\alpha;\beta) \quad (11\text{-}3)$$

$$E\{\tau;\alpha,\beta\} = \alpha\beta$$

$$\text{StDv}\{\tau;\alpha,\beta\} = \alpha\sqrt{\beta}$$

$$\text{CV}\{\tau;\alpha,\beta\} = 1/\sqrt{\beta}$$

$$CM_3\{\tau;\alpha,\beta\} = 2\alpha^3\beta$$

$$NCM_3\{\tau;\alpha,\beta\} = 2/\sqrt{\beta}$$

$$\frac{E\{\tau^n;\alpha,\beta\}}{[E\{\tau;\alpha,\beta\}]^n} = (1+1/\beta)(1+2/\beta)\cdots \times\left(1+\frac{n-1}{\beta}\right),\ n \geq 1$$

$$\text{mode}\{\tau;\alpha,\beta\} = \begin{cases} \alpha(\beta-1), & \beta \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{median}\{\tau;\alpha,\beta\} \approx \alpha(\beta-0.3)$$

Miscellaneous formulas are

$$gamf(\tau/\alpha;\beta+1) = gamf(\tau/\alpha;\beta) + x^{\beta}e^{-x}/[\Gamma(\beta+1)] \quad (11\text{-}4)$$

$$gamf(\chi^2/2;\nu/2) = csqf(\chi^2;\nu)$$

for ν a positive integer (11-5)

$$gamf\{\mu;r\} = \sum_{i=r}^{\infty}(e^{-\mu})\frac{\mu^i}{i!},$$

for r a non-negative integer (11-6)

Fig. 11-1 shows a few values of the *pdf*. Fig. 11-2 is a contour plot of the *pdf*. Many of the formulas in this chapter are adapted from Ref. 1.

11-3 TABLES

Tables of the gamma distribution are not very common. Refs. 2 and 3 are two of the few extensive tables available. Of course, chi-square tables can be used when β is a multiple or ½ (see Eq. 11-5). No tables are included in this volume.

Eq. 11-7 and its inverse are useful for approximating the gamma distribution

$$z_Q \simeq \left[(u_{Q,B/\beta})^{1/3} - \left(1-\frac{1}{9\beta}\right)\right] \Big/ \sqrt{1/(9\beta)} \quad (11\text{-}7)$$

where

$$gaufc(z_Q) = Q$$

$$gamfc(u_{Q,\beta};\beta) = Q$$

11-4 PARAMETER ESTIMATION

Rarely does one wish to estimate both parameters of the gamma distribution. If β is known, the estimation is straightforward—see the exponential distribution (Chapter 7).

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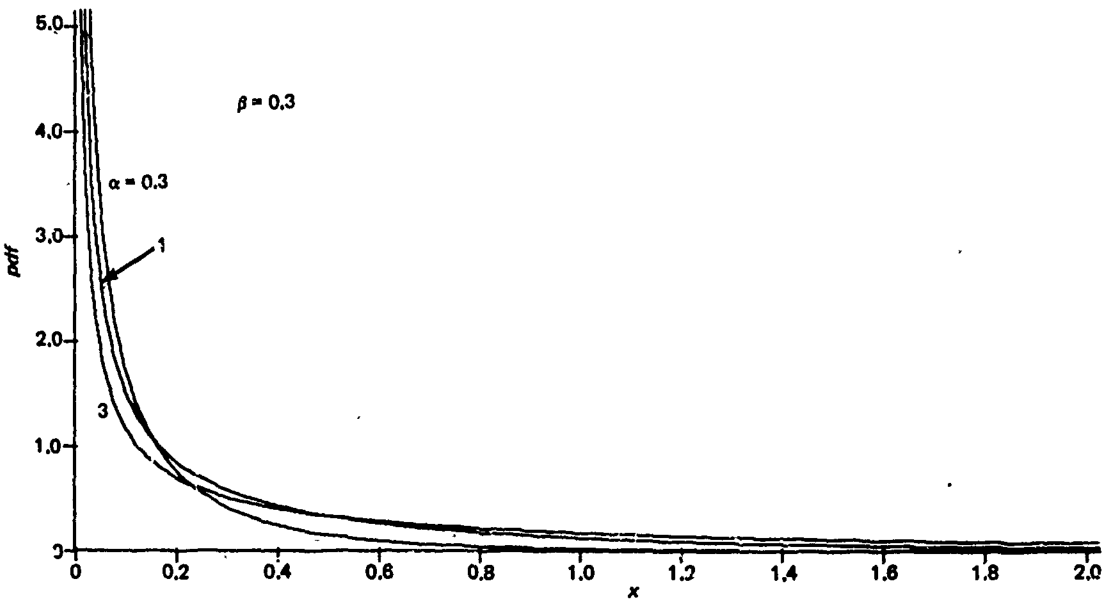
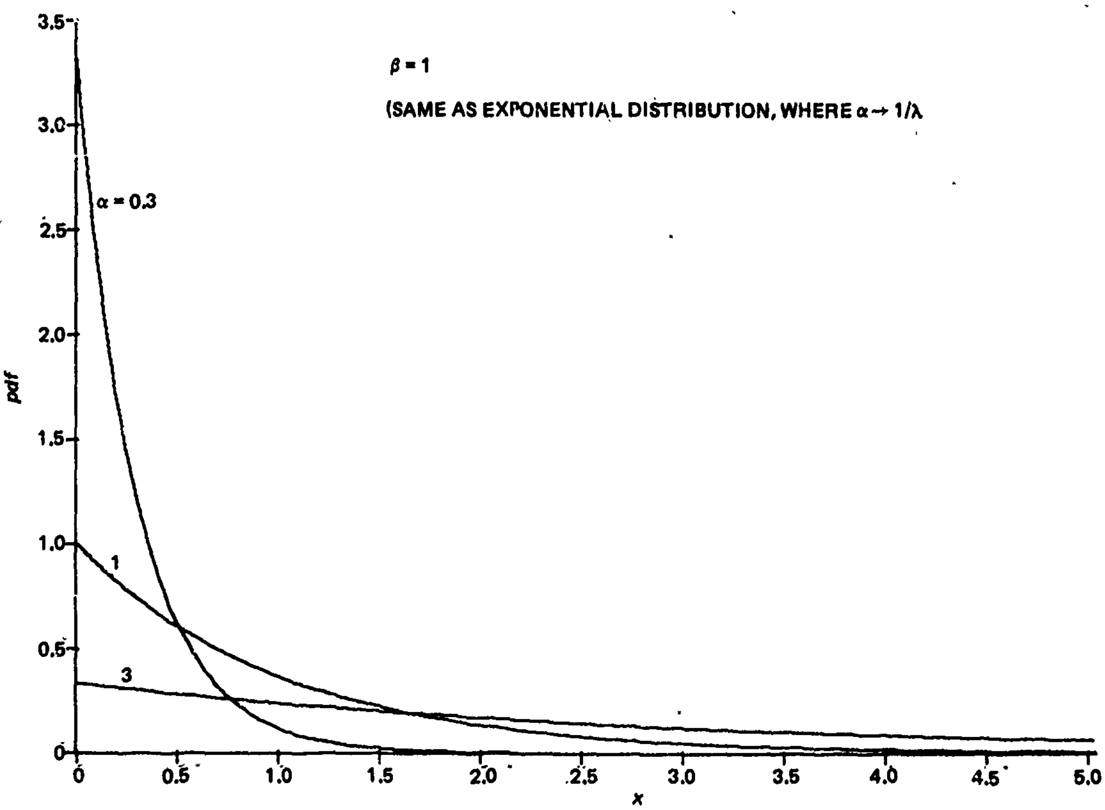


(A) SHAPE PARAMETER $\beta = 0.3$. TYPICAL OF ALL $\beta < 1$. (AS $x \rightarrow 0^+$, THE *pdf's* $\rightarrow \infty$).



(B) SHAPE PARAMETER $\beta = 1$. THIS IS THE EXPONENTIAL DISTRIBUTION; SEE FIG. 7-1.

Figure 11-1. Gamma Distribution, pdf

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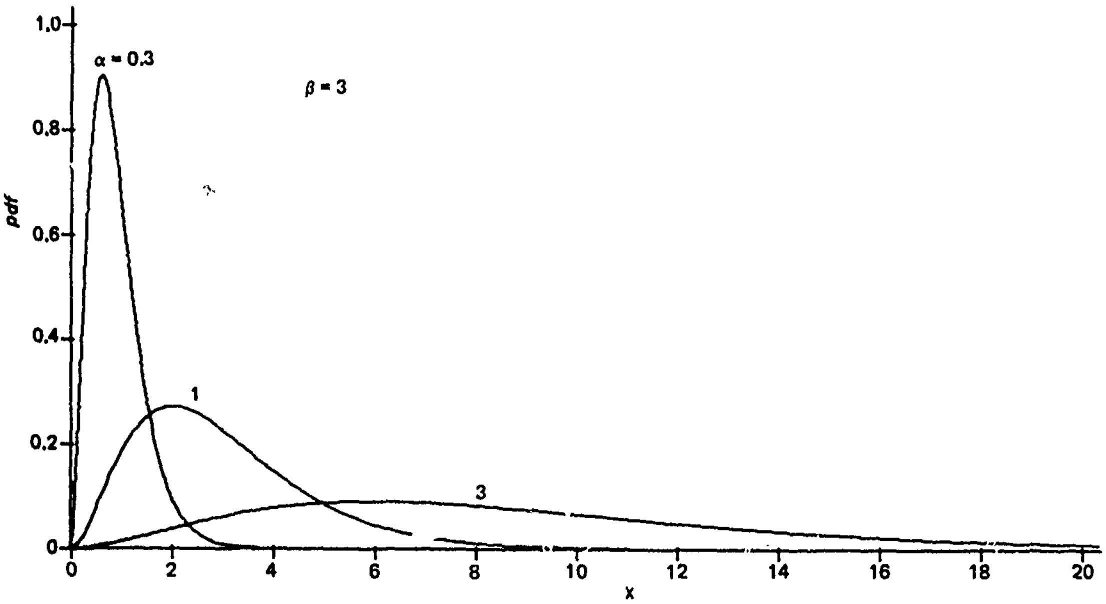

(C) SHAPE PARAMETER β = 3. TYPICAL FOR ALL β > 1.

Figure 11-1. Gamma Distribution, pdf (Continued)

If both parameters must be estimated, consult a statistician. Be sure to estimate the uncertainties in the parameter estimates.

11-5 GAMMA FUNCTION

The gamma function appears many places.

$$\Gamma(\beta) \equiv \int_0^\infty e^{-u} u^{\beta-1}\, du \tag{11-8}$$

See Ref. 1 (Chapter 6) for many characteristics and tables of this and related functions. Table 11-1 can be used to find $\Gamma(\beta)$.

REFERENCES

1. Abramowitz and Stegun, Eds., *Handbook of Mathematical Functions*, AM555, NBS, USGPO, June 1964 with subsequent corrected printings.
2. H. L. Harter, *New Tables of the Incomplete Gamma Function Ratio and of Percentage Points of the Chi-square and Beta Distributions*, USGPO, 1964.
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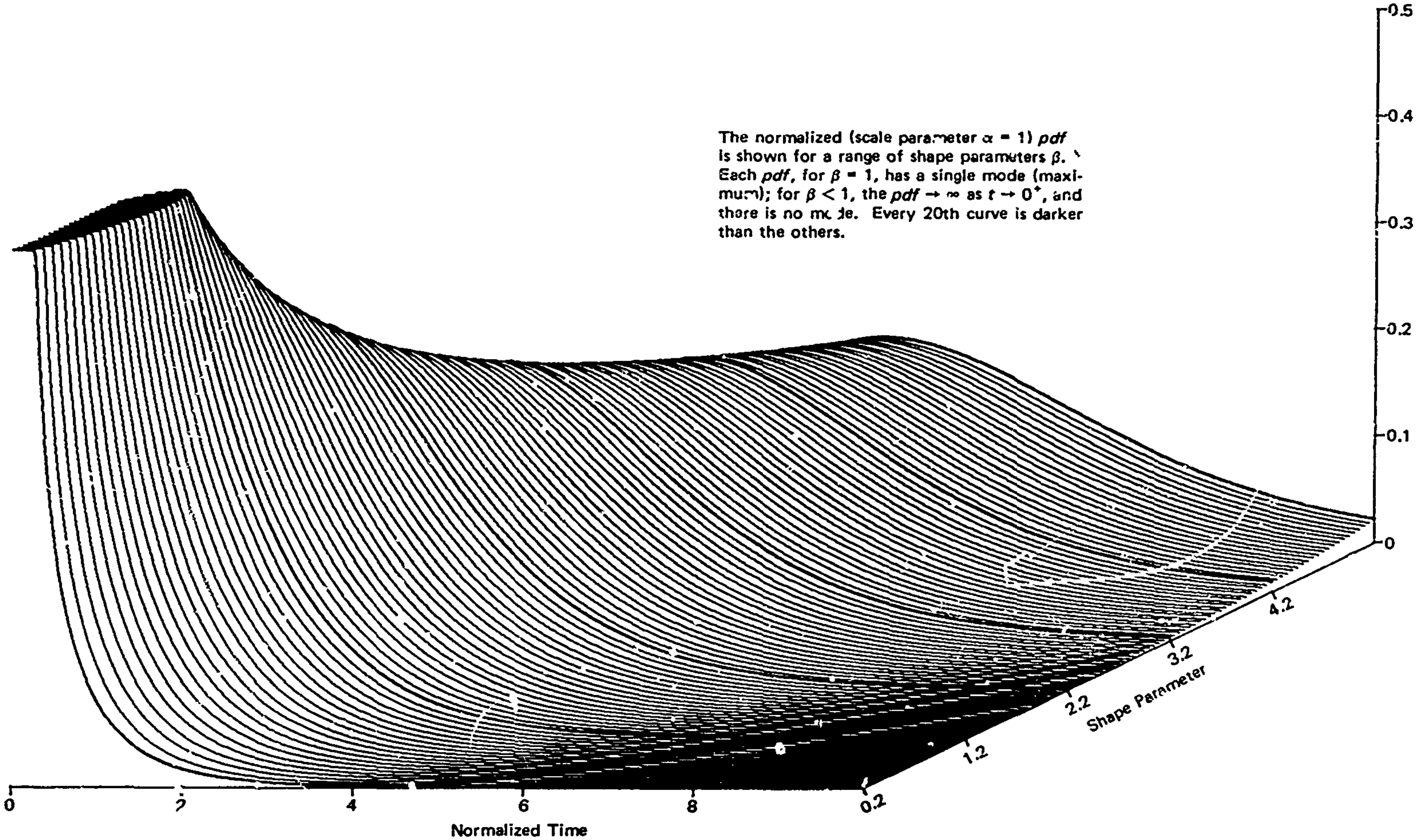

Figure 11-2. Gamma Distribution, Contour Plot of pdf

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TABLE 11-1

GAMMA FUNCTION

$$n! = \Gamma(n+1), \text{ for } n \text{ an integer}$$
$$\Gamma(\beta+1) = \beta\Gamma(\beta)$$
$$\Gamma(\beta+n) = (\beta+n-1)(\beta+n-2)\ldots\beta\Gamma(\beta)$$

β	$\Gamma(\beta)$	β	$\Gamma(\beta)$	β	$\Gamma(\beta)$	β	$\Gamma(\beta)$
1.00	1.00000	1.25	0.90640	1.50	0.88623	1.75	0.91906
1.01	0.99433	1.26	0.90440	1.51	0.88659	1.76	0.92137
1.02	0.98884	1.27	0.90250	1.52	0.88704	1.77	0.92376
1.03	0.98355	1.28	0.90072	1.53	0.88757	1.78	0.92623
1.04	0.97844	1.29	0.89904	1.54	0.88818	1.79	0.92877
1.05	0.97350	1.30	0.89747	1.55	0.88887	1.80	0.93138
1.06	0.96874	1.31	0.89600	1.56	0.88964	1.81	0.93408
1.07	0.96516	1.32	0.89464	1.57	0.89049	1.82	0.93685
1.08	0.95973	1.33	0.89338	1.58	0.89142	1.83	0.93969
1.09	0.95546	1.34	0.89222	1.59	0.89243	1.84	0.94261
1.10	0.95135	1.35	0.89115	1.60	0.89352	1.85	0.94561
1.11	0.94739	1.36	0.89018	1.61	0.89468	1.86	0.94869
1.12	0.94359	1.37	0.88031	1.62	0.89592	1.87	0.95184
1.13	0.93993	1.38	0.88854	1.63	0.89724	1.88	0.95507
1.14	0.93642	1.39	0.88785	1.64	0.89864	1.89	0.95838
1.15	0.93304	1.40	0.88726	1.65	0.90012	1.90	0.96177
1.16	0.92980	1.41	0.88676	1.66	0.90167	1.91	0.96523
1.17	0.92670	1.42	0.88636	1.67	0.90330	1.92	0.96878
1.18	0.92373	1.43	0.88604	1.68	0.90500	1.93	0.97240
1.19	0.92088	1.44	0.88580	1.69	0.90678	1.94	0.97610
1.20	0.91817	1.45	0.88565	1.70	0.90864	1.95	0.97988
1.21	0.91558	1.46	0.88560	1.71	0.91057	1.96	0.98374
1.22	0.91311	1.47	0.88563	1.72	0.91258	1.97	0.98768
1.23	0.91075	1.48	0.88575	1.73	0.91466	1.98	0.99171
1.24	0.90852	1.49	0.88595	1.74	0.91683	1.99	0.99581
						2.00	1.00000

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CHAPTER 12

s-CONFIDENCE

12-0 LIST OF SYMBOLS

C	=	s-Confidence level; or subscript that implies such a level
$\bar{C}$	=	$1 - C$
Cdf	=	Cumulative distribution function
Conf{ }	=	s-Confidence level
glb	=	greatest lower bound
LCL	=	lower s-confidence limit
L, U	=	subscripts implying lower and upper s-confidence levels and limits
lub	=	least upper bound
N	=	sample size
pdf	=	probability density function
pmf	=	probability mass function
PrD	=	probability distribution
r	=	integer random variable
s-	=	denotes statistical definition
Sf	=	Survivor function
t	=	t statistic
UCL	=	upper s-confidence limit
η	=	random variable from uniform distribution
$\bar{\eta}$	=	$1 - \eta$
$\phi\ \theta, \mu$	=	parameters about which a s-confidence statement is to be made
ν	=	degrees of freedom
χ^2/ν	=	chi-square/nu statistic
^	=	implies a point estimate of a parameter

12-1 INTRODUCTION

s-Confidence is one of the first difficult statistical concepts an engineer needs to grasp. s-Confidence is to statistics as entropy is to physics. s-Confidence and entropy are basic concepts that must become familiar in their own right; it is almost impossible to understand them in terms of more-basic concepts.

This chapter applies to the common situation where the parameters in a PrD are not random variables themselves; rather they are fixed, usually-unknown quantities. If the parameters are random variables (irrespective of the meaning attached to probability), the situation is handled by a Bayesian technique (see Chapter 15).

Parameters in a PrD are considered to be fixed nonrandom quantities. Probability statements that contain a parameter cannot imply that the parameter itself is a random variable.

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A s-confidence statement is generated by an algorithm (computational procedure) which is specified in advance of the experiment. Examples of such procedures are given in the paragraphs that follow. It is presumed that all functions and *PrD*'s are well-behaved enough for the concept of s-confidence to be easily and unambiguously applied. Such is the case for the *PrD*'s usually used in reliability work.

Eq. 12-1 shows one kind of expression which can be used to generate useful s-confidence statements.

$$(\bar{x} - \mu)/s = t_\nu \tag{12-1a}$$

$$s^2/\sigma^2 = (\chi^2/\nu)_\nu \tag{12-1b}$$

where $\bar{x}$ and s are the sample-mean and s statistic, respectively, of a fixed-size sample from a s-normal distribution with mean μ and standard deviation σ;

$$\hat{\theta}/\theta = (\chi^2/\nu)_\nu \tag{12-1c}$$

where θ is the sample mean-life of a fixed-size (fixed number of failures) sample from an exponential distribution with scale parameter (mean life) θ.

Each expression in Eq. 12-1 contains on the left:

(1) Only 1 unknown parameter from the *PrD*, and

(2) Sample statistics, i.e., quantities calculated from the sample data.

Each quantity on the right in Eq. 12-1 is a statistic whose *PrD* can be determined without regard to the *PrD* parameters or to the random data although it may depend on such things as the fixed sample size. Both t and χ^2/ν have well-known *PrD*'s (see Chapter 6) and depend on the sample size through ν.

Eq. 12-2 is another kind of expression that can be used to generate useful s-confidence statements.

$$r = c_\mu \tag{12-2}$$

where r is the number of failures observed in a sample from a Poisson distribution with parameter μ, and c_μ has the Poisson distribution with parameter μ.

The remainder of this chapter is limited to those cases for which:

(1) There is only 1 unknown parameter.

(2) There is a single s-sufficient statistic. (Examples of s-sufficient statistics are the number of failures in N tries for a binomial distribution, the number of events for a Poisson distribution, and total-test-time to achieve r_0 failures for the exponential distribution.)

(3) The situation is well-behaved enough that no difficulties are involved in making s-confidence statements.

The cases considered here have a simple geometric and analytic interpretation. Consult a competent statistician or a good reference, e.g., Ref. 2 (Chapter 20), for a discussion of other cases.

Suppose the single parameter in a *PrD* is ϕ and an estimate of ϕ, from a sample, is $\hat{\phi}$. Examples are

(1) Exponential distribution: parameter = scale parameter θ; estimate, $\hat{\theta} = T/r_0$ where T is total-test-time for r_0 failures.

(2) Poisson distribution: parameter = mean number of events, μ; estimate, $\hat{\mu} = r$ where r is the observed number of events.

(3) Binomial distribution: parameter = probability of an event p; estimate, $\hat{p} = r/N$ where r is the observed number of events in N trials.

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$\hat{\phi}$ is a random variable because it depends on the sample data. The procedure for making a *s*-confidence statement is as follows:

(1) Take the random sample and calculate $\hat{\phi}$ from the data.

(2) Use the algorithm to generate another statistic ϕ_C which will be a *s*-confidence limit. ϕ_C is a random variable; it is calculated from the sample data.

(3) Make a *s*-confidence statement. It will use Conf $\{\cdot\}$ rather than *Pr* $\{\cdot\}$ to denote probability. The Conf $\{\cdot\}$ implies that the parameter is unknown and that a value of the random variable ϕ_C has been calculated from the data. Conf $\{\cdot\}$ is interpreted as the fraction of times such a statement will be true when these 3 steps are followed.
Conf $\{\phi \leqslant \phi_C\} = C$.

The procedure often is extended by calculating 2 values of ϕ_C (ϕ_L and ϕ_U) such that the *s*-confidence statements are

$$\text{Conf}\{\phi \leq \phi_U\} = C_U \tag{12-3a}$$

$$\text{Conf}\{\phi \leq \phi_L\} = C_L \tag{12-3b}$$

$$\text{Conf}\{\phi_L \leq \phi \leq \phi_U\} = C_U - C_L\,. \tag{12-3c}$$

C_L is usually small (say 5%) and C_U is usually large (say 95%). Notation for *s*-confidence statements is not at all standard; so particular attention must be paid to the example forms (Eq. 12-3).

Suppose one is estimating the exponential parameter θ by means of a continuous random variable $\hat{\theta} \equiv T/r_0$. ($T$ is the total operating time to the predetermined fixed number of failures r_0.) Suppose a *s*-confidence limit θ_C is calculated at a *s*-confidence level C. $\theta_C = \theta_C(\hat{\theta}, C)$; then, θ_C is a function of $\hat{\theta}$ and C. The statement

$$\text{"Conf}\{\theta \leq \theta_C(\hat{\theta}, C)\} = C\text{"} \tag{12-4a}$$

means that if a $\hat{\theta}$ is found, and C is given, the statement $\theta \leqslant \theta_C(\hat{\theta},C)$ will be a true fraction C of the time, regardless of the true value of θ. The reason this can be so is that θ_C is selected so that the statement "$\theta_0 \leqslant \theta_C(\hat{\theta}, C)$" is true a fraction C of the time for any particular θ_0 and C, for all random samples from the *PrD* with that parameter θ_0. Since it is true for any particular θ_0, it is true regardless of the value of θ.

When the *PrD* is discrete, the *s*-confidence statements are usually of the forms

$$\text{Conf}\{\phi \leq \phi_{C+}(\hat{\phi}, C)\} \geq C \tag{12-4b}$$

$$\text{Conf}\{\phi \leq \phi_{C-}(\hat{\phi}, C)\} \leq C\,. \tag{12-4c}$$

This situation is discussed further in Par. 12-3.

In statistical papers, the concept of *s*-confidence often is modified slightly so that Eq. 12-4b is written Conf $\{\phi < \phi_{C+}(\hat{\phi}, C)\} = C$. This is confusing to nonengineers. This handbook series always makes the inexact *s*-confidence statements in the form of Eqs. 12-4b and 12-4c.

12-2 CONTINUOUS RANDOM VARIABLES

Fig. 12-1 graphically shows how *s*-confidence limits are derived.

s-Confidence statements need not have the inequality as in Eq. 12-3a, b. They can be of the form in Eq. 12-5. But the form in Eq. 12-5 is not used in 1-sided *s*-confidence statements in this chapter because it would further complicate an already complicated notation.

$$\text{Conf}\{\theta \geqslant \hat{\theta}_C(\theta, C)\} = 1 - C \tag{12-5}$$

Fig. 12-1 is a contour plot of $Sf\{\hat{\theta};\theta\}$; 3 contours are shown, $Sf = 10\%$, $Sf = C$, $Sf = 90\%$. The algorithm for finding $\theta_C(\hat{\theta}, C)$ is to choose $\theta_C(\hat{\theta}, C)$ such that $Sf\{\hat{\theta};\theta_C\} = C$.

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$$\text{Conf}\ \{\theta \leq \theta_C(\theta, C); \theta, C\} = Sf\ \{\hat{\theta}; \theta_C\} = C$$

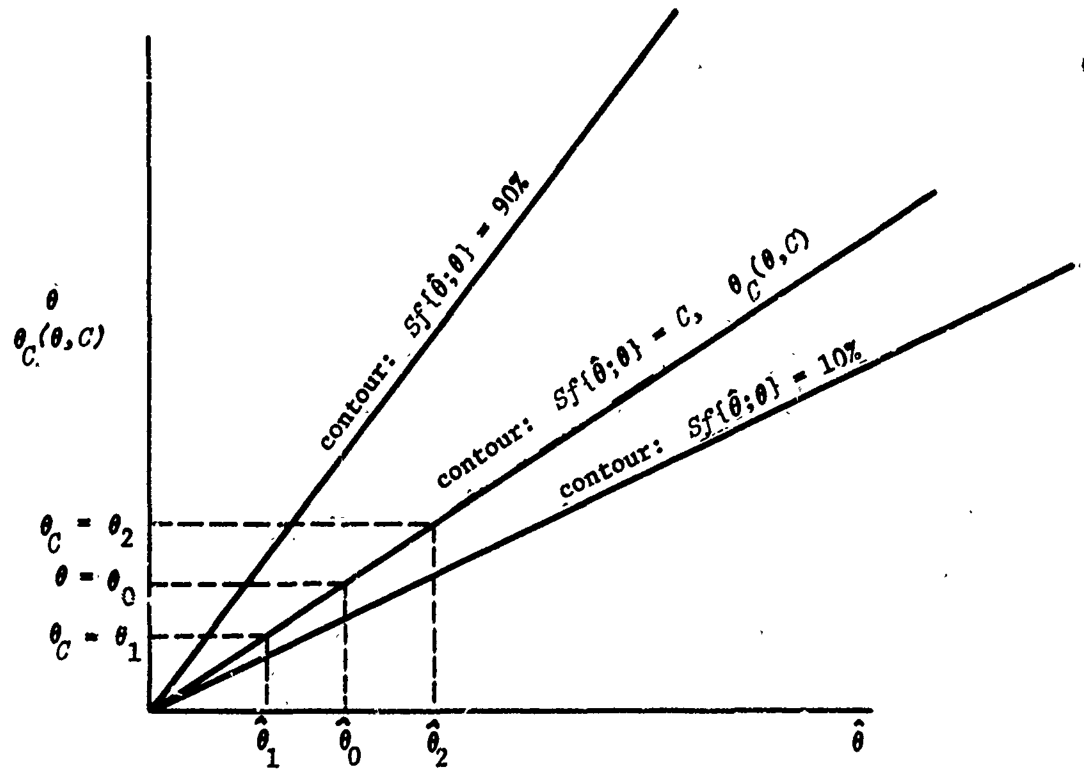

Figure 12-1. s-Confidence Diagram: Continuous Random Variable $\hat{\theta}$ (for well-behaved situations)

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That is, pick θ_C equal to the θ that corresponds to $\hat{\theta}$ on the $Sf = C$ contour.

θ_C is a random variable; it is calculated anew each time a $\hat{\theta}$ is estimated.

The algorithm can be checked in the following way from Fig. 12-1. Suppose θ is fixed at θ_0 and that many samples are drawn—each sample yields a $\hat{\theta}$. If $\hat{\theta} < \hat{\theta}_0$, say $\hat{\theta} = \hat{\theta}_1$, the statement "$\theta_0 \leq \theta_C$" will be false because $\theta_C = \theta_1 < \theta_0$. If $\hat{\theta} \geq \theta$, say $\hat{\theta} = \hat{\theta}_2$, the statement "$\theta_0 \leq \theta_C$" will be true because $\theta_C = \theta_2 > \theta_0$. By the nature of the construction "$\hat{\theta} \geq \hat{\theta}_0$" is true a fraction C of the time, i.e., $Sf\{\hat{\theta}_0; \theta_C\} = C$. Since the *s*-confidence statement is true for any θ_0, no matter what its value, it is true for any θ. Thus the *s*-confidence statement is:

$$\mathrm{Conf}\{\theta \leq \theta_C(\hat{\theta}, C)\} = C \tag{12-4a}$$

where $\theta_C(\hat{\theta}, C)$ is a random variable. A common misunderstanding is to assume that θ_C is not a random variable and to want the statements to hold true, assuming some sort of distribution of θ.

s-Confidence is a difficult concept. It doesn't really answer an engineer's question, "How much engineering confidence do I have in the answer?", but it does respond to a question the statistician can answer.

Suppose there are 2 *s*-confidence limits, θ_U and θ_L such that $\theta_U > \theta_L$; then

$$\mathrm{Conf}\{\theta_L(\hat{\theta}, C_L) < \theta < \theta_U(\theta, C_U)\} = C_U - C_L \tag{12-6}$$

This can be seen either graphically by the construction in Fig. 12-1 or analytically.

12-3 DISCRETE RANDOM VARIABLES

s-Confidence statements (and their derivations) are more difficult for discrete random variables than for continuous ones. The Poisson distribution (parameter μ) is a good example of a discrete random variable, and is the basis for the explanation in this paragraph. The discussion is valid for other discrete distributions, e.g., the binomial distribution.

Fig. 12-2(A) shows contour plots of the *Cdf* and *Sf*. The *PrD* is defined only for the discrete values of r. The dashed lines serve only to guide the eye from one point to a related point. The spacing between consecutive r's is irrelevant, as is the shape of the dashed lines. No *Sf* contour (other than 100%) is defined at the left boundary, and no *Cdf* contour (other than 100%) is defined at the right boundary. The Poisson variable r has no right boundary; the binomial variable r has a right boundary at $r = N$. N will be used to denote maximum value of r; for the Poisson distribution, $N \to \infty$. μ_{max} will be used to denote maximum value of μ.

For a given μ and C, $r_{C\mu}$ is defined such that

$$(1)\ Cdf\{r_{C\mu}; \mu\} > C, \tag{12-7a}$$

and

$$(2)\ Sf\{r_{C\mu}; \mu\} \geq C. \tag{12-7b}$$

It helps to visualize $r_{C\mu}$ from Fig. 12-2(A). Just find the value of r for which μ lies between the 2 contours. If $r_{C\mu}$ is a boundary, one of the 2 contours will not be defined there; but Eq. 12-7 will be satisfied since at the right boundary the $Cdf\ \{r, \mu\} = 1$ for any (r, μ); and at the left boundary the $Sf\ \{r, \mu\} = 1$ for any r, μ.

$$\text{Define } \mu_{C+} \equiv \mathrm{lub}\{\mu; r_{C\mu}\} \tag{12-8a}$$

$$\mu_{C-} \equiv \mathrm{glb}\{\mu; r_{C\mu}\} \tag{12-8b}$$

where lub means "least upper bound" and glb means "greatest lower bound".

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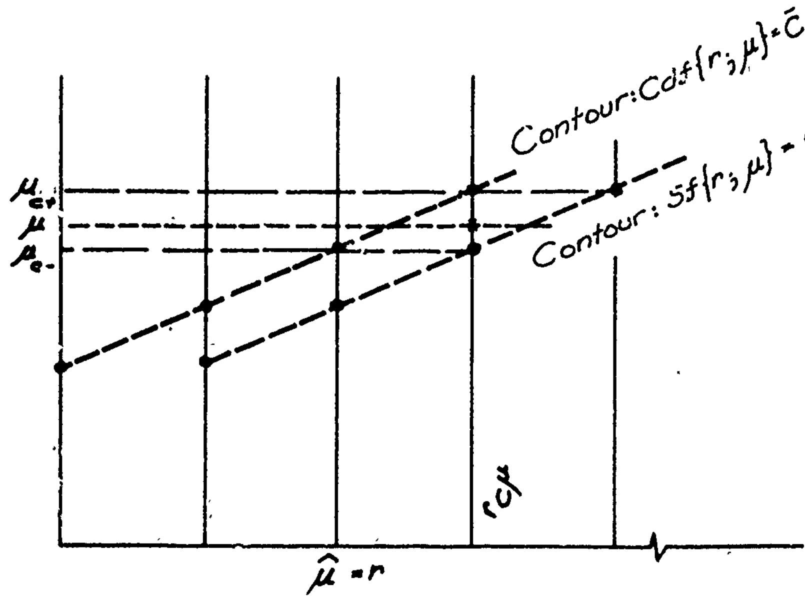
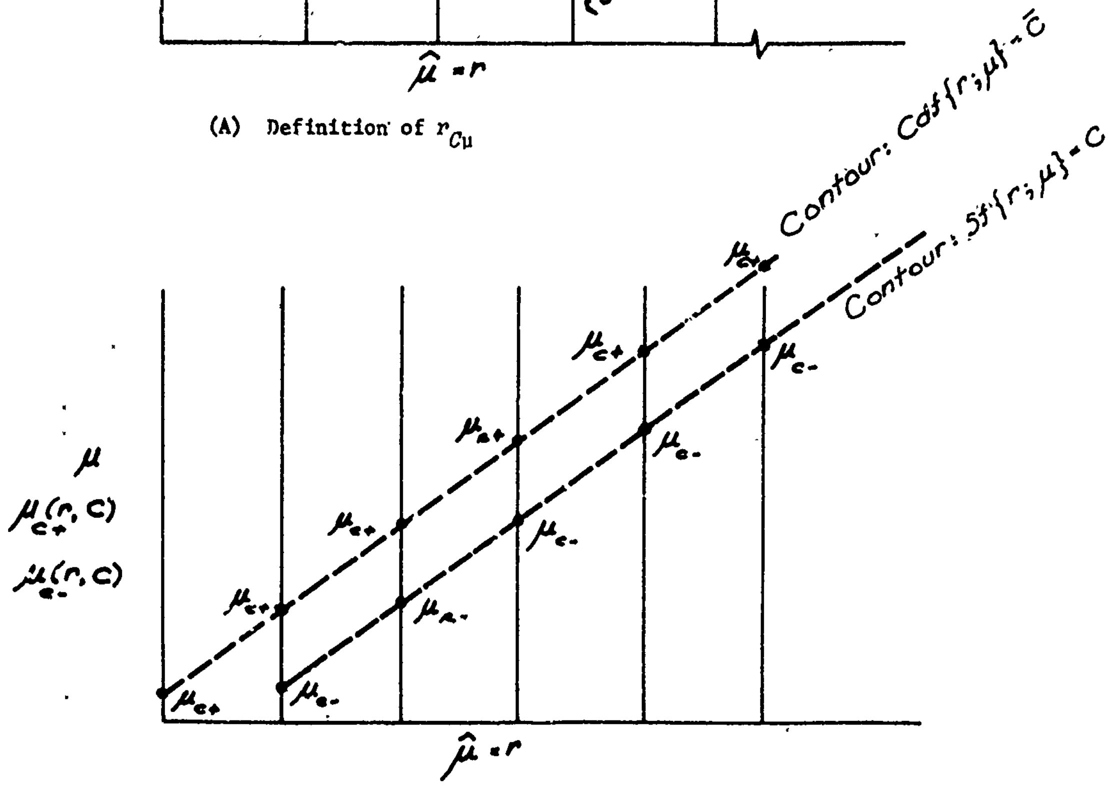


(A) Definition of $r_{C\mu}$



(B) s-Confidence

Figure 12-2. s-Confidence Diagram: Discrete Random Variable

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On Fig. 12-2(A), Eq. 12-8 means to move μ up until it intersects the "$Cdf = \overline{C}$ contour" (for μ_{C+}) and then down until it intersects the "$Sf = C$ contour" (for μ_{C-}).

It can readily be shown that μ_{C-} and μ_{C+} are defined, for each r, by

$$Cdf\{r; \mu_{C+}\} = \overline{C}, \text{ for } r \neq N \qquad (12\text{-}9a)$$

$$\mu_{C+} = \mu_{max}, \text{ for } r = N$$

$$Sf\{r; \mu_{C-}\} = C, \text{ for } r \neq 0 \qquad (12\text{-}9b)$$

$$\mu_{C-} = 0, \text{ for } r = 0$$

where each r is interpreted as the $r_{C\mu}$ in Fig. 12-2(A).

Fig. 12-2(B) shows how the μ_{C-} and μ_{C+} look for several values of r. The μ_{C-} for one value of r is the μ_{C+} for the previous value, $r - 1$, except at the endpoints.

By the nature of the construction of Fig. 12-2, the following s-confidence statements are appropriate.

$$\text{Conf}\{\mu \leq \mu_{C+}(r, C)\} \geq C \qquad (12\text{-}10a)$$

$$\text{Conf}\{\mu \leq \mu_{C-}(r, C)\} \leq C \qquad (12\text{-}10b)$$

Thus the + on μ_{C+} shows an excess of s-confidence the $-$ on μ_{C-} shows a deficit of s-confidence.

Suppose there are 2 s-confidence levels, C_U and C_L such that $C_U > C_L$. Then the levels can be combined as follows:

$$\text{Conf}\{\mu_{L-}(r, C_L) \leq \mu \leq \mu_{U+}(r, C_U)\} \geq C_U - C_L \qquad (12\text{-}11a)$$

$$\text{Conf}\{\mu_{L+}(r, C_L) \leq \mu \leq \mu_{U-}(r, C_U)\} \leq C_U - C_L \qquad (12\text{-}11b)$$

Eq. 12-11a is favored by statisticians because the s-confidence is at least the desired quantity. It is the one given in virtually all reliability texts and articles.

Eq. 12-9 shows that when the Cdf and Sf are complementary, the s-confidence bounds (+ and $-$) come together as in Par. 12-2.

12-4 DISCRETE RANDOM VARIABLES, EXACT CONFIDENCE BOUNDS

It appears to be foolish to use the worst case s-confidence bounds always, because obviously they are always further apart than need be. A method, generally attributed to Ref. 1, is available for generating a s-confidence interval that is exact. The basic idea is to generate a random variable and then, according to its value, choose a μ_C between μ_{C-} and μ_{C+} such that the s-confidence statement is exact.

For a given μ (and $r_{C\mu} \neq 0, N$) the statement $\mu < \mu_C$ (for $\mu_{C-} < \mu_C < \mu_{C+}$) will be true for $r = 1, \ldots, r_{C\mu} - 1$ and sometimes for $r = r_{C\mu}$, as shown in Fig. 12-2(A). Suppose for a fixed μ we calculate the number $\eta_{C\mu}$ between 0 and 1 such that

$$\begin{aligned}
\overline{C} &= pmf\{0; \mu\} + pmf\{1; \mu\} + pmf\{2; \mu\} \\
&\quad + \cdots + pmf\{r_{C\mu} - 1; \mu\} + \eta_{C\mu} pmf\{r_{C\mu}; \mu\} \\
&= pmf\{0; \mu\} + \cdots + pmf\{r_{C\mu}; \mu\} \\
&\quad - (1 - \eta_{C\mu}) pmf\{r_{C\mu}; \mu\} \\
&= Cdf\{r_{C\mu}; \mu\} - (1 - \eta_{C\mu}) pmf\{r_{C\mu}; \mu\},
\end{aligned}$$

$$\text{for } r \neq 0, N; \qquad (12\text{-}12a)$$

$$C = Sf\{r_{C\mu}; \mu\} - \eta_{C\mu} pmf\{r_{C\mu}; \mu\},$$

$$\text{for } r \neq 0, N. \qquad (12\text{-}12b)$$

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It is easy to solve for $\eta_{C\mu}$.

$$\eta_{C\mu} = 1 - \frac{Cdf\{r_{C\mu};\mu\} - \overline{C}}{pmf\{r_{C\mu};\mu\}}, \text{ for } r_{C\mu} \neq 0, N \tag{12-13a}$$

$$= \frac{Sf\{r_{C\mu};\mu\} - C}{pmf\{r_{C\mu};\mu\}}, \text{ for } r_{C\mu} \neq 0, N \tag{12-13b}$$

To use this for a *s*-confidence interval statement, choose an η from the uniform distribution on [0, 1). Then use Eq. 12-14 or 15 to calculate a μ_C (r, C, η).

$$\eta = \frac{Cdf\{r;\mu_C\} - \overline{C}}{pmf\{r;\mu_C\}} \text{ (defines } \mu_C\text{)}, \quad \text{for } r \neq 0, N \tag{12-14a}$$

$$\eta = \frac{Sf\{r;\mu_C\} - C}{pmf\{r;\mu_C\}} \text{ (defines } \mu_C\text{)}, \quad \text{for } r \neq 0, N \tag{12-15a}$$

The reason either Eq. 12-14a or 15a can be used is that the *pdf* of η and $1 - \eta$ are the same.

For $r_{C\mu} = 0, N$; Eqs. 12-12 and 12-13 become

$$\left.\begin{array}{ll} \overline{C} = \eta_{C\mu} pmf\{0;\mu\}, & \text{for } \eta_{C\mu} \geq \overline{C} \\ \mu = 0, & \text{for } \eta_{C\mu} < \overline{C} \end{array}\right\} \text{for } r = 0; \tag{12-12c}$$

$$\left.\begin{array}{ll} C = \overline{\eta}_{C\mu}\, bmf\{N;\mu\}, & \text{for } \eta_{C\mu} \leq \overline{C} \\ \mu = \mu_{\max}, & \text{for } \eta_{C\mu} > \overline{C} \end{array}\right\} \text{for } r = N. \tag{12-12d}$$

For $r = 0, N$; Eq. 12-15a becomes

$$\left.\begin{array}{ll} \eta = \dfrac{\overline{C}}{pmf\{0;\mu_C\}}, & \text{for } \eta \geq \overline{C} \\ \mu_C = 0, & \text{for } \eta < \overline{C} \end{array}\right\} \text{for } r = 0; \tag{12-15b}$$

$$\left.\begin{array}{ll} \overline{\eta} = \dfrac{C}{pmf\{N;\mu\}}, & \text{for } \eta \leq \overline{C} \\ \mu_C = \mu_{\max}, & \text{for } \eta > \overline{C} \end{array}\right\} \text{for } r = N. \tag{12-15c}$$

Equations corresponding to Eq. 12-15b and 12-15c follow from Eq. 12-14a.

To show that the μ_C (r, C, η) from Eq. 12-15 satisfies the proper *s*-confidence statement, we calculate Conf $\{\mu \leqslant \mu_C\}$. For $r \neq 0, N$:

$$\text{“}\mu \leq \mu_C(r, C, \eta)\text{” if “}[r > r_{C\mu}] \text{ or } [[r = r_{C\mu}] \text{ and } [\eta \geq \eta_{C\mu}]]\text{”} \tag{12-16}$$

$$\begin{aligned} \text{Conf}\{\mu \leq \mu_C\} &= [Sf\{r_{C\mu};\mu\} - pmf\{r_{C\mu};\mu\}] \\ &\quad + [pmf\{r_{C\mu};\mu\} \times Sf\{\eta_{C\mu}\}] \\ &= Sf\{r_{C\mu};\mu\} - pmf\{r_{C\mu};\mu\} \times Cdf\{\eta_{C\mu}\} \\ &= Sf\{r_{C\mu};\mu\} - pmf\{r_{C\mu};\mu\} \times \eta_{C\mu} \\ &\quad \text{(because } \eta \text{ has the uniform distribution)} \\ &= C \quad \text{(from Eqs. 12-12b and 12-13).} \end{aligned} \tag{12-17}$$

If $r = 0$, N: Conf $\{\mu \leqslant \mu_C\} = C$ follows directly from Eqs. 12-12c, 12-12d, 12-15b, and 12-15c.

Unfortunately Eqs. 12-14 or 15 are not simple to solve. Thus special tables must be generated for this method to be useful. Such tables have been generated for 2-sided

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s-confidence intervals for the binomial parameter p, and the Poisson parameter μ.

The traditional literature that explains this concept tends to use $(r + \eta_{C\mu})$ as the variable. This renders the understanding more difficult. Although if μ is plotted vs $(r + \eta_{C\mu})$, a continuous curve results which resembles Fig. 12-1.

12-5 MORE COMPLICATED *s*-CONFIDENCE SITUATIONS

Easy *s*-confidence statements are feasible for more complicated situations such as Eqs. 12-1a and 12-1b because it is easy to generate the appropriate random variable.

Joint *s*-confidence statements for several parameters in a *PrD* are very complicated, and are virtually impossible to make in any practical situation.

It is fairly easy to find pathological situations where the explanation of *s*-confidence given in this chapter does not apply. Those situations rarely, if ever, arise in reliability engineering.

REFERENCES

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2. Kendall and Stuart, *The Advanced Theory of Statistics,* Vol. 2, *Inference and Relationship* 3rd ed., Hafner Publishing Co., NY, 1973.

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CHAPTER 13

PLOTTING POSITIONS

13-0 LIST OF SYMBOLS

Cdf = Cumulative distribution function

Conf $\{\cdot\}$ = *s*-Confidence level of $\{\cdot\}$

H = cumulative hazard

i = ordered failure number, $i = 1, \ldots, N$

j = reverse order statistic, $j = N, \ldots, 1$

K-S = Kolmogorov-Smirnoff

N = sample size

PrD = Probability distribution

s- = denotes statistical definition

Sf = survivor function

13-1 INTRODUCTION

Any graphical method of analyzing sample data requires a plotting position for each sample point, i.e., the probability to be associated with each data point must be determined. There is no "right" method. Some are more convenient than others; some show the uncertainties in the data better; and some have been shown to have special statistical properties.

Graphical methods are not precise; they contain a great deal of subjectivity and nonrepeatability. These characteristics are not necessarily bad, but they must be recognized. For example, if greatly different answers are obtained from each of the popular plotting-position methods, then the data analysis is in trouble regardless of the plotting-position method. Generally speaking, the uncertainties due to small sample size swamp out the uncertainties due to the various plotting positions.

Plotting positions that use only point estimates ought to be avoided, since a most important use of statistics is to estimate the uncertainty in an answer. The two methods (mentioned here) which encourage estimates of the uncertainty are the sample *Cdf* (with K-S limits) and the percentile ranges.

When sample distributions are plotted it is often convenient to use one of the special probability papers such as *s*-normal or Weibull. Even if the theoretical *PrD* will not be a straight line on the paper, the special paper usually makes the theoretical *PrD* straighter than it would have been on linear paper.

13-2 SAMPLE *Cdf*

Notation:

N = number of items put on test

i = the ith order statistic

d = Kolmogorov-Smirnoff statistic

Failure i is plotted at a probability of i/N. For example, if $N = 10$, the second failure ($i = 2$) is plotted at 20%.

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The uncertainty in plotting position is introduced by means of the K-S statistic. See par. 14-3 for an explanation of this statistic and tables for its use. In Chapter 13 the plotting position is considered to lie between "$(i/N) + d$" and "$(i/N) - d$". If $N = 10$, and the desired *s*-confidence level is 90%, then $d = 0.368$. Thus failure #4 would be plotted in the range "$0.400 - 0.368 = 0.032$" and "$0.400 + 0.368 = 0.768$". The following *s*-confidence statement is true for all i, irrespective of the actual *PrD*.

$$\text{Conf}\left\{\frac{i}{N} - d_{C,N} \leq Cdf\,\{x_i\} \leq \frac{i}{N} + d_{C,N}\right\} = C \tag{13-1}$$

where

x_i = ith order statistic

C = *s*-confidence level

$d_{C,N}$ = K-S statistic for sample size N and *s*-confidence C

Because $i/N - d_{C,N}$ and $i/N + d_{C,N}$ often can lie outside the interval [0, 1], the limits often are written as

$$\text{lower limit} = \max\left\{0,\ \frac{i}{N} - d_{C,N}\right\} \tag{13-2a}$$

$$\text{upper limit} = \min\left\{1,\ \frac{i}{N} + d_{C,N}\right\} \tag{13-2b}$$

As shown in the example in this paragraph, for $N = 10$, $i = 4$, the 90% *s*-confidence limits are 0.032 and 0.768. They are discouragingly wide. They show why it doesn't pay to fool around trying to get the best fit to the sample data—the sample data rarely fall on the true *PrD* very well at all.

Statisticians occasionally recommend not using this test because it is so broad and is especially discouraging in the tail region. But their alternatives assume that much more is known about the data than is usually the case.

The sample *Cdf* with K-S limits is always a very sobering method for plotting probability data. See *Part Three* for more detail.

13-3 PERCENTILE RANGES

The *PrD* of order statistics is well known (see Chapter 10). It is useful to use percentiles of this *PrD* for plotting positions. Table 13-1 gives these plotting positions for the 5th, 50th, 95th percentiles. It covers many values of n. These percentiles refer to the *s*-confidence that any one true value will fall within the range. They are not joint *s*-confidence levels.

These bands of uncertainty are discouragingly wide. They illustrate how little is known from a sample and how important it is to make interval estimates. This method requires large tables; so, often the sample *Cdf* with K-S limits (see par. 13-2) is preferred.

The median (50%) plotting position often is used by itself when quick plotting must be done. For example, one may wish to get a starting value for an iterative analytic solution. The median plotting position is given approximately by Eq. 13-3.

$$pp_{50\%} \approx (i - 0.3)/(N + 0.4) \tag{13-3}$$

where

N = sample size

i = ordered failure number

Eq. 13-3 is much more accurate than $(i - 0.5)/N$ which is occasionally used.

13-4 MEAN

The *PrD* of the ith order statistic (mentioned in Par. 13-3) has a very simple mean that often is used as the plotting position when quick results are desired (usually to be followed by a more precise complete analytic solution). The mean plotting position is given by Eq. 13-4.

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TABLE 13-1.
PERCENTILE RANGES FOR PLOTTING POINTS
Beta Bounds Method Percentage Plotting Points of the *k*-th ordered-failure, Out of a Total Sample of N

N/k	3	4	5	6	7	8	9	10	11
1	1.7/21/63	1.2/16/53	1.0/13/45	0.85/11/39	0.74/9.4/35	0.65/8.3/31	0.57/7.4/28	0.51/6.7/26	0.47/6.1/24
2	14/50/86	9.8/39/75	7.6/31/66	6.3/27/58	5.3/23/52	4.7/20/47	4.1/18/43	3.7/16/39	3.3/15/36
3			19/50/81	15/42/73	13/36/66	11/32/60	9.8/29/55	8.7/26/51	8.0/24/47
4					23/50/77	19/44/71	17/39/66	15/36/61	14/32/56
5							25/50/75	22/45/70	20/41/65
6									27/50/73

N/k	12	13	14	15	16	17	18	19	20
1	0.43/5.6/22	0.40/5.2/21	0.37/4.8/19	0.34/4.5/18	0.32/4.2/17	0.30/4.0/16	0.29/3.0/16	0.28/3.6/15	0.26/3.4/14
2	3.1/14/34	2.8/13/32	2.6/12/30	2.5/11/28	2.3/10/26	2.2/9.8/25	2.1/9.2/24	1.9/8.7/23	1.8/8.3/22
3	7.2/22/44	6.7/20/41	6.1/19/39	5.7/18/36	5.4/16/34	5.0/16/33	4.8/15/31	4.5/14/30	4.3/13/29
4	12/30/53	11/28/49	10/26/47	9.7/24/44	9.1/23/42	8.5/21/40	8.0/20/38	7.6/19/36	7.3/18/35
5	18/38/61	17/35/57	15/33/54	14/31/51	13/29/48	12/27/46	12/26/44	11/24/42	11/23/40
6	25/46/68	23/43/65	21/40/61	19/37/58	18/35/55	17/33/52	16/31/50	15/29/48	14/28/46
7		29/50/71	27/47/67	25/44/64	23/41/61	21/39/58	20/36/55	19/35/53	18/33/51
8				30/50/70	28/47/67	26/44/65	24/42/61	23/40/58	22/38/56
9						31/50/69	29/47/66	27/45/63	26/43/60
10								32/50/68	30/48/65

N/k	22	24	26	28	30	35	40	45	50
1	0.23/3.1/13	0.21/2.9/12	0.20/2.6/11	0.18/2.5/10	0.17/2.3/9.5	0.15/2.0/8.2	0.13/1.7/7.2	0.11/1.5/6.4	0.10/1.4/5.8
2	1.6/7.5/20	1.5/6.9/18	1.4/6.4/17	1.3/5.9/16	1.2/5.5/15	1.0/4.8/13	0.90/4.2/11	0.79/3.7/10	0.71/3.3/9.1
3	3.8/12/26	3.5/11/24	3.2/10/22	3.0/9.4/21	2.8/8.8/20	2.4/7.6/17	2.1/6.6/15	1.8/5.9/13	1.7/5.3/12
4	6.5/16/32	5.9/15/29	5.4/14/27	5.0/13/25	4.7/12/24	4.0/10/21	3.5/9.1/18	3.1/8.1/16	2.8/7.3/15
5	9.4/21/37	8.6/19/34	7.9/18/32	7.3/16/30	6.8/15/28	5.8/13/24	5.1/12/21	4.5/10/19	4.0/9.3/17
6	13/25/42	11/23/39	11/22/36	9.8/20/34	9.1/19/32	7.7/16/28	6.7/14/25	6.0/13/22	5.4/11/20
7	16/30/47	15/27/43	13/25/40	12/24/38	12/22/36	9.8/19/31	8.5/17/27	7.5/15/25	6.8/13/22
8	20/34/52	18/32/48	16/29/45	15/27/42	14/25/39	12/22/34	10/19/30	9.2/17/27	8.3/15/25
9	23/39/56	21/36/52	19/33/49	18/31/46	17/29/43	14/25/37	12/21/33	11/19/30	9.7/17/27
10	27/43/60	25/40/56	23/37/53	21/34/49	19/32/47	16/27/41	14/24/36	13/21/32	11/19/29
11	31/48/65	28/44/60	26/41/56	24/38/53	22/35/50	19/30/44	16/26/39	14/24/35	13/21/32
12		32/48/64	29/44/60	27/41/57	25/38/53	22/34/47	18/29/41	16/26/37	14/23/34

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TABLE 13-1 (Continued)

	22	24	26	28	30	35	40	45	50
13			33/48/64	30/45/60	28/42/57	24/36/50	20/31/44	18/28/40	16/25/36
14				33/48/63	31/45/60	26/39/52	22/34/47	20/30/42	18/27/38
15					34/48/63	29/42/55	25/36/49	22/32/44	19/29/40
16						31/44/58	27/39/52	24/35/47	21/31/42
17						34/47/61	29/41/54	26/37/49	23/33/44
18						36/50/64	31/44/57	28/39/51	25/35/47
19							34/46/59	30/41/53	26/37/49
20							36/49/62	32/43/56	28/39/51
21								34/46/58	30/41/53
22								36/48/60	32/43/55
23								38/50/62	34/45/57
24									35/47/59
25									38/49/60

The body of the table lists for each (k, N) the 5%/50%/95% points for plotting purposes. To obtain the 5%/50%/95% plotting points for $(N+1-k, N)$ reverse the order from the (k, N) and subtract each from 100%. For example, for $(k, N) = (2, 5)$ the percentage plotting points are 7.6/31/66. For $(N+1-k, N) = (4,5)$, the percentage plotting points are $(100-66)/(100-31)/(100-7.6) = 34/69/92.4$.

Points through $n = 20$ are adapted from Ref. 4

Points above $n = 20$ are adapted from Ref. 26.

All are rounded off to 2 significant figures.

Interpolation for values of N not shown: For k small, interpolate (roughly) on a horizontal line. For values of k near $(N/2)$, interpolate on a diagonal ($k/N \approx$ constant); in that region they are roughly of the form: median plotting-point ± deviation. The deviation is easily calculated from the tabulated values, and the median plotting-point is easily estimated from Eq. 2-5.

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$$pp_{\text{mean}} = i/(N+1) \tag{13-4}$$

13-5 CENSORED DATA (HAZARD PLOTTING)

If the data are simply censored by stopping the test, then Pars. 13-2, 13-3, or 13-4 can be used for plotting positions. If the censoring occurs among the failures, then it is extremely difficult to find the *PrD* of failure times.

Hazard plotting was developed for this situation (credit is usually given to Wayne Nelson). The items are listed in order of their censoring and/or failure times (intermingled). They are given the reverse order statistics j (from N to 1). Each failure is then assigned the observed hazard rate $1/j$. The cumulative hazard H_j is calculated for each failure by summing the $1/j$ (for failures only) up to and including the failure. The plotting position is then

$$Cdf = 1 - \exp(-H_j), \quad Sf = \exp(-H_j). \tag{13-5}$$

Special probability paper can be printed that is labeled with H instead of *Cdf* (or *Sf*). It is difficult to assign an uncertainty to the plotting position, but $H \pm \sqrt{H}$ sometimes is used because of the relationship of H to the Poisson and exponential distributions.

An example of this plotting method is given in *Part Three.*

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CHAPTER 14

GOODNESS-OF-FIT TESTS

14-0 LIST OF SYMBOLS

Cdf = Cumulative distribution function

csqf = *Cdf* of the chi-square distribution

k = number of cells

K-S = Kolmogorov-Smirnoff

n_i = actual number in cell i

N = sample size

X^2 = statistic calculated from the data

s- = denotes statistical definition

μ_i = mean number in cell i

ν = degrees of freedom

$\chi^2_{\nu,P}$ = value of chi-square such that $csqf(\chi^2_{\nu,P};\nu) = P$

14-1 INTRODUCTION

Statisticians are divided on the utility of goodness-of-fit tests, although there is no question about their statistical validity. The question is on their utility. In almost any sampling situation, two extremes are possible:

(1) Take so few data that no hypothesis will be rejected.

(2) Take so many data that any hypothesis will be rejected.

Generally it is considered unwise to use goodness-of-fit tests as anything more than a very crude means to decide which *PrD* family to use. Samples are so varied, even from the same *PrD*, that one can look very foolish by trying to get more information from a sample than is there.

The two goodness-of-fit tests discussed in this chapter calculate a statistic from the data and compare it with the *PrD* of that statistic. Usual procedure is to see if the sample statistic is too large; if it is too large, the fit is regarded as inadequate. It is very worthwhile checking to see if the fit is "too good". If the fit is fortuitously very good (sample statistic is very small), there is a reasonable possibility that the sampling procedure was not as random as was planned. For example, someone may have massaged the data to make them look better.

14-2 CHI-SQUARE

The data are put into cells. The actual number in each cell is compared with the *s*-expected number for that cell. The numbers are combined into a statistic which has, asymptotically, a chi-square distribution.

14-2.1 DISCRETE RANDOM VARIABLES

The data fall naturally into cells—the discrete values of the random variable. For a large sample, the number in cell i, n_i, can

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reasonably be represented as a Poisson situation with mean μ_i for cell i. The standard deviation of n_i is $\sqrt{\mu_i}$. If μ_i is large, n_i has a s-normal distribution with mean μ_i and standard deviation $\sqrt{\mu_i}$. The n_i is converted to a standard s-normal variate by the transformation

$$z_i \equiv (n_i - \mu_i)/\sqrt{\mu_i} \qquad (14\text{-}1)$$

The sum of squares of ν s-independent standard, s-normal variables has a chi-square distribution with ν degrees of freedom. Suppose there are k cells (values of the discrete random variable), e.g., for the usual pair of dice, there are 11 cells: the numbers 2, 3, 4, ..., 11, 12. If sample size is fixed and known, only $(k - 1)$ of the z_i are s-independent, because if the first $k - 1$ are known, the kth can be calculated from the data. Therefore $\nu = k - 1$, and the statistic

$$X^2 \equiv \sum_{i=1}^{k} z_i^2 = \sum_{i=1}^{k} (n_i - \mu_i)^2/\mu_i \qquad (14\text{-}2)$$

has a chi-square distribution with $\nu = (k - 1)$ degrees of freedom. If any of the *PrD* parameters are estimated from the data, ν usually is reduced by the number of parameters so estimated.

Conventional wisdom suggests that $\mu_i \geq 5$ for all cells, otherwise cells ought to be combined. Simulation has shown that this is too strict. If fewer than 1/5 of the μ_i are less than 5 and none are less than 1, reasonable results will be obtained.

The previous heuristic description of the source of the statistic is not rigorous, but it helps in remembering how to calculate the statistic and what its limitations are.

X^2 in Eq. 14-2 is compared with $\chi^2_{\nu,P}$ where $csqf(\chi^2_{\nu,P};\ \nu) = P$, and P is some reasonably large percentage, e.g., 95%. If $X^2 > \chi^2_{\nu,P}$, the fit is regarded as too poor. X^2 also is compared with a $\chi^2_{\nu,P}$ where P is a reasonably small percentage, e.g., 5%. If $X^2 < \chi^2_{\nu,P}$, the fit is regarded as suspiciously good and the source of the data is investigated.

Example No. 1. A single coin was flipped 10 times; the results were 2 heads, 8 tails. Was the combination of coin and flipping-method a fair one; i.e., is this result reasonable when the expectation is 50%-50%?

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Procedure	Example
1. Calculate number in each cell. State degrees of freedom.	1. $n_H = 2$ $n_T = 8$ $\nu = 2 - 1 = 1$
2. Calculate X^2 from Eq. 14-2.	2. see table below
3. In Table 6-1, find $\chi^2_{1,95\%}$ and $\chi^2_{1,5\%}$	3. $\chi^2_{1,95\%} = 3.85$ $\chi^2_{1,5\%} = .0039$
4. State a conclusion.	4. The data are not "too good"; so there is no difficulty there. The data are poor, but they are that poor over 5% of the time. So one might be suspicious of the fairness of the procedure, but that is all.

Example, step 2:

	n	μ	$(n-\mu)^2/\mu$
H	2	5	1.8
T	8	5	1.8
Total	10	10	3.6

(The data actually were acquired with a nominally fair coin and method of flipping.)

Example No. 2. A pair of dice were rolled 72 times; the results are given in column 1 of Table 14-1. Is the combination of dice and rolling method fair; i.e., is this a reasonable result if the expected values are as shown in column 2 of the table?

TABLE 14-1

DATA FOR EXAMPLE NO. 2

all	#1 n_i	#2 μ_i	#3 $(n_i-\mu_i)^2/\mu_i$	all	#1 n_i	#2 μ_i	#3 $(n_i-\mu_i)^2/\mu_i$
2	2	2	0.00	8	10	10	0.00
3	5	4	0.25	9	5	8	1.13
4	8	6	0.67	10	7	6	0.17
5	8	8	0.00	11	5	4	0.25
6	12	10	0.40	12	1	2	0.50
7	9	12	0.75	Total	72	72	4.12

$\nu = 11 - 1 = 10$

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Procedure	Example
1. Put the data and s-expected values in a table.	1. See Table 14-1, columns 1 and 2. The usual assumptions about dice were made. $\nu = 11 - 1 = 10$
2. Calculate the individual terms in Eq. 14-2; then find X^2.	2. See column 3. $X^2 = 4.12$
3. In Table 6-1, find $\chi^2_{10,95\%}$ and $\chi^2_{10,5\%}$.	3. $\chi^2_{10,95\%} = 18.3$ $\chi^2_{10,5\%} = 3.94$
4. State a conclusion.	4. Since the X^2 is so low, the data are almost suspiciously good. Certainly, X^2 is not too large

(The data actually were acquired with a nominally fair pair of dice and rolling method.)

14-2.2 CONTINUOUS RANDOM VARIABLE

The basic theory is similar to that for a discrete random variable in par. 14-2.1, except that artificial cells must be set up. In the absence of an otherwise obvious method, the equal probability method has much to recommend it. The cell intervals are adjusted so that a random observation is equally likely to fall in any cell. With this method, the μ_i can be less than 5.0, perhaps even as low as 1.0 or 2.0. An adequate policy is to choose the number of cells (which must be an integer) so that μ_i is just less than 5; e.g., if there are 43 data, calculate 43/5 = 8.6 and round upwards to 9 cells. In this example, it wouldn't hurt to choose $k = 10$ ($\mu_i = 4.3$) because 10 is such an easy number to work with.

If the equal probability method is used, Eq. 14-2 becomes

$$X^2 = \frac{k}{N}\left(\sum_{i=1}^{k} x_i^2\right) - N \tag{14-3}$$

Eq. 14-2 can also be used if it is more convenient.

Example No. 3. A table in the literature is asserted to be random standard s-normal deviates. Pick the first 50 numbers and check that assertion with a chi-square test for goodness-of-fit.

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Procedure	Example
1. Choose the intervals.	1. With $N = 50$ numbers, it is convenient to pick 10 cells, $\nu = 10 - 1 = 9$. The cell boundaries correspond to $gauf(z) = 0.00, 0.10, 0.20, 0.30, \ldots, 0.90, 1.00$. The cell boundaries are shown in column 1 of Step 2.
2a. Prepare a table which shows how many fall in each interval. b. Calculate x_i^2 for each cell, and complete the table.	2. (table below)

Interval	x_i	x_i^2
$-\infty$		
	5	25
−1.28		
	5	25
−0.842		
	8	64
−0.524		
	6	36
−0.253		
	5	25
0		
	0	0
+0.253		
	4	16
+0.524		
	5	25
+0.842		
	6	36
+1.28		
	6	36
$+\infty$		
Total	50	288

$k = 10, N = 50, \nu = 9$

Procedure	Example
3. Calculate X^2 from Eq. 14-3.	3. $X^2 = \frac{10}{50} \times 288 - 50 = 7.60$
4. Find $csqf(X^2; \nu)$ from Table 5-1, and state the conclusion.	4. $csqf(7.60; 9) \approx 43\%$
	5. This is a very average value of X^2. On the basis of this test, it would be difficult to fault the table.

A detailed discussion of the chi-square test is also given in Part Four, par. 2-4.1 of this Handbook series. For sample sizes larger than, say, 20 or 30, this is a reasonably good test, although the K-S test is also quite good (for any sample size).

14-3 KOLMOGOROV-SMIRNOFF

This test for goodness-of-fit compares the sample *Cdf* with the hypothesized *Cdf*. It finds the maximum difference (+ or −) between the two and compares it to a sample statistic. Table 14-2 is a tabulation of the critical values. 1-sided tests can be made, but for most engineering purposes the 2-sided test (given here) is better. It is an excellent test and is rarely, if ever, inappropriate. If parameters of the hypothesized distribution are estimated from the data, the intervals ought to be narrower. See Ref. 1; it is a good general reference on the topic.

Example No. 4. Table 14-3 gives 10 values of a random variable, presumed to be from the uniform distribution on [0, 1]. Are they reasonable values?

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TABLE 14-2

CRITICAL VALUES OF THE KOLMOGOROV-SMIRNOFF TEST STATISTIC

N = sample size, C = s-confidence level, S = s-significance level

N	C = 80% S = 20%	90% 10%	95% 5%	98% 2%	99% 1%
1	.900	.950	.975	.990	.995
2	.684	.776	.842	.900	.929
3	.565	.636	.708	.785	.829
4	.493	.565	.624	.689	.734
5	.447	.509	.563	.627	.669
6	.410	.468	.519	.577	.617
7	.381	.436	.483	.538	.576
8	.358	.410	.454	.507	.542
9	.339	.387	.430	.480	.513
10	.323	.369	.409	.457	.489
11	.308	.352	.391	.437	.468
12	.296	.338	.375	.419	.449
13	.285	.325	.361	.404	.432
14	.275	.314	.349	.390	.418
15	.266	.304	.338	.377	.404
16	.258	.295	.327	.366	.392
17	.250	.286	.318	.355	.381
18	.244	.279	.309	.346	.371
19	.237	.271	.301	.337	.361
20	.232	.265	.294	.329	.352
22	.221	.253	.281	.314	.337
24	.212	.242	.269	.301	.323
26	.204	.233	.259	.290	.311
28	.197	.225	.250	.279	.300
30	.190	.218	.242	.270	.290
32	.184	.211	.234	.262	.281
34	.179	.205	.227	.254	.273
36	.174	.199	.221	.247	.265
38	.170	.194	.215	.241	.258
40	.165	.189	.210	.235	.252
approximation for N > 10	$\frac{1.07}{\sqrt{N+1}}$	$\frac{1.22}{\sqrt{N+1}}$	$\frac{1.36}{\sqrt{N+1}}$	$\frac{1.52}{\sqrt{N+1}}$	$\frac{1.63}{\sqrt{N+1}}$

Notes:

(1) The approximate formula has an error less than about ± 2% of the actual value.

(2) This K-S statistic is compared to the $U_{max} \equiv \max |Cdf_{actual} - Cdf_{hypothesis}|$ for all sample points. If the K-S statistic is no more than U_{max}, the hypothesis is accepted at the appropriate s-confidence level. The Table gives the 2-sided statistic.

(3) This K-S statistic can also be used to put a s-confidence band around a hypothesized Cdf.

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Procedure	Example
1. Prepare the data in a table.	1. See Table 14-3, Cols. 1 and 2.
2. Calculate the sample *Cdf*, and the difference of the Cols. 2 and 3.	2. See Table 14-3, Cols. 3 and 4.
3. Find the max $\{\lvert\text{difference}\rvert\}$.	3. max. diff. = 0.1708
4. From Table 14-2, find the K-S statistic for several *s*-confidence levels.	4. *s*-Confidence level for $N = 10$ 80% 0.323 90% 0.369 95% 0.409
5. Draw a conclusion.	5. The maximum deviation is well within bounds.

TABLE 14-3

DATA FOR EXAMPLE NO. 4

order number	random number	sample *Cdf*	difference
1	0.1080	0.1000	0.0080
2	0.3153	0.2000	0.1153
3	0.4708	0.3000	0.1708
4	0.4885	0.4000	0.0885
5	0.6018	0.5000	0.1018
6	0.6795	0.6000	0.0795
7	0.7548	0.7000	0.0548
8	0.8791	0.8000	0.0791
9	0.9032	0.9000	0.0032
10	0.9961	1.0000	0.0039

Note: For the uniform distribution, $Cdf\{x\} = x$; so column 2 is both x and $Cdf\{x\}$.

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Example No. 5. Pick 8 random samples of 10 points each from the uniform distribution and plot their actual *Cdf* vs uniform *Cdf*. Draw the 90% K-S lines on the graph. See Fig. 14-1. Of the 8 samples, none crossed the 90% K-S lines, although on the average, 1 out of 10 samples will go outside the limits. The best linear fit to any of the lines is probably not one going through the origin with 45 deg slope (the population line). Certainly, all of the lines are quite crooked. Choosing a curved line to go through a set of points would be most inappropriate.

See also the example in Part Four, par. 2-2.6.

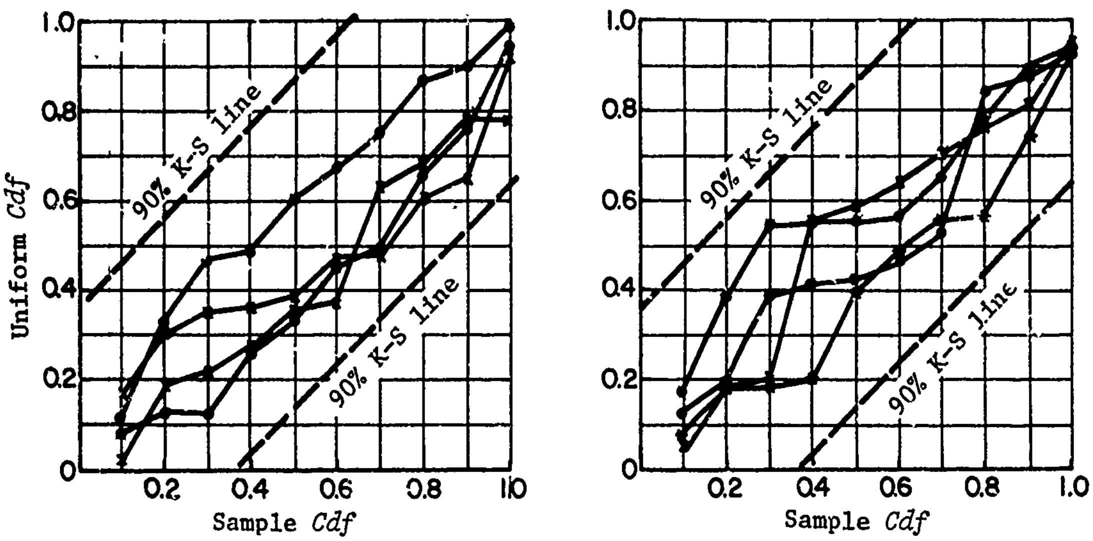

Figure 14-1. Random Samples of 10 from the Uniform Distribution on [0, 1]

REFERENCE

1. L. H. Miller, "Table of Percentage Points of Kolmogorov Statistics", *Journal of Amer. Statistical Assoc.*, Vol. 51, pp. 111-121 (1956).

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CHAPTER 15

TESTS FOR MONOTONIC FAILURE RATES

If it is known that a *PrD* has an increasing failure rate (IFR) or has a decreasing failure rate (DFR), then various other characteristics of these distributions can be proved. This is a field of current research.

Tests for IFR and DFR, and further references are given in Ref. 1 (Sec. 3.4.6). The arithmetic in applying these tests is tedious but straightforward. There are difficulties in interpretation:

(1) The conclusion applies only to the time interval within which data are taken. There is no guarantee that the conclusion applies to the *PrD* for very long times, but it is at very long times that the conclusion is of most interest.

(2) The alternate hypotheses involve only monotonic failure rates. Failure rates that increase then decrease (e.g., a lognormal distribution), or vice versa, are not considered.

(3) It is not clear why a reliability engineer would really want to know this information. Even if he were sure, for example, that a *PrD* had an IFR, he wouldn't know how fast it was increasing. Most of the theorems in the literature are more interesting to the reliability theorist than to reliability engineers.

Before using a test for monotonic failure rates, a statistician ought to be consulted to be sure that the test is not blindly applied and interpreted.

REFERENCE

1. W. Yurkowsky, *Nonelectronic Reliability Notebook,* March 1970, RADC-TR-69-458, AD-868 372.

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CHAPTER 16

BAYESIAN STATISTICS

16-1 INTRODUCTION

This chapter discusses the various Bayesian techniques and their caveats and controversies; it does not give detailed information on their use. A statistician ought to be consulted. There is never any quarrel with the Bayes formula (also called Bayes theorem). It is strongly associated with the very definition of conditional probability.

There are 4 main categories of Bayesian activity (the categories are not necessarily mutually exclusive):

(1) Prior distribution is real and known (no controversy).

(2) Empirical Bayes. Prior distribution is real, but unknown.

(3) Subjective Probability. Probability is used as a measure of degree-of-belief (d-of-b); the prior distribution is one of d-of-b before a particular test is run. This is quite controversial.

(4) Bayesian Decision Theory. This is very difficult to use in practice; so it rarely is used in anything but simple examples in textbooks and articles. If it were used often, it would be very controversial.

16-2 BAYES FORMULA

Suppose the set of possible values for a parameter is discrete and finite: A_i, $i = 1, \ldots, a$. Suppose the possible outcomes of an experiment are the set B_j, $j = 1, \ldots, b$. Then Bayes formula is

$$Pr\{A_i | B_j\} = c\,Pr\{B_j | A_i\} Pr\{A_i\} \qquad (16\text{-}1)$$

where c is a normalizing constant such that

$$\sum\nolimits_i Pr\{A_i | B_j\} = 1$$

$\sum_i$ implies the sum over all i.

If the set of possible values for the parameter is the continuous random variable x, then Bayes formula is

$$pdf\{x | B_j\} = c\,Pr\{B_j | x\}\, pdf\{x\} \qquad (16\text{-}2)$$

where c is a normalizing constant such that $\int_x pdf\{x|B_j\}\,dx = 1$, $\int_x$ implies the integral over all x.

If the possible outcomes of the experiment are the continuous random variable y, then Bayes formula is

$$pdf\{x | y\} = c\,pdf\{y | x\}\, pdf\{x\} \qquad (16\text{-}3)$$

where c is a normalizing constant such that

$$\int_x pdf\{x | y\}dx = 1.$$

There is nothing controversial about any of these formulas, they are straightforward, well-known applications of probability theory. It is in their use that controversy arises.

16-3 INTERPRETATION OF PROBABILITY

Probability is a mathematical concept and

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as such can be applied to anything that fits the constraints of the theory. The two main interpretations are

(1) Relative frequency

(2) Degree-of-belief.

Relative frequency is a straightforward concept and is the classical statistical approach. Degree-of-belief often is associated with Bayesian theory and is a controversial approach. The controversy stems about its subjective nature. A prudent person (by definition) will adjust his d-of-b to correspond to relative frequency where the relative frequency is known. For example, a person who is concerned about the outcomes of honest throws of honest dice would be wise to have his d-of-b the same as the well-known relative frequency for dice.

The proponents of d-of-b argue further that there are many situations where relative frequency is not appropriate since it will never be known. For example, it is d-of-b that one has concerning whether a pair of dice is honest or not. It is d-of-b that can be refined by actually throwing the dice and observing the outcomes.

Degree-of-belief before any tests are run ("prior" d-of-b) is subjective and not reproducible from person to person or even time to time for the same person. To opponents this is a disadvantage; to proponents it is an advantage because it recognizes a fact of life. In many complicated situations it is very difficult to know one's d-of-b. One may even believe mutually contradictory things, especially when the contradiction is not apparent.

Many engineers view the results of their own labors very optimistically. In order for them to use d-of-b fruitfully, they must impose a discipline upon themselves. Otherwise, they will commit the sins that opponents of d-of-b like to talk about. Degree-of-belief is useful, but unless one actively practices the necessary discipline, d-of-b must be avoided.

The discipline has the following steps:

(1) Write down the prior d-of-b.

(2) Run many hypothetical experiments. Calculate the new d-of-b after each hypothetical experiment.

(3) Analyze whether or not each new d-of-b seems reasonable in view of the hypothetical data.

(4) (a) If it does, repeat Steps 2 and 3 until virtually all possible outcomes have been hypothesized.

(b) If it does not, revise the prior d-of-b and go back to Step 2.

Example. Suppose an equipment is being designed and an engineer describes his prior d-of-b about its failure rate in a 1000-hr test.

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Procedure	Example
1. Step 1, state the prior d-of-b.	1. See Table 16-1 rows 1 and 2 For row 1 we presume that the engineer has decided it is reasonable to distinguish between these three failure rates. In practice, one would probably use more; e.g., 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}. For row 2 we presume that the engineer is rather optimistic about his handiwork, i.e., he is just positive it is almost perfect—a very common state of affairs, unfortunately.
2. Step 2, try several hypothetical experiments. Use Eq. 16-1 for calculations.	2. Suppose 5 samples are put on test. Hypothesize 0 failures 1 failure 2 failures. See rows 3, 4, and 5 of Table 16-1 for the calculations. We have presumed that the tests are of the pass-fail type. Thus the terms in the binomial distribution give the probability of the observed results for row 3.
3. Analyze the results (Step 3).	3. For the 0 failure case, the new d-of-b's are reasonable. For the 1 failure case, it is not likely that anyone would still have a 13% d-of-b that the failure-rate was 10^{-5} For the 2 failure case, the results are reasonable; all the d-of-b has shifted to the worst failure rate.
4. Step 4b; try again.	4. See Table 16-2, rows 1 and 2. This time, for row 2 the engineer is less blindly enthusiastic about his work, because he has seen the bad logical consequences of his former (Table 16-1) allocation of d-of-b about the failure rates.
5. Repeat Step 2 of this procedure.	5. 5 samples are put on test. Hypothesize 0 failure 1 failure 2 failures See rows 3, 4, and 5 of Table 16-2 for the calculations.
6. Analyze the results.	6. For 0 failures, the new d-of-b's are reasonable. For 1 failure, the new d-of-b in a 10^{-5} failure rate is lower than before, but still seems too high.

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7. Try again.

7. See Table 16-3, rows 1 and 2. This time, for row 2 the engineer is downright humble about his work, again because he has seen the bad logical consequences of his former (Tables 16-1 and 16-2) allocations of d-of-b about the failure rates.

8. Repeat Step 2 of this procedure.

8. See Table 16-3, rows 3 and 4.

9. Analyze the results.

9. For 0 failures, the new d-of-b's are reasonable. For 1 failure, the new d-of-b's are more reasonable than they were. Perhaps one could live with them. These methods are the way a rational (in the Bayesian sense) person converts prior d-of-b and the test results into a new d-of-b. One cannot go back and change the prior d-of-b after seeing the real actual data. That is why extensive simulation is so necessary. The exact same prior d-of-b that is used to convert the good test results also must be used to convert the bad ones. Anyone who suggests differently is, at best, ill informed. Unfortunately, the Bayesian reliability literature abounds with those bad suggestions about changing the prior d-of-b after seeing the actual data.

The example will not be pursued further, but in practice, the simulation ought to be much more extensive. If one is not prepared to perform extensive simulation on his prior degree-of-belief, he ought to avoid subjective Bayesian analysis altogether.

16-4 PRIOR DISTRIBUTION IS REAL AND KNOWN

This case presents no difficulties, it often is used as an example in textbooks to demonstrate apparent paradoxes about probabilities. It is probably rare that the prior distribution is known, although some work has been done on the *PrD* of actual reliability vs predicted reliability for some military systems. In these applications the validity of the prior distribution can be questioned as far as its future utility is concerned; but this is no different than in many applications of probability. A statistician ought to be consulted. The blind application of formulas can be very misleading.

16-5 EMPIRICAL BAYES

The prior distribution is presumed to be real, but unknown. As samples are taken, they are presumed to illustrate that real prior distribution. Since the real *Cdf* of the prior distribution is only coarsely defined by the data (i.e., by the sample *Cdf*) a smoothing function is employed to estimate the real prior *Cdf*. Once this real prior *Cdf* has been estimated, it is used with the sample data in the same way as a real, known *Cdf* would be.

A statistician ought to be consulted. The choice of a smoothing function is an art, not a science. One may wish to test the hypothesis that the samples do come from *PrD*'s with different parameters. A great deal of engineering and statistical judgment is necessary, and it ought to be made as explicit as possible.

16-6 BAYESIAN DECISION THEORY

The basic tenet is that eventually one wants

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TABLE 16-1

DATA AND RESULTS FOR EXAMPLE — TRIAL NO. 1

1.	a. failure-rate, per 1000 hr	10^{-1}	10^{-3}	10^{-5}
	b. $\bar{R}$, %	9.52	0.100	0.00100
2.	prior d-of-b, %	0.1	0.1	99.8
3.	a. prob. of outcome:			
	0 failures in 5 tries, R^5	0.607	0.995	1.000
	b. product of rows 2 and 3a	6.07×10^{-4}	9.95×10^{-4}	0.998
	c. new d-of-b, %	0.06	0.10	99.84
4.	a. prob. of outcome:			
	1 failure in 5 tries, $5R^4\bar{R}$	0.319	4.98×10^{-3}	5.00×10^{-5}
	b. product of rows 2 and 4a	3.19×10^{-4}	4.98×10^{-6}	4.99×10^{-5}
	c. new d-of-b, %	85.32	1.33	13.35
5.	a. prob. of outcome:			
	2 failures in 5 tries; $10R^3\bar{R}^2$	6.71×10^{-2}	9.96×10^{-6}	1.00×10^{-9}
	b. product of rows 2 and 5a	6.71×10^{-5}	9.96×10^{-9}	9.98×10^{-10}
	c. new d-of-b, %	99.984	0.0148	0.0015

Notation:

d-of-b = degree-of-belief

R = s-reliability, $R = \exp(-\lambda t)$, λ is failure rate, t is 1000 hr for the test

$\bar{R}$ = $1 - R$

Notes:

1. All calculations are made and kept to 10 significant figures, even though they are all rounded off for recording in the table
2. The sum of the d-of-b's is not always exactly 100%, due to rounding errors from Note 1.

TABLE 16-2

DATA AND RESULTS FOR EXAMPLE — TRIAL NO. 2

1.	a. failure rate, per 1000 hr	10^{-1}	10^{-3}	10^{-5}
	b. $\bar{R}$, %	9.52	0.100	0.00100
2.	prior d-of-b, %	0.4	0.6	99.0
3.	a. prob. of outcome:			
	0 failures in 5 tries, $\bar{R}^5$	0.607	0.995	1.000
	b. product of rows 2 and 3a	2.43×10^{-3}	5.97×10^{-3}	0.990
	c. new d-of-b, %	0.24	0.60	99.16
4.	a. prob. of outcome:			
	1 failure in 5 tries, $5R^4\bar{R}$	0.319	4.98×10^{-3}	5.00×10^{-5}
	b. product of rows 2 and 4a	1.28×10^{-3}	2.99×10^{-5}	4.95×10^{-5}
	c. new d-of-b, %	94.14	2.20	3.65
5.	a. prob. of outcome:			
	2 failures in 5 tries, $10R^3\bar{R}^2$	6.71×10^{-2}	9.96×10^{-6}	1.00×10^{-9}
	b. product of rows 2 and 5a	2.68×10^{-4}	5.98×10^{-8}	9.90×10^{-10}
	c. new d-of-b, %	99.977	0.022	0.0004

Notation & Notes: Same as in Table 16-1

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TABLE 16-3

DATA AND RESULTS FOR EXAMPLE — TRIAL NO. 3

1.	a.	failure rate, per 1000 hr	10^{-1}	10^{-3}	10^{-5}
	b.	$\bar{R}$, %	9.52	0.100	0.00100
2.		prior d-of-b, %	1	1	98
3.	a.	prob. of outcome: 0 failures in 5 tries, R^5	0.607	0.995	1.000
	b.	product of rows 2 and 3a	6.07×10^{-3}	9.95×10^{-3}	0.980
	c.	new d-of-b, %	0.61	1.00	98.39
4.	a.	prob. of outcome: 1 failure in 5 tries, $5R^4\bar{R}$	0.319	4.98×10^{-3}	5.00×10^{-5}
	b.	product of rows 2 and 4a	3.19×10^{-3}	4.98×10^{-5}	4.90×10^{-5}
	c.	new d-of-b, %	97.00	1.51	1.49
5.	a.	prob. of outcome: 2 failures in 5 tries, $10R^3\bar{R}^2$	6.71×10^{-2}	9.96×10^{-6}	1.00×10^{-9}
	b.	product of rows 2 and 5a	6.71×10^{-4}	9.96×10^{-8}	9.80×10^{-10}
	c.	new d-of-b, %	99.985	0.0148	0.00015

Notation & Notes: Same as in Table 16-1

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to make a decision that is based on the experimental results. Those results are not of interest in themselves; so why analyze them in detail. The procedure is to list the possible states of nature (e.g., the 3 failure rates in Table 16-1). Then the loss or gain involved in choosing each state when some state is true is estimated. Then a criterion for good decisions is hypothesized (e.g., minimize the worst possible loss, or maximize the s-expected value) and the decision is chosen (given the experimental data) according to the criterion for a good decision.

The argument against this whole process is that there are too many arbitrary assumptions that get lost in the shuffle. The final result appears quite emphatic, but the arbitrariness is hidden from view (perhaps unintentionally) and there is no measure of the uncertainties involved. It is argued that except for the most simple-minded situations of the kind used in textbook examples, applying Bayesian decision theory is impossible.

The arguments for Bayesian decision theory are that it gets all the assumptions out where they can be viewed. The value of more information can be calculated, and a variation analysis can be performed to find the critical variables in the decision. Far from hiding things, it makes everything explicit. The complications merely reflect reality.

One certainly ought not to attempt to use Bayesian decision theory without the services of a very competent statistician (who understands it) and a very competent engineer (who understands it). The odds against its being really productive, rather than pointless or misleading are quite high. So unless there are lots of resources, leave it alone.

16-7 SUBJECTIVE PROBABILITY

The 2 main approaches are:

(1) Use a discrete *PrD* for the random variable about which degree-of-belief statements are to be made.

(2) Use the conjugate prior distribution (continuous). It transforms the simpler situations into very straightforward calculations. For example, for the constant failure rate case, choosing a prior distribution is equivalent to choosing a prior test time and prior number of failures. The pair of sample data (failures, test time) are appropriately added to the prior pair to give the new pair, which will represent the new degree-of-belief. The binomial situation, is similar, except that total-number-tested replaces test-time.

The conjugate failure distribution method ought to be used with caution. The family of prior distributions is quite rich, but it is difficult for an engineer to quantify his information in the necessary way. As mentioned in par. 16-3, extensive simulation of experimental outcomes is necessary. It is quite easy, if no simulation is performed, to make seemingly realistic assumptions about one's prior beliefs, which turn out to be grossly misleading.

The discrete prior distribution is more straightforward, although tedious to calculate. It has many advantages in terms of the visibility of the results. In cases where there are double peaks in the new d-of-b distribution, the engineer is alerted to the fact that choosing a single number for his "best" belief might be misleading.

If either of these approaches is used, a statistician ought to be consulted. The arithmetic is easy enough to do (although sometimes tedious) but the results may be difficult to interpret. Merely because the calculations can be made does not mean they ought to be made.

16-8 RECOMMENDATIONS

Engineers have a great deal of prior

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knowledge. If they did not, no production line would ever work. But engineers do not design, predict, or produce perfectly. A large part of their training is in the directions to try for improvement. Prediction techniques that somehow use an engineer's prior knowledge, without being overly optimistic (or even blindly optimistic), are needed. Under some conditions, and with help from competent statisticians, one of the Bayesian techniques might be fruitful. This is an area of research, not an area for blind calculations.

Cookbook formulas and procedures have been omitted from this chapter on purpose. They are too easy to misuse.

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CHAPTER 17

SAMPLING PLANS

MIL-STD-781 and MIL-STD-105 contain sampling plans that are useful in reliability and quality control. They are not repeated here. Before any sampling plan is used, its operating characteristics ought to be investigated rather thoroughly. If the sample size is not fixed, then the average sample-size and maximum sample-size characterestics ought to be investigated. Much of this is already done in the MIL-STD's.

There is rarely a need to invent new sampling plans. One doesn't really know exactly what operating characteristic he is willing to settle for, and a great deal of arbitrariness exists—enough so that it will usually encompass an already analyzed plan.

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CHAPTER 18

MISCELLANEOUS DESIGN AIDS

Several references have collected many reliability-mathematics design-aids. They are not reprinted here since each requires extensive explanation of its procedures and limitations. Some of the aids become obsolete as the techniques and materiel to which they refer are replaced by newer technologies and analyses. Refs. 1, 2, and 3 are good sources of these aids.

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